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# THE ROLE OF COMMUTATORS IN A NON-CANCELLATION PHENOMENON

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Abstract. We construct an explicit (from the transition function point of view) diffeomorphism between the cartesian products with the 3-sphere of two 10-dimensional loop spaces that are not homotopy equivalent to each other. Our method employs specific models for some $S^3$-principal bundles over $S^7$ and relates the study of this type of non-cancellation phenomena to commutators of groups. Our formulas depend only on specifying homotopies of powers of commutators to constants.

## 1. Introduction

The term non-cancellation phenomenon is used here in the following sense:

Differentiable manifolds $M_1$, $M_2$ and $N$ are given to satisfy

(i) $M_1 \times N$ is diffeomorphic to $M_2 \times N$;
(ii) $M_1 \not\simeq M_2$, where $\simeq$ means same homotopy type.

Charlap in 1965, furnished one of the early examples of such phenomenon in [Ch], where $M_1$, $M_2$ are flat riemannian manifolds and $N = S^1$. Such an example is obtained as a consequence of his classification for $\mathbb{Z}_p$-manifolds (riemannian manifolds with holonomy group equal to $\mathbb{Z}_p$) with $p$ prime.

In 1969, Hilton and Roitberg [HR2] considered the case where $M_1$ and $M_2$ are total spaces of principal bundles and $N$ the corresponding structural group, more precisely, they consider principal $S^3$-bundles over spheres $S^n$. The principal $S^3$-bundles over $S^n$ are classified by $\pi_n(BS^3) \cong \pi_{n-1}(S^3)$ (cf. [St]). So, we have for each $\alpha \in \pi_{n-1}(S^3)$ the corresponding $S^3$-bundle $S^3 \cdots E_\alpha \xrightarrow{p_\alpha} S^n$ classified by the adjoint $\alpha_0 \in \pi_n(BS^3)$ of $\alpha$. Given $\alpha, \beta \in \pi_{n-1}(S^3)$ let $E_{\alpha\beta}$ be the principal $S^3$-bundle over $E_\alpha$ induced from the bundle $E_\beta$ by the projection $p_\alpha : E_\alpha \longrightarrow S^n$. We have in this way the following commutative diagram:

$$\begin{array}{ccccc}
S^3 & & S^3 & & \\
\vdots & & \vdots & & \\
E_{\alpha\beta} & \longrightarrow & E_\beta & & \\
\downarrow & & \downarrow{\scriptstyle p_\beta} & & \\
E_\alpha & \xrightarrow{p_\alpha} & S^n & \xrightarrow{\beta_0} & BS^3.
\end{array}$$

diagram 1

**Theorem 1** (Hilton-Roitberg [HR2]).

i) $E_\alpha \simeq E_\beta \Longleftrightarrow \alpha = \pm\beta$;
ii) *Let* $\alpha \in \pi_{n-1}(S^3)$ *be an element of order* $k$ *and* $\beta = l\alpha$, $l \in \mathbb{Z}$. *If there exists* $l'$, $l' \equiv l \bmod k$, *such that*

$$\frac{l'(l'-1)}{2}\omega \circ \Sigma^3\alpha = 0 \in \pi_{n+2}(S^3),$$

*where* $\omega \in \pi_6(S^3)$ *is the generator,* $\Sigma^3$ *is the* 3*-fold suspension of* $\alpha$, *then the bundle* $S^3 \cdots E_{\alpha\beta} \to E_\alpha$ *is trivial.*

We observe that by the construction of the induced bundle it follows easily that $E_{\alpha\beta} = E_{\beta\alpha}$. This observation together with the above theorem give the following example of non-cancellation phenomena:

Consider $S^3$-bundles over $S^7$, $M_1 = E_\alpha$, where $\alpha = \omega \in \pi_6(S^3) \cong \mathbb{Z}_{12}$ is a generator, and $M_2 = E_\beta$ with $\beta = 7\alpha$. Since $\frac{7(7-1)}{2}\omega \circ \Sigma^3\alpha = 21\omega \circ \Sigma^3\alpha = 0$ by $\pi_9(S^3) \cong \mathbb{Z}_3$, similarly as $\alpha = 7\beta$ we also have $\frac{7(7-1)}{2}\omega \circ \Sigma^3\beta = 0$. As $E_\omega$ is the canonical $S^3$-bundle $Sp(2)$ over $S^7$ we have now

$$Sp(2) \times S^3 = E_{7\omega} \times S^3 \quad \text{and} \quad Sp(2) \not\simeq E_{7\omega}. \tag{1}$$

This example is also relevant in a different context:

$Sp(2)$ and $E_{7\omega}$ are the only total spaces of principal $S^3$-bundles over $S^7$, up to orientation, that admit a loop-space structure (cf. [HMR2], [CM] or [Z]). This, together with the second part of (1), tells us that $Sp(2)$ and $E_{7\omega}$ are $H$-spaces with distinct $H$-structures.

In the examples above $M_1$, $M_2$ and $N$ are at most 2-connected. In 1972 Hilton, Mislin and Roitberg [HMR1] provided examples of $M_1$, $M_2$ and $N$ all arbitrarily highly connected.

These examples of non-cancellation phenomena are obtained by indirect ways, that is, the diffeomorphism (1) is not explicitated.

Hilton and Roitberg ([HR1], [HR2]) consider the cellular decomposition of the spaces $E_\alpha$ ($\alpha \in \pi_{n-1}(S^3)$):

$$E_\alpha = (S^3 \cup_\alpha e^n) \cup e^{n+3} = C_\alpha \cup e^{n+3}.$$

It is shown that under the same conditions in which we have $E_\alpha \times S^3 = E_\beta \times S^3$ we also have $C_\alpha \vee S^3 \simeq C_\beta \vee S^3$ (where $\vee$ denotes the one point union) although, in general $E_\alpha \not\simeq E_\beta$ and $C_\alpha \not\simeq C_\beta$. They suggest then a more careful analysis of the diffeomorphism between $E_\alpha \times S^3$ and $E_\beta \times S^3$.

The subject of this paper is to analyse the example of Hilton-Roitberg above, trying to give an idea of the complexity of the diffeomorphism between $Sp(2) \times S^3$ and $E_{7\omega} \times S^3$.

To do this, we worked with the models for principal $S^3$-bundles over $S^7$ denoted in [R] by $\tilde{P}_n$. Such bundles are represented by 10-dimensional submanifolds of $Sp(n)$ and have transition functions $g^n_{VU} : U \cap V \longrightarrow S^3$ relative to an open covering of $S^7$ by just two sets $U = \left\{ \binom{a}{b} \in S^7; a \neq 0 \right\}$ and $V = \left\{ \binom{a}{b} \in S^7; b \neq 0 \right\}$, given by $g^n_{VU}\binom{a}{b} = \frac{b^{n-1}(a\bar{b})^{n-1}\bar{a}^{n-1}}{(|a||b|)^{2(n-1)}}$ and the method used in [R] shows that

$$\text{(2)} \qquad g^n_{VU} \overset{H}{\simeq} 1 \Longrightarrow \begin{array}{l}\text{The bundle } \tilde{P}_n \text{ is trivial and a global section} \\ \text{can be constructed explicitly by means of } H.\end{array}$$

There exists a diffeomorphism $\beta : S^3 \times S^3 \times (0, \frac{\pi}{2}) \longrightarrow U \cap V$ such that $g^n_{VU} \circ \beta(A, B, \theta) = B^{n-1}(A\bar{B})^{n-1}\bar{A}^{n-1}$. Thus, working in the group of homotopy classes of maps $[S^3 \times S^3, S^3]$ we obtained that $B^8(A\bar{B})^8\bar{A}^8 \simeq 1$ and the implication (2) can be realized for $n = 9$, which coincides with the classification of the bundles $\tilde{P}_n$ given in [B2]. We use the same idea to construct a diffeomorphism between $Sp(2) \times S^3$ and $E_{7\omega} \times S^3$. In this case the bundles $E_{\alpha\beta}$ are modeled through the $\tilde{P}_n$'s via the pull-back construction providing the principal $S^3$-bundles $\tilde{P}_{n,m}$ over $\tilde{P}_n$, in such a way that the homotopy classes of transition functions of these bundles are in one to one correspondence with $[S^3 \times S^3 \times S^3, S^3]$, so the method applied above works here but the calculations are much more complicated.

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## 2. The bundles $\tilde{P}_n$ and $\tilde{P}_{n,m}$

Let $M_n = M_n(a, b, x_1, x_2, \ldots, x_n) \in Sp(n)$ $(n \geq 3)$ be given by

$$M_3 = \begin{pmatrix} a & -b|b|^2 & x_1 \\ b & b\bar{a}b & x_2 \\ 0 & a\sqrt{1+|b|^2} & x_3 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} a & -b|b|^2 L(4)^{-1} & 0 & x_1 \\ b & b\bar{a}bL(4)^{-1} & 0 & x_2 \\ 0 & a|a|^2 L(4)^{-1} & -b & x_3 \\ 0 & a\bar{b}aL(4)^{-1} & a & x_4 \end{pmatrix},$$

where $L(4) = \sqrt{|a|^4 + |b|^4}$, and for $n \geq 5$

$$M_n = \begin{pmatrix} a & \frac{-b|b|^2}{L(n)} & 0 & \dots & 0 & 0 & 0 & 0 & x_1 \\ b & \frac{b\bar{a}b}{L(n)} & 0 & \dots & 0 & 0 & 0 & 0 & x_2 \\ 0 & \frac{af_{n-5}}{L(n)} & \frac{-b}{L_{n-4}} & \dots & 0 & 0 & 0 & 0 & x_3 \\ 0 & \frac{(a\bar{b})af_{n-6}}{L(n)} & \frac{af_{n-6}}{L_{n-4}} & \dots & 0 & 0 & 0 & 0 & x_4 \\ 0 & \frac{(a\bar{b})^2af_{n-7}}{L(n)} & \frac{(a\bar{b})af_{n-7}}{L_{n-4}} & \dots & 0 & 0 & 0 & 0 & x_5 \\ 0 & \frac{(a\bar{b})^3af_{n-8}}{L(n)} & \frac{(a\bar{b})^2af_{n-8}}{L_{n-4}} & \dots & 0 & 0 & 0 & 0 & x_6 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{(a\bar{b})^{n-7}af_2}{L(n)} & \frac{(a\bar{b})^{n-8}af_2}{L_{n-4}} & \dots & \frac{-b}{L_3} & 0 & 0 & 0 & x_{n-4} \\ 0 & \frac{(a\bar{b})^{n-6}af_1}{L(n)} & \frac{(a\bar{b})^{n-7}af_1}{L_{n-4}} & \dots & \frac{af_1}{L_3} & \frac{-b}{L_2} & 0 & 0 & x_{n-3} \\ 0 & \frac{(a\bar{b})^{n-5}af_0}{L(n)} & \frac{(a\bar{b})^{n-6}af_0}{L_{n-4}} & \dots & \frac{(a\bar{b})af_0}{L_3} & \frac{af_0}{L_2} & \frac{-b}{L_1} & 0 & x_{n-2} \\ 0 & \frac{(a\bar{b})^{n-4}a|a|^2}{L(n)} & \frac{(a\bar{b})^{n-5}a|a|^2}{L_{n-4}} & \dots & \frac{(a\bar{b})^2a|a|^2}{L_3} & \frac{(a\bar{b})a|a|^2}{L_2} & \frac{a|a|^2}{L_1} & -b & x_{n-1} \\ 0 & \frac{(a\bar{b})^{n-3}a}{L(n)} & \frac{(a\bar{b})^{n-4}a}{L_{n-4}} & \dots & \frac{(a\bar{b})^3a}{L_3} & \frac{(a\bar{b})^2a}{L_2} & \frac{(a\bar{b})a}{L_1} & a & x_n \end{pmatrix},$$

where

$$\begin{aligned} L_1^2 &= |a|^4 + |b|^2, \\ L_k^2 &= |a|^{2(k+1)}(L_1L_2L_3\dots L_{k-1})^2 + |b|^2,\ k = 2,3,4,\dots,n-4, \\ L(n)^2 &= |a|^{2(n-2)}(L_1L_2L_3\dots L_{n-4})^2 + |b|^4, \\ f_0 &= |a|^4, \\ f_k &= |a|^{2(k+2)}(L_1L_2L_3\dots L_{k-1})^2,\ k = 2,3,4,\dots,n-4. \end{aligned}$$

The bundles $S^3 \cdots \tilde{P}_n \xrightarrow{\tilde{p}_n} S^7$ are such that $\tilde{P}_2$ is the canonical $S^3$-bundle $Sp(2)$ over $S^7$ and for $n > 2$, $\tilde{P}_n = \{M \in Sp(n); M = M_n(a,b,x_1,\dots,x_n)$, for certain $a, b, x_i$ in $\mathbb{H}\}$, $\tilde{p}_n(M_n(a,b,x_1,\dots,x_n)) = \binom{a}{b} \in S^7$ and $S^3$ acts on $\tilde{P}_n$ by multiplication on the right over the last column. We have the classification theorem.

**Theorem 2** ([B2]). *If $\omega \in \pi_6(S^3)$ is the preferred generator then*

$$\tilde{P}_n \cong E_{\varphi(n-1)\omega},$$

*where* $\varphi(n) = \binom{n+1}{2} = \frac{(n+1)n}{2}$.

We calculate here the transition functions of the bundles $\tilde{P}_n$ over $S^7$ and of the bundles $E_{\alpha\beta}$ over $E_\alpha$ with respect to a certain open covering of $S^7$ and $E_\alpha$ respectively containing just 2 open sets.

Let $S^7 = \left\{\binom{a}{b} \in \mathbb{H}^2; a\bar{a} + b\bar{b} = 1\right\}$ be the seven sphere and $U$, $V$ the open subsets defined by $U = \left\{\binom{a}{b} \in S^7; a \neq 0\right\}$, $V = \left\{\binom{a}{b} \in S^7; b \neq 0\right\}$. Thus for $n \geq 5$, $\tilde{p}_n^{-1}(U) = \{M_n(a, b, y_1, y_2, \ldots, y_n) \in Sp(n); a \neq 0\}$ and $\tilde{p}_n^{-1}(V) = \{M_n(a, b, z_1, z_2, \ldots, z_n) \in Sp(n); b \neq 0\}$, where

$$\begin{aligned}
y_1 &= -(a\bar{b})|a|^{-2}y_2,\\
|y_2| &= |y_2(a,b)| = |a|^{n-1}(L_1L_2\ldots L_{n-4})L(n)^{-1},\\
y_k &= -(a\bar{b})^{k-1}|a|^{-2(k-1)}(L_{n-(k+1)}L_{n-k}\ldots L_{n-4})^{-2}y_2,\ 3 \leq k \leq n-2,\\
y_{n-1} &= -(a\bar{b})^{n-2}|a|^{-2(n-2)}(L_1L_2\ldots L_{n-4})^{-2}y_2,\\
y_n &= -(a\bar{b})^{n-1}|a|^{-2(n-1)}(L_1L_2\ldots L_{n-4})^{-2}y_2,\\
z_1 &= (b\bar{a})^{n-2}|b|^{-2(n-2)}(L_1L_2\ldots L_{n-4})^{2}z_n,\\
z_2 &= -(b\bar{a})^{n-1}|b|^{-2(n-1)}(L_1L_2\ldots L_{n-4})^{2}z_n,\\
z_k &= (b\bar{a})^{n-k}|b|^{-2(n-k)}(L_1L_2\ldots L_{n-(k+2)})^{2}x_n,\ 3 \leq k \leq n-3,\\
z_{n-2} &= (b\bar{a})^2|b|^{-4}z_n,\\
z_{n-1} &= (b\bar{a})|b|^{-2}z_n,\\
|z_n| &= |z_n(a,b)| = |b|^{n-1}(L_1L_2\ldots L_{n-4}L(n))^{-1}
\end{aligned}$$

are obtained solving the equations $\overline{(\operatorname{col} n)} \cdot (\operatorname{col} i) = 0$ $(i = 1, 2, \ldots, n-1)$ and using the fact that $a \neq 0$ and $b \neq 0$ in $U$ and $V$ respectively (cf. [R]).

We note that an element of $\tilde{p}_n^{-1}(U)$ depends only on the values of $a$, $b$ and $y_2$, thus we can write $M_n(a, b, y_1, y_2, \ldots, y_n) = M_n(a, b, y_2) \in \tilde{p}_n^{-1}(U)$. Similarly, $\tilde{p}_n^{-1}(V)$ depends only on the values of $a$, $b$ and $z_n$, then $M_n(a, b, z_1, z_2, \ldots, z_n) = M_n(a, b, z_n) \in \tilde{p}_n^{-1}(V)$.

We define the partial sections $S_U^n : U \longrightarrow \tilde{p}_n^{-1}(U)$ over $U$ and $S_V^n : V \longrightarrow \tilde{p}_n^{-1}(V)$ over $V$ by

$$S_U^n\begin{pmatrix} a \\ b \end{pmatrix} = M_n(a, b, y_2), \qquad y_2 = \bar{a}^{n-1}(L_1L_2\ldots L_{n-4})L(n)^{-1},$$

$$S_V^n\begin{pmatrix} a \\ b \end{pmatrix} = M_n(a, b, z_n), \qquad z_n = -\bar{b}^{n-1}(L_1L_2\ldots L_{n-4}L(n))^{-1}.$$

Note that $y_2$ and $z_n$ are restricted only by their modules, i.e., they belong to certain $S^3$'s. Their values were chosen so that the transition function $g_{VU}^n$ can be factored through the $S^3 \wedge S^3 = S^6$ (cf. §3).

If $g_{VU}^n : U \cap V \longrightarrow S^3$ is a transition function of the bundle $\tilde{P}_n$ with respect to the open sets $U$ and $V$, then

$$S_V^n\begin{pmatrix} a \\ b \end{pmatrix} g_{VU}^n\begin{pmatrix} a \\ b \end{pmatrix} = S_U^n\begin{pmatrix} a \\ b \end{pmatrix}, \quad \forall \begin{pmatrix} a \\ b \end{pmatrix} \in U \cap V. \tag{3}$$

As the action of $S^3$ on $\tilde{P}_n$ is by multiplication from the right in the last column, we have

$$z_i g^n_{VU}\begin{pmatrix} a \\ b \end{pmatrix} = y_i, \ i = 1, 2, \ldots, n, \tag{4}$$

so, for example

$$z_n g^n_{VU}\begin{pmatrix} a \\ b \end{pmatrix}$$
$$= y_n = -(a\bar{b})^{n-1}|a|^{-2(n-1)}(L_1L_2\ldots L_{n-4})^{-2}\bar{a}^{n-1}(L_1L_2\ldots L_{n-4})L(n)^{-1},$$

hence

$$-\bar{b}^{n-1}(L_1L_2\ldots L_{n-4}L(n))^{-1}g^n_{VU}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{-(a\bar{b})^{n-1}\bar{a}^{n-1}}{(L_1L_2\ldots L_{n-4})|a|^{2(n-1)}L(n)},$$

thus

$$-b^{n-1}(-\bar{b}^{n-1})g^n_{VU}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{b^{n-1}(a\bar{b})^{n-1}\bar{a}^{n-1}}{|a|^{2(n-1)}},$$

therefore

$$g^n_{VU}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{b^{n-1}(a\bar{b})^{n-1}\bar{a}^{n-1}}{(|a||b|)^{2(n-1)}}. \tag{5}$$

Analogously we can define sections $S^k_U : U \longrightarrow \tilde{p}_k^{-1}(U)$ and $S^k_V : V \longrightarrow \tilde{p}_k^{-1}(V)$ $(k = 2, 3, 4)$ in such a way that $g^k_{VU}\binom{a}{b} = \frac{b^{k-1}(a\bar{b})^{k-1}\bar{a}^{k-1}}{(|a||b|)^{2(k-1)}}$ is a transition function of $\tilde{P}_k$ with respect to the open sets $U$ and $V$. Thus we have

**Lemma 1.** *If $S^7 = \left\{\binom{a}{b} \in I\!H; a\bar{a} + b\bar{b} = 1\right\}$ is the 7-sphere and $U$, $V$ are the open subsets $U = \left\{\binom{a}{b} \in S^7; a \neq 0\right\}$ and $V = \left\{\binom{a}{b} \in S^7; b \neq 0\right\}$ respectively, then a transition function of the bundle $S^3 \cdots \tilde{P}_n \longrightarrow S^7$ with respect to the open sets $U$ and $V$ is the function $g^n_{VU} : U \cap V \longrightarrow S^3$ given by (5) above.*

We now apply an analogous procedure to obtain transition functions of $E_{\alpha\beta}$.

Let $\tilde{P}_{n,m}$ be the principal $S^3$-bundle over $\tilde{P}_n$ induced from $\tilde{P}_m$ by the projection $\tilde{p}_n : \tilde{P}_n \longrightarrow S^7$. So, we have the commutative diagram:

$$\begin{array}{ccccc}
S^3 & & S^3 & & S^3 \\
\vdots & & \vdots & & \vdots \\
\tilde{P}_{n,m} & \longrightarrow & \tilde{P}_m & \longrightarrow & Sp(2) \\
\downarrow & & \downarrow{\scriptstyle \tilde{p}_m} & & \downarrow \\
\tilde{P}_n & \xrightarrow{\tilde{p}_n} & S^7 & \xrightarrow{\varphi(m-1)\iota_7} & S^7.
\end{array}$$

diagram 2

By the definition of the induced bundle we have $\tilde{P}_{n,m} = \{(M_n, M_m) \in \tilde{P}_n \times \tilde{P}_m; \tilde{p}_n(M_n) = \tilde{p}_m(M_m)\}$. This provides a model for $E_{\varphi(n-1)\omega,\varphi(m-1)\omega}$ where $\omega \in \pi_6(S^3)$ is the preferred generator.

Let us consider the open sets $\tilde{U}_n = \tilde{p}_n^{-1}(U) = \{M_n(a,b,y_2) \in \tilde{P}_n; a \neq 0\}$ and $\tilde{V}_n = \tilde{p}_n^{-1}(V) = \{M_n(a,b,z_n) \in \tilde{P}_n; b \neq 0\}$. For each $k,t \in \mathbb{Z}$ such that $\frac{m-tkn+tk-1}{t}$ is an integer, let us define partial sections $s^{kt}_{\tilde{U}_n} : \tilde{U}_n \longrightarrow \tilde{P}_{n,m}$ over $\tilde{U}_n$ and $s^{kt}_{\tilde{V}_n} : \tilde{V}_n \longrightarrow \tilde{P}_{n,m}$ over $\tilde{V}_n$ by

$$s^{kt}_{\tilde{U}_n}(M_n(a,b,y_2)) = (M_n(a,b,y_2), M_m(a,b,Y_2(k,t))),$$
$$s^{kt}_{\tilde{V}_n}(M_n(a,b,z_n)) = (M_n(a,b,z_n), M_m(a,b,Z_m(k,t))),$$

where

$$Y_2(k,t) = (y_2^k \bar{a}^{\frac{m-tkn+tk-1}{t}})^t \frac{L_1L_2\dots L_{m-4}L(n)^{tk}}{(L_1L_2\dots L_{n-4})^{tk}L(m)},$$
$$Z_m(k,t) = (-1)^{tk+1}(z_n^k \bar{b}^{\frac{m-tkn+tk-1}{t}})^t \frac{(L_1L_2\dots L_{n-4}L(n))^{kt}}{L_1L_2\dots L_{m-4}L(m)}.$$

By using the expression of $y_n$ on page 77, if $M_n(a,b,y_2) = M_n(a,b,z_n)$ over $\tilde{U}_n \cap \tilde{V}_n$ then

$$z_n = \frac{-(a\bar{b})^{n-1}y_2}{|a|^{2(n-1)}(L_1L_2\dots L_{n-4})^2} \Longrightarrow z_n^k = \frac{(-1)^k((a\bar{b})^{n-1}y_2)^k}{|a|^{2k(n-1)}(L_1L_2\dots L_{n-4})^{2k}}. \tag{6}$$

A transition function $g_{n,m,k,t} : \tilde{U}_n \cap \tilde{V}_n \longrightarrow S^3$ of $\tilde{P}_{n,m}$ with respect to the open sets $\tilde{U}_n$ and $\tilde{V}_n$ can be given solving the equation

$$s^{kt}_{\tilde{V}_n}(M).g_{n,m,k,t}(M) = s^{kt}_{\tilde{U}_n}(M), \quad \forall M \in \tilde{U}_n \cap \tilde{V}_n. \tag{7}$$

It follows from the expressions on page 77 that

$$Y_m(k,t) = \frac{-(a\bar{b})^{m-1}(y_2^k \bar{a}^{\frac{m-tkn+tk-1}{t}})^t L(n)^{tk}}{|a|^{2(m-1)}(L_1L_2\dots L_{m-4})(L_1L_2\dots L_{n-4})^{tk}L(m)}. \tag{8}$$

Setting $M_n = M_n(a,b,y_2)$ it follows from (7) that

$$Z_m(k,t)g_{n,m,k,t}(M_n) = Y_m(k,t), \tag{9}$$

hence

$$(-1)^{tk+1}(z_n^k \bar{b}^{\frac{m-tkn+tk-1}{t}})^t \frac{(L_1L_2\dots L_{n-4}L(n))^{kt}}{L_1L_2\dots L_{m-4}L(m)}.g_{n,m,k,t}(M_n)$$
$$= \frac{-(a\bar{b})^{m-1}(y_2^k \bar{a}^{\frac{m-tkn+tk-1}{t}})^t L(n)^{tk}}{|a|^{2(m-1)}(L_1L_2\dots L_{m-4})(L_1L_2\dots L_{n-4})^{tk}L(m)},$$

using (6) we obtain

$$\frac{(-1)^{tk+1}((-1)^k((a\bar{b})^{n-1}y_2)^k\bar{b}^{\frac{m-tkn+tk-1}{t}})^t}{(|a|^{2k(m-1)}(L_1L_2\dots L_{n-4})^{2k})^t}g_{n,m,k,t}(M_n)$$
$$=\frac{-(a\bar{b})^{m-1}(y_2^k\bar{a}^{\frac{m-tkn+tk-1}{t}})^t}{|a|^{2(m-1)}(L_1L_2\dots L_{n-4})^{2tk}},$$

thus, setting $p=\frac{m-ktn+tk-1}{t}$ we have

$$\frac{(-1)^{2kt+1}(|a||b|)^{2kt(n-1)}|y_2|^{2kt}|b|^{2pt}}{|a|^{2kt(n-1)}}g_{n,m,k,t}(M_n)$$
$$=\frac{-(b^p(\bar{y}_2(b\bar{a})^{n-1})^k)^t(a\bar{b})^{m-1}(y_2^k\bar{a}^p)^t}{|a|^{2(m-1)}},$$

therefore

$$g_{n,m,k,t}(M_n)=\frac{(b^{\frac{m-tkn+tk-1}{t}}(\bar{y}_2(b\bar{a})^{n-1})^k)^t(a\bar{b})^{m-1}(y_2^k\bar{a}^{\frac{m-tkn+tk-1}{t}})^t}{|ab|^{2(m-1)}|y_2|^{2tk}}. \tag{10}$$

Evidently, from the definition of the induced bundle we have that

$$\tilde{g}_{n,m}(M_n(a,b,y_2))=g^m_{VU}\circ\tilde{p}_n(M_n(a,b,y_2))=\frac{b^{m-1}(a\bar{b})^{m-1}\bar{a}^{m-1}}{(|a||b|)^{2(m-1)}} \tag{11}$$

is also a transition function of $\tilde{P}_{n,m}$, which coincides with $g_{n,m,0,1}$.

We have in this way the following:

**Lemma 2.** *Let $n$, $m$ be integers greater than* 1*. If $\tilde{U}_n=\{M_n(a,b,y_2)\in\tilde{P}_n; a\neq 0\}$, $\tilde{V}_n=\{M_n(a,b,z_n)\in\tilde{P}_n; b\neq 0\}$ then for all $k,t\in\mathbb{Z}$ such that $\frac{m-tkn+tk-1}{t}\in\mathbb{Z}$ the functions $g_{n,m,k,t}:\tilde{U}_n\cap\tilde{V}_n\longrightarrow S^3$ given by*

$$g_{n,m,k,t}(M_n(a,b,y_2))=\frac{(b^{\frac{m-tkn+tk-1}{t}}(\bar{y}_2(b\bar{a})^{n-1})^k)^t(a\bar{b})^{m-1}(y_2^k\bar{a}^{\frac{m-tkn+tk-1}{t}})^t}{|ab|^{2(m-1)}|y_2|^{2tk}}$$

*are equivalent transition functions (in the sense of equivalent coordinate transformation, Lemma* 2.10 *from Steenrod* [St]*) of the bundle $S^3\cdots\tilde{P}_{n,m}\longrightarrow\tilde{P}_n$.*

## 3. Trivialization of $\tilde{P}_9$

Here we construct a global section of $\tilde{P}_9$ up to a homotopy. To do this we use some elementary results from the theory of nilpotent groups, which we list below, and finally we show how this method can be applied to construct a diffeomorphism between $Sp(2)\times S^3$ and $E_{7\omega}\times S^3$ up to homotopies of powers of commutators.

Let $\Gamma$ be a multiplicative group, given elements $x$ and $y$ in $\Gamma$ we recall that (cf. [Ha]) the **commutator** of $x$ and $y$ is defined by $[x, y] = x^{-1}y^{-1}xy$ and for $x, y$ and $z \in \Gamma$ we have the following properties:

p$_1$: $[x, y]^{-1} = [y, x]$,
p$_2$: $[xy, z] = [x, z][[x, z], y][y, z]$,
p$_3$: $[x, yz] = [x, z][x, y][[x, y], z]$,
p$_4$: $xy = yx[x, y]$,
p$_5$: $xy = [x^{-1}, y^{-1}]yx$.

Given subsets $X$ and $Y$ of a group $\Gamma$, we define $[X, Y]$ as the subgroup of $\Gamma$ generated by every $[x, y] \in \Gamma$ such that $x \in X$ and $y \in Y$.

A group $\Gamma$ is called **nilpotent of class $\leq$ *r*** if there are subgroups $\Gamma_0, \Gamma_1, \dots, \Gamma_r$ of $\Gamma$ such that

$$\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_r = \{1\}, \tag{12}$$

$$\Gamma_i \text{ is a normal subgroup of } \Gamma,\ i = 1, 2, \dots, r, \tag{13}$$

$$[\Gamma_i, \Gamma] \subseteq \Gamma_{i+1},\ i = 0, 1, 2, \dots, r-1. \tag{14}$$

The series of subgroups (12) satisfying (13) and (14) is called **central series** or **central chain**. Observe that (14) is equivalent to $\frac{\Gamma_i}{\Gamma_{i+1}} \subseteq \text{center}(\frac{G}{\Gamma_{i+1}})$, in particular

$$\Gamma \text{ is nilpotent of class} \leq r \Longrightarrow \Gamma_{r-1} \subseteq \text{center}(\Gamma). \tag{15}$$

If $G2$ and $G3$ are nilpotent groups of classes $\leq 2$ and $\leq 3$ respectively with central chains $G2 = \Omega_0 \supseteq \Omega_1 \supseteq \Omega_2 = \{1\}$ and $G3 = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 = \{1\}$, then the following formulas are easily obtained from properties p$_1, \dots,$p$_5$:

N1: $[x, y] = [y^{-1}, x] = [y, x^{-1}] = [x^{-1}, y^{-1}] \quad \forall x, y \in G2$,
N2: $[x, y]^n = [x, y^n] = [x^n, y] \quad \forall x, y \in G2$ and $\forall n \in \mathbb{Z}$,
N3: $(xy)^n = y^n[x, y]^{\varphi(n)}x^n \quad \forall x, y \in G2$, where $\varphi(n) = \binom{n+1}{2} = \frac{n(n+1)}{2}$,
N4: $[x, y^n] = [x, y]^n[[x, y], y]^{\varphi(n-1)} \quad \forall x, y \in G3$ and $n \in \mathbb{N}$,
N5: $[x^n, y] = [x, y]^n[x, [y, x]]^{\varphi(n-1)} \quad \forall x, y \in G3$ and $n \in \mathbb{N}$,
N6: Given $x, y \in G3$ and $n \in \mathbb{Z}$ we have
i) if $n \geq 0$
$(xy)^n = y^n[x^{-1}, [y, x]]^{a(n)}[[x, y], y]^{a(n-1)}[x, y]^{\varphi(n)}x^n$,
ii) if $n < 0$
$(xy)^n = y^n[y^{-1}, [x^{-1}, y^{-1}]]^{a(n)}[[x^{-1}, y^{-1}], x^{-1}]^{a(n-1)}[x^{-1}, y^{-1}]^{\varphi(n)}x^n$,
where $a(n) = \binom{n+2}{3} = \frac{(n+2)(n+1)n}{6}$,
N7: $[[x, y], z][[y, z], x][[z, x], y] = 1 \quad \forall x, y, z \in G3$.

**Remark 1.** The function $\varphi$ in N4, N5 and N6 is the same as that of N3.

To trivialize $\tilde{P}_9$ let us consider the diffeomorphisms $\alpha : U \cap V \longrightarrow S^3 \times S^3 \times (0, \frac{\pi}{2})$ and $\beta : S^3 \times S^3 \times (0, \frac{\pi}{2}) \longrightarrow U \cap V$ given by $\alpha\binom{a}{b} = (\frac{a}{|a|}, \frac{b}{|b|}, \cos^{-1}|a|)$ and $\beta(A, B, \theta) = \binom{\cos\theta A}{\sin\theta B}$, which are mutual inverses (cf. [R]).

Let $S^9_U : U \longrightarrow \tilde{p}_9^{-1}(U)$ and $S^9_V : V \longrightarrow \tilde{p}_9^{-1}(V)$ be the sections given in page 77 above (setting $n = 9$), that is,

$$S^9_U\binom{a}{b} = M_9(a, b, \bar{a}^8(L_1L_2L_3L_4L_5)L(9)^{-1}),$$
$$S^9_V\binom{a}{b} = M_9(a, b, -\bar{b}^8(L_1L_2L_3L_4L_5L(9))^{-1}).$$

$\tilde{P}_9$ has as transition function $g^9_{VU} : U \cap V \longrightarrow S^3$, $g^9_{VU}\binom{a}{b} = \frac{b^8(a\bar{b})^8\bar{a}^8}{(|a||b|)^{16}}$.

We have that $g^9_{VU} \circ \beta(A, B, \theta) = B^8(A\bar{B})^8\bar{A}^8$. As $S^3 \times S^3 \times (0, \frac{\pi}{2}) \simeq S^3 \times S^3$ it follows that $[S^3 \times S^3 \times (0, \frac{\pi}{2}), S^3] \cong [S^3 \times S^3, S^3]$, where $[X, Y]$ denotes the set of homotopy classes of maps from $X$ to $Y$.

For arbitrary positive integers $n_i$ $(i = 1, 2, \ldots, k)$ consider the group $\Gamma = [S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}, G]$ where $S^{n_i}$ is the $n_i$-dimensional sphere and $G$ is a topological group. For $i = 1, 2, \ldots, k$ let $P_i$ be the set of all points in $S^{n_1} \times \cdots \times S^{n_k}$ with at least $k - i$ coordinates equal to the base point and $\Gamma_i$ be the group of homotopy classes of maps $f \in \Gamma$ such that $f|_{P_i}$ is nullhomotopic, then we have

**Theorem 3** (G. W. Whitehead [W1]). *The group $[S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}, G]$ has the central chain $\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_k = \{1\}$ and*

$$\frac{\Gamma_{i-1}}{\Gamma_i} \cong \prod_{|I|=i} \pi_{n(I)}(G),$$

*where $I \subseteq \{1, 2, 3, \ldots, k\}$, $|I| =$ cardinality of $I$ and $n(I) = \sum_{i \in I} n_i$.*

If $\Gamma = [S^3 \times S^3, S^3]$, then $G = S^3$, $k = 2$, $n_1 = n_2 = 3$ and by the result above $\Gamma$ is nilpotent of class $\leq 2$ and has a central chain $\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 = \{1\}$ such that

$$\frac{\Gamma_0}{\Gamma_1} \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad \frac{\Gamma_1}{\Gamma_2} \cong \Gamma_1 \cong \mathbb{Z}_{12}.$$

**Remark 2.** Recently M. Mimura and H. Ōshima (cf. [MO]) described the group structure of $[S^n \times S^m, S^k, \mu_r]$ for $m, n, k \in \{1, 3, 7\}$ and of $[E_\alpha, E_\beta, \mu_0^{(r)}]$ for $\alpha, \beta \in \pi_6(S^3)$, where $\mu_r(x, y) = xy[x, y]^r$ and $\mu_0(x, y) = xy$ are the usual complex, quaternionic or Cayley multiplications and $\mu_0^{(r)}$ is defined similarly for convenient multiplications $\mu_0^{(0)}$. Thus, for example $[S^3 \times S^3, S^3, \mu_0]$ is the group generated by the projections $p_1, p_2 : S^3 \times S^3 \longrightarrow S^3$ and with relations $p_1[p_1, p_2] = [p_1, p_2]p_1$, $p_2[p_1, p_2] = [p_1, p_2]p_2$, $[p_1, p_2]^{12} = 1$.

Given $f : S^3 \times S^3 \longrightarrow S^3$ we denote by $\bar{f} : S^3 \times S^3 \longrightarrow S^3$ the map $\bar{f}(x,y) = \overline{f(x,y)}$ (quaternionic conjugation of $f$) and as in the remark above $p_i : S^3 \times S^3 \longrightarrow S^3$ $(i = 1,2)$ are the projections.

We know that, if $f, g \in \Gamma = [S^3 \times S^3, S^3]$, then $f.g$ is the homotopy class of the product of $f$ by $g$ in $S^3$ with this we observe that $\bar{p}_i = p_i^{-1}$ in $\Gamma$ $(i = 1,2)$ and from N3 we have that $(p_1\bar{p}_2)^n = \bar{p}_2^n[\bar{p}_2, \bar{p}_1]^{\varphi(n)} p_1^n$ so $p_2^n(p_1\bar{p}_2)^n\bar{p}_1^n = [\bar{p}_2, \bar{p}_1]^{\varphi(n)}$ in $\Gamma$.

We observe that $p_2^{n-1}(p_1\bar{p}_2)^{n-1}\bar{p}_1^{n-1} = g_{VU}^n \circ \beta$, and so we conclude

$$g_{VU}^n \circ \beta \simeq [\bar{p}_2, \bar{p}_1]^{\varphi(n-1)}. \tag{16}$$

Thus, $g_{VU}^9 \circ \beta \simeq [\bar{p}_2, \bar{p}_1]^{36}$, as $[\bar{p}_2, \bar{p}_1] \in \Gamma_1 \cong \mathbb{Z}_{12}$ it follows that $g_{VU}^9 \circ \beta \simeq 1$.

Let $F : S^3 \times S^3 \times [0, \frac{\pi}{2}] \longrightarrow S^3$ be a smooth homotopy such that

$$F(A,B,\theta) = \begin{cases} B^8(A\bar{B})^8\bar{A}^8 & \text{if } \theta \in [0, \frac{\pi}{6}] \\ 1 & \text{if } \theta \in [\frac{\pi}{3}, \frac{\pi}{2}], \end{cases}$$

then $S : S^7 \longrightarrow \tilde{P}_9$ given by

$$S\begin{pmatrix} a \\ b \end{pmatrix} = \begin{cases} S_V^9\binom{a}{b} & \text{if } \frac{5\pi}{12} \le \cos^{-1}|a| \le \frac{\pi}{2} \\ S_V^9\binom{a}{b}(F\circ\alpha)\binom{a}{b} & \text{if } \frac{\pi}{12} \le \cos^{-1}|a| \le \frac{5\pi}{12} \\ S_U^9\binom{a}{b} & \text{if } 0 \le \cos^{-1}|a| \le \frac{\pi}{12} \end{cases}$$

is a global section of $\tilde{P}_9$.

A diffeomorphism $\Phi : S^7 \times S^3 \longrightarrow \tilde{P}_9$ is given by

$$\Phi\left(\begin{pmatrix} a \\ b \end{pmatrix}, \nu\right) = M_9(a, b, w_1\nu, w_2\nu, w_3\nu, w_4\nu, w_5\nu, w_6\nu, w_7\nu, w_8\nu, w_9\nu)$$

where

$$w_1 = \begin{cases} -(b\bar{a})^7\bar{b}^8|b|^{-14}(L_1L_2L_3L_4L_5)L(9)^{-1} & \text{if } \frac{5\pi}{12} \le \cos^{-1}|a| \le \frac{\pi}{2} \\ -(b\bar{a})^7\bar{b}^8|b|^{-14}(L_1L_2L_3L_4L_5)L(9)^{-1}(F\circ\alpha)\binom{a}{b} & \text{if } \frac{\pi}{12} \le \cos^{-1}|a| \le \frac{5\pi}{12} \\ -(a\bar{b})\bar{a}^8|a|^{-2}(L_1L_2L_3L_4L_5)L(9)^{-1} & \text{if } 0 \le \cos^{-1}|a| \le \frac{\pi}{12}, \end{cases}$$

$$w_2 = \begin{cases} (b\bar{a})^8\bar{b}^8|b|^{-16}(L_1L_2L_3L_4L_5)L(9)^{-1} & \text{if } \frac{5\pi}{12} \le \cos^{-1}|a| \le \frac{\pi}{2} \\ (b\bar{a})^8\bar{b}^8|b|^{-16}(L_1L_2L_3L_4L_5)L(9)^{-1}(F\circ\alpha)\binom{a}{b} & \text{if } \frac{\pi}{12} \le \cos^{-1}|a| \le \frac{5\pi}{12} \\ \bar{a}^8(L_1L_2L_3L_4L_5)L(9)^{-1} & \text{if } 0 \le \cos^{-1}|a| \le \frac{\pi}{12}, \end{cases}$$

$$w_k = \begin{cases} -(b\bar{a})^{9-k}\bar{b}^8|b|^{-2(9-k)}\frac{(L_1\dots L_{9-(k+2)})}{L_{9-(k+1)}\dots L_5L(9)} & \text{if } \frac{5\pi}{12} \le \cos^{-1}|a| \le \frac{\pi}{2} \\ -(b\bar{a})^{9-k}\bar{b}^8|b|^{-2(9-k)}\frac{(L_1\dots L_{9-(k+2)})}{L_{9-(k+1)}\dots L_5L(9)}(F\circ\alpha)\binom{a}{b} & \text{if } \frac{\pi}{12} \le \cos^{-1}|a| \le \frac{5\pi}{12} \\ -(a\bar{b})^{k-1}\bar{a}^8|a|^{-2(k-1)}\frac{(L_1\dots L_{9-(k+2)})}{L_{9-(k+1)}\dots L_5L(9)} & \text{if } 0 \le \cos^{-1}|a| \le \frac{\pi}{12} \end{cases}$$

for $3 \leq k \leq 6$,

$$w_7 = \begin{cases} -(b\bar{a})^2\bar{b}^8|b|^{-4}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ -(b\bar{a})^2\bar{b}^8|b|^{-4}(L_1L_2L_3L_4L_5L(9))^{-1}(F\circ\alpha)\binom{a}{b} & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})^6\bar{a}^8|a|^{-12}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

$$w_8 = \begin{cases} -(b\bar{a})\bar{b}^8|b|^{-2}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ -(b\bar{a})\bar{b}^8|b|^{-2}(L_1L_2L_3L_4L_5L(9))^{-1}(F\circ\alpha)\binom{a}{b} & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})^7\bar{a}^8|a|^{-14}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

$$w_9 = \begin{cases} -\bar{b}^8(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ -\bar{b}^8(L_1L_2L_3L_4L_5L(9))^{-1}(F\circ\alpha)\binom{a}{b} & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})^8\bar{a}^8|a|^{-16}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}. \end{cases}$$

The reasoning employed above shows us that if the transition function $g^n_{VU}$ is null-homotopic then the bundle $\tilde{P}_n$ is trivial.

Let us consider the following commutative diagram:

$$\begin{tikzcd} S^3 \times S^3 \arrow[r, "{[\bar{p}_2,\bar{p}_1]}"] \arrow[d, "\wedge"'] & S^3 \\ S^6 = S^3 \wedge S^3 \arrow[r, "\mathrm{id}"] & S^3 \wedge S^3, \arrow[u, "\omega"'] \end{tikzcd}$$

diagram 3

that is, $\omega$ is defined here by $\omega(A \wedge B) = [\bar{B}, \bar{A}] = BA\bar{B}\bar{A}$.

**Remark 3.** It is well known that $\omega$ defined above is a generator of $\pi_6(S^3)$ (cf. [J], [Mc] or remark 2). With the aid of this fact and (16) above we note that the transition functions $g^n_{VU} : U \cap V \longrightarrow S^3$, $g^n_{VU}\binom{a}{b} = \frac{b^{n-1}(a\bar{b})^{n-1}\bar{a}^{n-1}}{(|a||b|)^{2(n-1)}}$ of $\tilde{P}_n$ are such that $g^n_{VU} \circ \beta : S^3 \times S^3 \longrightarrow S^3$ all factor through $S^3 \wedge S^3$ where $\beta$ is the diffeomorphism between $S^3 \times S^3 \times (0, \frac{\pi}{2})$ and $U \cap V$ given above, that is, there exists $\omega_n : S^3 \wedge S^3 \longrightarrow S^3$ such that $g^n_{UV} \circ \beta = \omega_n \circ \wedge$. Moreover if the equivalence class of the transition function $g^n_{VU}$ classifies the bundle $\xi$ then the homotopy class of $\omega_n$ in $\pi_6(S^3)$ also classifies the same bundle $\xi$. We have thus the following homotopy-commutative diagram:

$$\begin{tikzcd} S^3 \times S^3 \arrow[r, "g^n_{VU}\circ\delta"] \arrow[d, "\wedge"] & S^3 \arrow[d, equal] \\ S^3 \wedge S^3 \arrow[r, "\omega_n"] & S^3, \end{tikzcd}$$

diagram 4

where $\omega_n(A \wedge B) \simeq [\bar{B}, \bar{A}]^{\varphi(n-1)}$.

**Remark 4.** Recently Carlos E. Duran [D] has exhibited the following a priori smooth formula for the Blakers-Massey element $\omega : S^6 \subseteq \mathrm{Im}(\mathbb{H}) \oplus \mathbb{H} \longrightarrow S^3$,

$$\omega\begin{pmatrix} p \\ u \end{pmatrix} = \begin{cases} \frac{\bar{u}}{|u|}\exp(\pi p)\frac{u}{|u|} & \text{if } u \neq 0 \\ -1 & \text{if } u = 0, \end{cases}$$

where $\exp(\theta p) = \cos(\theta|p|) + \sin(\theta|p|)\frac{p}{|p|}$.

## 4. The Diffeomorphism $Sp(2) \times S^3 = E_{7\omega} \times S^3$

Now we try writing explicitly (in terms of transition functions) a diffeomorphism $Sp(2) \times S^3 = E_{7\omega} \times S^3$ following the same steps as above.

**Lemma 3.** *There exists a diffeomorphism* $\delta_n : S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2}) \longrightarrow \tilde{\mathbf{U}}_{\mathbf{n}} \cap \tilde{\mathbf{V}}_{\mathbf{n}}$ *for all* $n \in \mathbb{N}$, $n \geq 2$.

*Proof.* We define $\delta_n$ as follows:

$$\delta_n(A, B, C, \theta) = M_n(\cos\theta A, \sin\theta B, (\cos^{n-1}\theta)(l_1 l_2 \dots l_{n-4})l(n)^{-1}C),$$

where $l_k = L_k$ changing $a$ by $\cos\theta A$, $b$ by $\sin\theta B$ $(1 \leq k \leq n-4)$ and $l(n) = L(n)$ making the same changes. As examples we have

$$\delta_2(A, B, C, \theta) = \begin{pmatrix} \cos\theta A & -\sin\theta A\bar{B}C \\ \sin\theta B & \cos\theta C \end{pmatrix},$$

$$\delta_3(A, B, C, \theta) = \begin{pmatrix} \cos\theta A & -\sin^3\theta B & -\cos\theta\sin\theta\sqrt{1+\sin^2\theta}A\bar{B}C \\ \sin\theta B & \cos\theta\sin^2\theta B\bar{A}B & \cos^2\theta\sqrt{1+\sin^2\theta}C \\ 0 & \cos\theta\sqrt{1+\sin^2\theta}A & -\cos^{-1}\theta\sin\theta A\bar{B}C \end{pmatrix},$$

$$\delta_4(A, B, C, \theta) = \begin{pmatrix} \cos\theta A & -\sin^3\theta Bl^{-1} & 0 & -\cos^2\theta\sin\theta A\bar{B}Cl^{-1} \\ \sin\theta B & \sin^2\theta\cos\theta B\bar{A}Bl^{-1} & 0 & \cos^3\theta Cl^{-1} \\ 0 & \cos^3\theta Al^{-1} & -\sin\theta B & -\cos\theta\sin^2\theta(A\bar{B})^2Cl^{-1} \\ 0 & \cos^2\theta\sin\theta A\bar{B}Al^{-1} & \cos\theta A & -\sin^3\theta(A\bar{B})^3Cl^{-1} \end{pmatrix},$$

where $l = l(4) = \sqrt{\sin^4\theta + \cos^4\theta}$.

We can easily verify then

$$\delta_n^{-1}(M_n(a, b, y_2)) = \left(\frac{a}{|a|}, \frac{b}{|b|}, \frac{y_2}{|y_2|}, \cos^{-1}|a|\right)$$

and with this the Lemma is proved. $\square$

**Lemma 4.** *Let* $f : S^3 \times S^3 \times S^3 \longrightarrow S^3$ *and* $p : S^3 \times S^3 \times S^3 \longrightarrow S^3 \times S^3$ *be continuous functions.*

i) *If* $f = c_2 \circ p$*, where* $c_2(x, y) = [x, y] = x^{-1}y^{-1}xy$*, then the order of* $f$ *in* $[S^3 \times S^3 \times S^3, S^3]$ *is a divisor of* 12*,*

ii) *If* $f = c_3 \circ p$*, where* $c_3(x, y) = [x, [x, y]]$ *or* $[y, [x, y]]$*, then* $f \simeq 1$*.*

*Proof.* It is suffices to observe that, by Theorem 3 applied to the group $[S^3 \times S^3, S^3]$ we have that $c_2$ represents a homotopy class of order 12, and $c_3 \simeq 1$. □

With this Lemma, we note that for $x, y \in G_3 = [S^3 \times S^3 \times S^3, S^3]$ we have

$$[x, y] = x^{-1}y^{-1}xy = x^{-1}xy^{-1}[y^{-1}, x]y = y^{-1}y[y^{-1}, x][[y^{-1}, x], y] = [y^{-1}, x].$$

In a similar manner as in N1 we obtain

**Lemma 5.** *Given* $x, y \in G_3 = [S^3 \times S^3 \times S^3, S^3]$*, then*

$$[x, y] = [y^{-1}, x] = [y, x^{-1}] = [x^{-1}, y^{-1}].$$

With the aid of the last two lemmas we observe that N4, N5 and N6 applied to the group $G_3 = [S^3 \times S^3 \times S^3, S^3]$ transform to Lemma 6 and Theorem 4 bellow.

**Lemma 6.** *Given* $x, y \in G_3 = [S^3 \times S^3 \times S^3, S^3]$*, then*

$$[x, y]^n = [x^n, y] = [x, y^n], \ n \in \mathbb{Z}.$$

*Proof.* For $n \geq 0$ the result follows directly from N1 and N5. If $n < 0$ we have $[x, y]^n = [y, x]^{-n} = [y^{-n}, x] = [y, x^{-n}]$, but $[y^{-n}, x] = [(y^n)^{-1}, x] = [x, y^n]$ and $[y, x^{-n}] = [y, (x^n)^{-1}] = [x^n, y]$, hence the result follows. □

**Theorem 4.** *Given* $x, y \in G_3 = [S^3 \times S^3 \times S^3, S^3]$*, then*

$$(xy)^n = y^n[x, y]^{\varphi(n)}x^n, \ n \in \mathbb{Z}.$$

We observe that $[S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2}), S^3] = [S^3 \times S^3 \times S^3, S^3]$. It follows from Theorem 3 that $\Gamma = [S^3 \times S^3 \times S^3, S^3]$ is a nilpotent group of class $\leq 3$ with central chain $\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 = \{1\}$ such that

$$\frac{\Gamma_0}{\Gamma_1} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad \frac{\Gamma_1}{\Gamma_2} \cong \mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{12}, \quad \frac{\Gamma_2}{\Gamma_3} = \Gamma_2 \cong \mathbb{Z}_3.$$

We denote by $A$, $B$, $C : S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2}) \longrightarrow S^3$ the projections $A(x_1, x_2, x_3, \theta) = x_1$, $B(x_1, x_2, x_3, \theta) = x_2$, $C(x_1, x_2, x_3, \theta) = x_3$, and by $\bar{A}$, $\bar{B}$, $\bar{C}$ the quaternionic conjugates of $A$, $B$, $C$ respectively. To simplify the notation we set $p = \frac{m-tkn+tk-1}{t}$ and using the results above we obtain

**Theorem 5.** *The transition function $g_{n,m,k,t}\circ\delta_n$ can be written as a product of simple and double commutators, namely,*

$$g_{n,m,k,t}\circ\delta_n \simeq [C,B]^{e_1}[B,A]^{e_2}[A,C]^{e_1}[[B,C],A]^{f} \text{ for certain } e_1,e_2,f\in\mathbb{Z}.$$

*Proof.*

$$\begin{aligned}
&g_{n,m,k,t}\circ\delta_n\\
&\quad= (B^p(\bar{C}(B\bar{A})^{n-1})^k)^t(A\bar{B})^{m-1}(C^k\bar{A}^p)^t\\
&\underset{\text{(T. 4)}}{\simeq} (\bar{C}(B\bar{A})^{n-1})^{tk}[B^p,(\bar{C}(B\bar{A})^{n-1})^k]^{\varphi(t)}B^{pt}(A\bar{B})^{m-1}\bar{A}^{pt}\\
&\qquad [C^k,\bar{A}^p]^{\varphi(t)}C^{tk}\\
&\underset{\text{(L. 4)}}{\simeq} [B^p,(\bar{C}(B\bar{A})^{n-1})^k]^{\varphi(t)}(\bar{C}(B\bar{A})^{n-1})^{tk}B^{pt}(A\bar{B})^{m-1}\bar{A}^{pt}\\
&\qquad [C^k,\bar{A}^p]^{\varphi(t)}C^{kt}\\
&\underset{\text{(T. 4 and L. 6)}}{\simeq} [B,\bar{C}(B\bar{A})^{n-1}]^{k\varphi(t)p}(B\bar{A})^{tk(n-1)}[\bar{C},(B\bar{A})^{n-1}]^{\varphi(tk)}\bar{C}^{tk}B^{pt}\\
&\qquad (A\bar{B})^{m-1}\bar{A}^{pt}[C,\bar{A}]^{k\varphi(t)p}C^{tk}\\
&\underset{\text{(T. 4)}}{\simeq} [B,\bar{C}(B\bar{A})^{n-1}]^{k\varphi(t)p}[\bar{C},(B\bar{A})^{n-1}]^{\varphi(tk)}(B\bar{A})^{tk(n-1)}\bar{C}^{tk}B^{pt}\\
&\qquad (A\bar{B})^{m-1}\bar{A}^{pt}[C,\bar{A}]^{k\varphi(t)p}C^{tk}\\
&\underset{(p_4)}{\simeq} [B,\bar{C}(B\bar{A})^{n-1}]^{k\varphi(t)p}[\bar{C},(B\bar{A})^{n-1}]^{\varphi(tk)}\bar{C}^{tk}(B\bar{A})^{tk(n-1)}\\
&\qquad [(B\bar{A})^{tk(n-1)},\bar{C}^{tk}]B^{pt}(A\bar{B})^{m-1}\bar{A}^{pt}[C,\bar{A}]^{k\varphi(t)p}C^{tk}\\
&\underset{\text{(T. 4)}}{\simeq} [B,\bar{C}(B\bar{A})^{n-1}]^{k\varphi(t)p}[\bar{C},(B\bar{A})^{n-1}]^{\varphi(tk)}\bar{C}^{tk}[(B\bar{A})^{tk(n-1)},\bar{C}^{tk}]\\
&\qquad (B\bar{A})^{tk(n-1)}B^{pt}(A\bar{B})^{m-1}\bar{A}^{pt}[C,\bar{A}]^{k\varphi(t)p}C^{tk}\\
&\underset{(p_3,\ p_4\text{ and L. 6})}{\simeq} ([B,(B\bar{A})^{n-1}][B,\bar{C}][[B,\bar{C}],(B\bar{A})^{n-1}])^{k\varphi(t)p}[\bar{C},B\bar{A}]^{\varphi(tk)(n-1)}\\
&\qquad \bar{C}^{tk}[B\bar{A},\bar{C}]^{(tk)^2(n-1)}B^{pt}(B\bar{A})^{tk(n-1)}[(B\bar{A})^{tk(n-1)},B^{pt}]\\
&\qquad (A\bar{B})^{m-1}\bar{A}^{pt}[C,\bar{A}]^{k\varphi(t)p}C^{tk}\\
&\underset{\text{(15 and T. 4)}}{\simeq} [B,B\bar{A}]^{(n-1)k\varphi(t)p}[B,\bar{C}]^{k\varphi(t)p}[[B,\bar{C}],B\bar{A}]^{(n-1)k\varphi(t)p}\\
&\qquad [\bar{C},B\bar{A}]^{\varphi(tk)(n-1)}\bar{C}^{tk}[B\bar{A},\bar{C}]^{(tk)^2(n-1)}[(B\bar{A})^{tk(n-1)},B^{pt}]\\
&\qquad B^{pt}(A\bar{B})^{pt}\bar{A}^{pt}[C,\bar{A}]^{k\varphi(t)p}C^{tk}\\
&\underset{(p_2,\ p_3\text{ and T. 4})}{\simeq} [B,\bar{A}]^{(n-1)k\varphi(t)p}[B,\bar{C}]^{k\varphi(t)p}[[B,\bar{C}],\bar{A}]^{(n-1)k\varphi(t)p}\\
&\qquad [B\bar{A},\bar{C}]^{\varphi(tk-1)(n-1)}\bar{C}^{tk}[\bar{A},B]^{tk(n-1)pt}[A,\bar{B}]^{\varphi(pt)}[C,\bar{A}]^{k\varphi(t)p}C^{tk}\\
&\underset{(p_2\text{ and }p_4)}{\simeq} [B,\bar{A}]^{(n-1)k\varphi(t)p}[B,\bar{C}]^{k\varphi(t)p}[[B,\bar{C}],\bar{A}]^{(n-1)k\varphi(t)p}\\
&\qquad ([B,\bar{C}][[B,\bar{C}],\bar{A}][\bar{A},\bar{C}])^{\varphi(tk-1)(n-1)}[\bar{A},B]^{\varphi(pt)+tk(n-1)pt}\bar{C}^{tk}\\
&\qquad [\bar{C},[\bar{A},B]]^{tk\varphi(pt)+(tk)^2(n-1)pt}[C,\bar{A}]^{k\varphi(t)p}C^{tk}\\
&\underset{\text{(15 and L. 6)}}{\simeq} [B,\bar{C}]^{k\varphi(t)p+\varphi(tk-1)(n-1)}[A,\bar{B}]^{\varphi(pt)+tk(n-1)pt-(n-1)k\varphi(t)p}\\
&\qquad [C,\bar{A}]^{k\varphi(t)p+\varphi(tk-1)(n-1)}[[B,\bar{C}],\bar{A}]^{(n-1)k\varphi(t)p+\varphi(tk-1)(n-1)}\\
&\qquad [\bar{C},[\bar{A},B]]^{tk\varphi(pt)+(tk)^2(n-1)pt}\\
&\underset{(p_1\text{ and L. 6})}{\simeq} [C,B]^{k\varphi(t)p+\varphi(tk-1)(n-1)}[B,A]^{\varphi(pt)+tk(n-1)pt-(n-1)k\varphi(t)p}\\
&\qquad [A,C]^{k\varphi(t)p+\varphi(tk-1)(n-1)}[[B,C],A]^{(n-1)k\varphi(t)p+\varphi(tk-1)(n-1)}\\
&\qquad [[A,B],C]^{-\{tk\varphi(pt)+(tk)^2(n-1)pt\}}.
\end{aligned}$$

We showed thus that for $k,t\in\mathbb{Z}$ such that $\frac{m-tkn+tk-1}{t}\in\mathbb{Z}$, we have

$$g_{n,m,k,t}\circ\delta_n \simeq [C,B]^{e_1}[B,A]^{e_2}[A,C]^{e_1}[[B,C],A]^{e_3}[[A,B],C]^{-e_4},$$

where $e_1 = \frac{(m-n+p)tk}{2}$, $e_2 = \frac{(m-kn+k)tp}{2}$, $e_3 = \frac{(p+t(p+k)-1)tk(n-1)}{2}$, $e_4 = \frac{(2m-tp-1)t^2pk}{2}$ and $p = p(m,n,k,t) = \frac{m-tkn+tk-1}{t}$.

Now if $\wedge : S^3 \times S^3 \times S^3 \longrightarrow S^3 \wedge S^3 \wedge S^3 = S^9$ is the natural projection, then $[[B,C],A] = \wedge^*(\omega \circ \Sigma^3\omega)$, where $\omega$ is the generator of $\pi_6(S^3)$ given in Remark 3 above (cf. [B1]), and as $\omega \circ \Sigma^3\omega$ is a generator of $\pi_9(S^3) \cong \mathbb{Z}_3$ it follows that $[[B,C],A]$ in $[S^3 \times S^3 \times S^3, S^3]$ has order 3. The same occurs with $[[C,A],B]$ and $[[A,B],C]$ and as none of them is nullhomotopic we conclude with the aid of N7 that $[[B,C],A] = [[C,A],B] = [[A,B],C]$ in $[S^3 \times S^3 \times S^3, S^3]$. Thus we have finally

$$g_{n,m,k,t} \circ \delta_n \simeq [C,B]^{e_1}[B,A]^{e_2}[A,C]^{e_1}[[B,C],A]^f,$$

where $f = e_3 - e_4$. $\square$

**Remark 5.** The choice of the partial sections $S_V^n$ and $S_U^n$ for the bundles $S^3 \cdots \tilde{P}_n \longrightarrow S^7$ given in page 77 enabled us to write the homotopy class of the corresponding transition functions as a power of a unique commutator of weight 2. We would like to have partial sections for the bundles $S^3 \cdots \tilde{P}_{n,m} \longrightarrow \tilde{P}_n$ for which the homotopy class of the corresponding transition functions could be expressed as power of a unique commutator of weight 3 as is suggested by the obstruction in the Hilton-Roitberg formula (Theorem 1). This however, cannot be realized with our choice of transition functions:

If we suppose that $[C,B]^{e_1}[B,A]^{e_2}[A,C]^{e_1}[[B,C],A]^f \simeq [[B,C],A]^r$ for some $r$ then $g = [C,B]^{e_1}[B,A]^{e_2}[A,C]^{e_1} \simeq [[B,C],A]^s$ $(s = r-f)$, and so $g \in [\Gamma_1, \Gamma] \subseteq \Gamma_2$ which implies that $g|_{P_2} \simeq 1$ where $P_2 = X_1 \cup X_2 \cup X_3$, $X_1 = \{1\} \times S^3 \times S^3$, $X_2 = S^3 \times \{1\} \times S^3$, $X_3 = S^3 \times S^3 \times \{1\}$ and this implies $g|_{X_i} \simeq 1$ $(i = 1,2,3)$ in other words, $[C,B]^{e_1}$, $[B,A]^{e_2}$, $[A,C]^{e_1}$ : $S^3 \times S^3 \longrightarrow S^3$ are all nullhomotopic, which gives $e_1 \equiv e_2 \equiv 0 \bmod 12$ for every $m,n,k,t$ such that $\frac{m-nkt+kt-1}{t} \in \mathbb{Z}$, but this is not true for example if $m=5$, $n=2$, $k=1$ and $t=2$ or $m=14$, $n=23$, $k=2$ and $t=13$. We do not know if there exist such transition functions.

It follows from Theorem 2 that $E_{1,7} = \tilde{P}_{23,14}$ and $E_{7,1} = \tilde{P}_{14,23}$.

From Theorem 3 and Theorem 5 choosing $k=19$ and $t=1$ we have that $g_{14,23,19,1} \circ \delta_{14}$ is homotopic to

$$([C,B]^{12})^{-171}([B,A]^{12})^{2100}([A,C]^{12})^{-171}([[B,C],A]^3)^{174591} \simeq 1. \tag{17}$$

Also, choosing $k=5$ and $t=-13$ we obtain that $g_{23,14,5,-13} \circ \delta_{23}$ is homotopic to

$$([C,B]^{12})^{325}([B,A]^{12})^{-5772}([A,C]^{12})^{325}([[B,C],A]^3)^{-22437350} \simeq 1. \tag{18}$$

Thus, following the same steps of the trivialization of $\tilde{P}_9$, we can exhibit, up to a homotopy of the above commutator powers to the constant 1, the diffeomorphisms

$$\tilde{P}_{23} \times S^3 = \tilde{P}_{23,14} \quad \text{and} \quad \tilde{P}_{14} \times S^3 = \tilde{P}_{14,23}.$$

Let $H : S^3 \times S^3 \times S^3 \times [0, \frac{\pi}{2}] \longrightarrow S^3$ be a smooth homotopy such that

$$H(A,B,C,\theta) = \begin{cases} g_{23,14,5,-13} \circ \delta_{23}(A,B,C,\theta) & \text{if } \theta \in [0, \frac{\pi}{6}] \\ 1 & \text{if } \theta \in [\frac{\pi}{3}, \frac{\pi}{2}]. \end{cases}$$

By remembering that $s^{5,-13}_{\tilde{U}_{23}} : \tilde{U}_{23} \longrightarrow \tilde{P}_{23,14}$ and $s^{5,-13}_{\tilde{V}_{23}} : \tilde{V}_{23} \longrightarrow \tilde{P}_{23,14}$ are partial sections of $\tilde{P}_{23,14}$ over $\tilde{U}_{23}$ and $\tilde{V}_{23}$ respectively given by

$$\begin{aligned} s^{5,-13}_{\tilde{U}_{23}}(M_{23}(a,b,y_2)) &= (M_{23}(a,b,y_2), M_{14}(a,b,Y_2(5,-13))), \\ s^{5,-13}_{\tilde{V}_{23}}(M_{23}(a,b,z_{23})) &= (M_{23}(a,b,z_{23}), M_{14}(a,b,Z_{14}(5,-13))), \end{aligned}$$

where

$$\begin{aligned} Y_2(5,-13) &= (y_2^5 \bar{A}^{-111})^{-13} \frac{L_1 L_2 \dots L_{10} L(23)^{-65}}{(L_1 L_2 \dots L_{19})^{-65} L(14)}, \\ Z_{14}(5,-13) &= (-1)^{-64} (z_{23}^5 \bar{B}^{-111})^{-13} \frac{(L_1 L_2 \dots L_{19} L(23))^{-65}}{L_1 L_2 \dots L_{10} L(14)}, \end{aligned}$$

we can then construct a global section $s_{23,14} : \tilde{P}_{23} \longrightarrow \tilde{P}_{23,14}$ given by

$$s_{23,14}(M_{23}) = \begin{cases} s^{5,-13}_{\tilde{V}_{23}}(M_{23}) & \text{if } \frac{5\pi}{12} \le \cos^{-1}|a| \le \frac{\pi}{2} \\ s^{5,-13}_{\tilde{V}_{23}}(M_{23})(H \circ \delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12} \le \cos^{-1}|a| \le \frac{5\pi}{12} \\ s^{5,-13}_{\tilde{U}_{23}}(M_{23}) & \text{if } 0 \le \cos^{-1}|a| \le \frac{\pi}{12}. \end{cases}$$

Therefore, a diffeomorphism $\Phi_{23,14} : \tilde{P}_{23} \times S^3 \longrightarrow \tilde{P}_{23,14}$ is given by

$$\begin{aligned} &\Phi_{23,14}(M_{23}(a,b,x_1,x_2,\dots,x_{23}),q) \\ &\quad = (M_{23}(a,b,x_1,x_2,\dots,x_{23}), M_{14}(a,b,y_1 q, y_2 q, \dots, y_{14} q)), \end{aligned}$$

where if $\theta = \theta(a) = \cos^{-1}|a|$ then

$$y_1 = \begin{cases} (b\bar{a})^{12}|b|^{-24}(L_1 L_2 \dots L_{10})^2 Z_{14} & \text{if } \frac{5\pi}{12} \le \theta \le \frac{\pi}{2} \\ (b\bar{a})^{12}|b|^{-24}(L_1 L_2 \dots L_{10})^2 Z_{14}.(H \circ \delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12} \le \theta \le \frac{5\pi}{12} \\ -(a\bar{b})|a|^{-2} Y_2 & \text{if } 0 \le \theta \le \frac{\pi}{12}, \end{cases}$$

$$y_2 = \begin{cases} -(b\bar{a})^{13}|b|^{-26}(L_1 L_2 \dots L_{10})^2 Z_{14} & \text{if } \frac{5\pi}{12} \le \theta \le \frac{\pi}{2} \\ -(b\bar{a})^{13}|b|^{-26}(L_1 L_2 \dots L_{10})^2 Z_{14}.(H \circ \delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12} \le \theta \le \frac{5\pi}{12} \\ Y_2 & \text{if } 0 \le \theta \le \frac{\pi}{12}, \end{cases}$$

$$y_k = \begin{cases} (b\bar{a})^{14-k}|b|^{2(14-k)}(L_1L_2\ldots L_{14-(k+2)})^2 Z_{14} & \text{if } \frac{5\pi}{12} \le \theta \le \frac{\pi}{2} \\ (b\bar{a})^{14-k}|b|^{2(14-k)}(L_1L_2\ldots L_{14-(k+2)})^2 Z_{14}.(H\circ\delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12} \le \theta \le \frac{5\pi}{12} \\ -(a\bar{b})^{k-1}|a|^{-2(k-1)}(L_{14-(k+1)}L_{14-k}\ldots L_{10})^{-2}Y_2 & \text{if } 0 \le \theta \le \frac{\pi}{12} \end{cases}$$

for $3 \le k \le 11$,

$$y_{12} = \begin{cases} (b\bar{a})^2|b|^{-4}Z_{14} & \text{if } \frac{5\pi}{12} \le \theta \le \frac{\pi}{2} \\ (b\bar{a})^2|b|^{-4}Z_{14}.(H\circ\delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12} \le \theta \le \frac{5\pi}{12} \\ -(a\bar{b})^{11}|a|^{-22}(L_1L_2\ldots L_{10})^{-2}Y_2 & \text{if } 0 \le \theta \le \frac{\pi}{12}, \end{cases}$$

$$y_{13} = \begin{cases} (b\bar{a})|b|^{-2}Z_{14} & \text{if } \frac{5\pi}{12} \le \theta \le \frac{\pi}{2} \\ (b\bar{a})|b|^{-2}Z_{14}.(H\circ\delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12} \le \theta \le \frac{5\pi}{12} \\ -(a\bar{b})^{12}|a|^{-24}(L_1L_2\ldots L_{10})^{-2}Y_2 & \text{if } 0 \le \theta \le \frac{\pi}{12}, \end{cases}$$

$$y_{14} = \begin{cases} Z_{14} & \text{if } \frac{5\pi}{12} \le \theta \le \frac{\pi}{2} \\ Z_{14}.(H\circ\delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12} \le \theta \le \frac{5\pi}{12} \\ -(a\bar{b})^{13}|a|^{-26}(L_1L_2\ldots L_{10})^{-2}Y_2 & \text{if } 0 \le \theta \le \frac{\pi}{12}, \end{cases}$$

$$\begin{aligned} Z_{14} &= (-1)^{-64}(x_{23}^5\bar{b}^{-111})^{-13}\tfrac{(L_1L_2\ldots L_{19}L(23))^{-65}}{(L_1L_2\ldots L_{10}L(14))}, \\ Y_2 &= (x_2^5\bar{a}^{-111})^{-13}\tfrac{L_1L_2\ldots L_{10}L(23)^{-65}}{(L_1L_2\ldots L_{19})^{-65}L(14)}. \end{aligned}$$

Let now $G : S^3 \times S^3 \times S^3 \times [0, \frac{\pi}{2}] \longrightarrow S^3$ be a smooth homotopy such that

$$G(A,B,C,\theta) = \begin{cases} g_{14,23,19,1} \circ \delta_{14}(A,B,C,\theta) & \text{if } \theta \in [0, \frac{\pi}{6}] \\ 1 & \text{if } \theta \in [\frac{\pi}{3}, \frac{\pi}{2}]. \end{cases}$$

Following the same steps of the construction of $\Phi_{23,14}$ we have

$$\begin{aligned} s^{19,1}_{\tilde{U}_{14}}(M_{14}(a,b,y_2)) &= (M_{14}(a,b,y_2), M_{23}(a,b,\mathcal{Y}_2(19,1))), \\ s^{19,1}_{\tilde{V}_{14}}(M_{14}(a,b,z_{14})) &= (M_{14}(a,b,z_{14}), M_{23}(a,b,\mathcal{Z}_{23}(19,1))) \end{aligned}$$

are partial sections of $\tilde{P}_{14,23}$ over $\tilde{U}_{14}$ and $\tilde{V}_{14}$ respectively, where

$$\begin{aligned} \mathcal{Y}_2(19,1) = \mathcal{Y}_2(19,1)(a,b,y_2) &= y_2^{19}\bar{a}^{-225}\frac{L_1L_2L_3\ldots L_{19}L(14)}{(L_1L_2\ldots L_{10})^{19}L(23)}, \\ \mathcal{Z}_{23}(19,1) = \mathcal{Z}_{23}(19,1)(a,b,z_{14}) &= -z_{14}^{19}\bar{b}^{-225}\frac{(L_1L_2\ldots L_{10}L(14))^{19}}{L_1L_2L_3\ldots L_{19}L(23)}. \end{aligned}$$

If $M_{14} = M_{14}(a, b, x_1, x_2, \ldots, x_{14}) \in \tilde{P}_{14}$ then a global section, $s_{14,23} : \tilde{P}_{14} \longrightarrow \tilde{P}_{14,23}$ is given by

$$s_{14,23}(M_{14}) = \begin{cases} s^{19,1}_{\tilde{V}_{14}}(M_{14}) & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ s^{19,1}_{\tilde{V}_{14}}(M_{14}).(G \circ \delta^{-1}_{14})(M_{14}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ s^{19,1}_{\tilde{U}_{14}}(M_{14}) & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

and a diffeomorphism $\Phi_{14,23} : \tilde{P}_{14} \times S^3 \longrightarrow \tilde{P}_{14,23}$ is given by

$$\begin{aligned} &\Phi(M_{14}(a, b, x_1, x_2, \ldots, x_{14}), q) \\ &\quad = (M_{14}(a, b, x_1, \ldots, x_{14}), M_{23}(a, b, r_1.q, r_2.q, \ldots, r_{23}.q)) \end{aligned}$$

with

$$r_1 = \begin{cases} (b\bar{a})^{21}|b|^{-42}(L_1L_2\ldots L_{19})^2\mathcal{Z}_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ (b\bar{a})^{21}|b|^{-42}(L_1L_2\ldots L_{19})^2\mathcal{Z}_{23}.(G \circ \delta^{-1}_{14})(M_{14}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})|a|^{-2}\mathcal{Y}_2 & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

$$r_2 = \begin{cases} -(b\bar{a})^{22}|b|^{-44}(L_1L_2\ldots L_{19})^2\mathcal{Z}_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ -(b\bar{a})^{22}|b|^{-44}(L_1L_2\ldots L_{19})^2\mathcal{Z}_{23}.(G \circ \delta^{-1}_{14})(M_{14}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ x_2^{19}\bar{a}^{-225}\frac{L_1L_2\ldots L_{19}L(14)^{19}}{(L_1L_2\ldots L_{10})^{19}L(23)} & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

$$r_k = \begin{cases} (b\bar{a})^{23-k}|b|^{-2(23-k)}(L_1L_2\ldots L_{21-k})^2\mathcal{Z}_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ \frac{(b\bar{a})^{23-k}}{|b|^{2(23-k)}}(L_1L_2\ldots L_{21-k})^2\mathcal{Z}_{23}.(G \circ \delta^{-1}_{14})(M_{14}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})^{k-1}|a|^{-2(k-1)}(L_{23-(k+1)}L_{23-k}\ldots L_{19})^{-2}\mathcal{Y}_2 & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12} \end{cases}$$

for $3 \leq k \leq 20$,

$$r_{21} = \begin{cases} (b\bar{a})^2|b|^{-4}\mathcal{Z}_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ (b\bar{a})^2|b|^{-4}\mathcal{Z}_{23}.(G \circ \delta^{-1}_{14})(M_{14}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})^{20}|a|^{-40}(L_1L_2\ldots L_{19})^{-2}\mathcal{Y}_2 & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

$$r_{22} = \begin{cases} (b\bar{a})|b|^{-2}\mathcal{Z}_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ (b\bar{a})|b|^{-2}\mathcal{Z}_{23}.(G \circ \delta^{-1}_{14})(M_{14}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})^{21}|a|^{-42}(L_1L_2\ldots L_{19})^{-2}\mathcal{Y}_2 & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

$$r_{23} = \begin{cases} -x_{14}^{19}\bar{b}^{-225}\frac{(L_1L_2\ldots L_{10}L(14))^{19}}{L_1L_2\ldots L_{19}L(23)} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ -x_{14}^{19}\bar{b}^{-225}\frac{(L_1L_2\ldots L_{10}L(14))^{19}}{L_1L_2\ldots L_{19}L(23)}.(G \circ \delta^{-1}_{14})(M_{14}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})^{22}|a|^{-44}(L_1L_2\ldots L_{19})^{-2}\mathcal{Y}_2 & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

$$\mathcal{Y}_2 = x_2^{19}\bar{a}^{-225}\tfrac{L_1L_2...L_{19}L(14)^{19}}{(L_1L_2...L_{10})^{19}L(23)},$$
$$\mathcal{Z}_{23} = -x_{14}^{19}\bar{b}^{-225}\tfrac{(L_1L_2...L_{10}L(14))^{19}}{L_1L_2...L_{19}L(23)}.$$

We have with this

$$\tilde{P}_{23} \times S^3 \xrightarrow{\Phi_{23,14}} \tilde{P}_{23,14} \xrightarrow{c} \tilde{P}_{14,23} \xrightarrow{\Phi^{-1}_{14,23}} \tilde{P}_{14} \times S^3,$$

where $c(M_2, M_{11}) = (M_{11}, M_2)$.

**Remark 6.** We also observe that the same procedure provides the trivialization of the bundle $\tilde{P}_{2,11}$, choosing $k = 11$ and $t = 5$.

## Exotic Actions

Non-cancellation phenomena related to products $M \times G = N \times G$ of non equivalent spaces $M$ and $N$ by a group $G$ can be seen as exotic actions of the group on, say, $M \times G$ with quotient $N$. We have treated here two such cases were $G = S^3$. We showed precisely how the specific diffeomorphisms and, equivalently, the corresponding exotic actions depend on the homotopy commutativity of certain powers of commutators in $S^3$.

In the case of the exotic actions treated above we have $S^3$ acting freely on $P \times S^3$, where $P$ can be considered as a parameter space on which $S^3$ acts in a standard way and that parametrizes a complicated action of $S^3$ on itself, the second factor, so that the action on the product is free and the quotient is not equivalent to $P$. Investigating properties of prospective $P$'s, like for example, how small such a compact $P$ can be, etc., seems like an interesting way of looking at some classical questions.

On the other hand, specifying explicit homotopies between powers of commutators and constants seems to be a problem of geometric nature [CR], [RC].

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