Math. J. Okayama Univ. **42** (2000), 29-54

SYMMETRY OF ALMOST HEREDITARY RINGS

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In [6] an almost N-projective module is defined as a generalization of a N-projective module to characterize the lifting property. This module is further studied in the succeeding papers [4], [7], [8]. And in [10] M. Harada called a module M to be *almost projective* if M is almost N-projective for any finitely generated module N. Semisimple rings, serial rings, QF-rings and H-rings are well-characterized by the property of an almost projective module in [10], [11]. Using this remarkable module, in [9] he defined a *right almost hereditary ring* R, i.e., R is an artinian ring with J_R almost projective, where J is the Jacobson radical of R. On the other hand, it is well known that an artinian hereditary ring R is characterized by the following equivalent conditions:

(1) J_R is projective;
(2) ${}_RJ$ is projective;
(3) $E/Socle(E)$ is injective for any injective right R-module E;
(4) $E/Socle(E)$ is injective for any injective left R-module E.

Therefore a right almost hereditary ring is a generalization of an artinian hereditary ring. In this paper, first we characterize a right almost hereditary ring using left ideals in section 3 (we note that M. Harada already gave a structure theorem of it using right ideals in [9]). Further in section 4 we generalize the above condition (3) as follows:

$(\#)_r$ A factor module of E by its socle is a direct sum of an injective module and finitely generated almost injective modules for any injective right R-module E (not necessarily finitely generated).

Symmetrically we consider the left version $(\#)_l$. And we show that a ring R is a right almost hereditary ring if and only if it satisfies $(\#)_l$ using a characterization of a right almost hereditary ring given in section 3. But M. Harada already showed that a right almost hereditary ring is not always a left almost hereditary ring in [9, p801]. That is, the equivalences (1) $\Leftrightarrow$ (4) and (2) $\Leftrightarrow$ (3) are generalized. But the other equivalences are not generalized.

1. Preliminaries

In this paper, we always assume that every ring is a basic artinian ring with identity and every module is unitary. Let R be a ring and let $P(R) =$

$\{e_i\}_{i=1}^n$ be a complete set of pairwise orthogonal primitive idempotents in R. We denote the *Jacobson radical*, an *injective hull* and the *composition length* of a module M by $J(M)$, $E(M)$ and $|M|$, respectively. Especially, we put $J := J(R_R)$. For a module M we denote the *socle* of M by $S(M)$ and the *k-th socle* of M by $S_k(M)$ (i.e., $S_k(M)$ is a submodule of M defined by $S_k(M)/S_{k-1}(M) = S(M/S_{k-1}(M))$ inductively).

Let M and N be modules. M is called N*-projective* (resp. N*-injective*) if for any homomorphism $\phi : M \to L$ (resp. $\phi' : L \to M$) and any epimorphism $\pi : N \to L$ (resp. monomorphism $\iota : L \to N$) there exists a homomorphism $\tilde{\phi} : M \to N$ (resp. $\tilde{\phi}' : N \to M$) such that $\phi = \pi\tilde{\phi}$ (resp. $\phi' = \tilde{\phi}'\iota$). And M is called *almost* N*-projective* (resp. *almost* N*-injective*) if for any homomorphism $\phi : M \to L$ (resp. $\phi' : L \to M$) and any epimorphism $\pi : N \to L$ (resp. monomorphism $\iota : L \to N$) either there exists a homomorphism $\tilde{\phi} : M \to N$ (resp. $\tilde{\phi}' : N \to M$) such that $\phi = \pi\tilde{\phi}$ (resp. $\phi' = \tilde{\phi}'\iota$) or there exist a nonzero direct summand N' of N and a homomorphism $\theta : N' \to M$ (resp. $\theta' : M \to N'$) such that $\phi\theta = \pi i$ (resp. $\theta'\phi' = p\iota$), where i is an inclusion of N' in N (resp. p is a projection on N' of N).

A ring R is called *right* (resp. *left*) *hereditary* if every submodule of a projective right (resp. left) R-module is also projective. It is well known that a perfect or neotherian ring is right hereditary iff it is left hereditary (see, for instance, [13, Chapter 9]). So we call a right hereditary ring a *hereditary* ring since rings are artinian in this paper. Further an artinian ring R is hereditary iff J_R is projective (see, for instance, [1, 18. Exercises 10 (2)]). Furthermore an artinian ring R is hereditary iff (a) $E/S(E)$ is injective for any injective right R-module E. We give a proof of it for reader's convenience. By [1, 18. Exercises 10 (1)] we see that R is hereditary iff (b) E/A is injective for any submodule A of an injective module E. So we only show that, if (a) holds, then (b) also holds. Let E be an injective module and A a submodule of E. Then $E = E' \oplus E(A)$ for some E'. So we may assume that $E = E(A)$. Since $S(E) = S(A)$, $E/S(E) = E/S(A) \supseteq A/S(A)$. And $E/S(E)$ is injective by assumption. Therefore we see that $E/S_2(A) \cong (E/S(A))/S(A/S(A))$ is also injective by the same way as the first argument. Thus (b) holds by induction on $S_i(A) = \{a \in A \mid aJ^i = 0\}$. Further M is called *almost projective* (resp. *almost injective*) if M is always almost N-projective (resp. almost N-injective) for any finitely generated R-module N. The following is an important characterization of an almost projective module given by M. Harada.

Lemma A ([10, Corollary $1^{\#}$]). *Suppose that M is an indecomposable finitely generated left R-module. Then M is almost injective but not injective if and only if there exist an indecomposable injective left R-module E*

and a positive integer k such that $M \cong J^kE$ and J^iE is projective for any $i = 0, \cdots, k-1$.

And we call an artinian ring R a *right almost hereditary ring* if J is almost projective as a right R-module. By [10, Theorem 1] this definition is equivalent to the condition: $J(P)$ is almost projective for any finitely generated projective right R-module P.

A module is called *uniserial* if its lattice of submodules is a finite chain, i.e., any two submodules are comparable. An artinian ring R is called a *right serial* (resp. *co-serial*) *ring* if every indecomposable projective (resp. injective) right R-module is uniserial. And we call a ring R a *serial ring* if R is a right and left serial ring. Let $f_1, f_2, \cdots, f_n$ be primitive idempotents in a serial ring R. Then a sequence $\{f_1R, f_2R, \cdots, f_nR\}$ (resp. $\{Rf_1, Rf_2, \cdots, Rf_n\}$) of indecomposable projective right (resp. left) R-modules is called a *Kupisch series* if $f_jJ/f_jJ^2 \cong f_{j+1}R/f_{j+1}J$ (resp. $Jf_j/J^2f_j \cong Rf_{j+1}/Jf_{j+1}$) holds for any $j = 1, \cdots, n-1$. Further $\{f_1R, f_2R, \cdots, f_nR\}$ (resp. $\{Rf_1, Rf_2, \cdots, Rf_n\}$) is called a *cyclic Kupisch series* if it is a Kupisch series and $f_nJ/f_nJ^2 \cong f_1R/f_1J$ (resp. $Jf_n/J^2f_n \cong Rf_1/Jf_1$) holds. Let R be a serial ring with a Kupisch series $\{f_1R, f_2R, \cdots, f_nR\}$. If $f_nJ = 0$ and $P(R) = \{f_1, \cdots, f_n\}$, then R is called a serial ring *in the first category.* And if $\{f_1R, f_2R, \cdots, f_nR\}$ is a cyclic Kupisch series and $P(R) = \{f_1, \cdots, f_n\}$, then R is called a serial ring *in the second category.*

For a set S of R-modules, a subset S' of S is called a *basic set* of S if the following two conditions are satisfied.

(1) For any $M, M' \in S'$, $M \approx M'$ as R-modules iff $M = M'$.

(2) For any $N \in S$, there exists $M \in S'$ such that $M \approx N$ as R-modules.

2. A structure theorem for an almost hereditary ring

The following is a structure theorem for a right almost hereditary ring given by M. Harada.

Theorem B ([9, Theorem 1])**.** *A ring is right almost hereditary if and only if it is a direct sum of the following rings:*

(i) *Hereditary rings;*

(ii) *serial rings;*

(iii) *rings R with $P(R) = \{h_1, \cdots, h_m, f_1^{(1)}, f_2^{(1)}, \cdots, f_{n_1}^{(1)}, f_1^{(2)}, \cdots, f_{n_2}^{(2)}, f_1^{(3)}, \cdots, f_{n_k}^{(k)}\}$ such that, for each $l = 1, \cdots, k$ we put $S_l := \sum_{j=1}^{n_l} f_j^{(l)}$ and $p_l := |f_1^{(l)}R_R|$, the following four conditions hold for any $l = 1, \cdots, k$ and $s = 1, \cdots, m$,*

(a) *S_lRS_l is a serial ring in the first category with $\{f_1^{(l)}RS_l, f_2^{(l)}RS_l, \cdots, f_{n_l}^{(l)}RS_l\}$ a Kupisch series of right S_lRS_l-modules,*

(b) *$S_lR(1-S_l)=0$, $(h_1+\cdots+h_m)R(f_1^{(l)}+\cdots+f_{p_l-1}^{(l)})\neq 0$ and $(h_1+\cdots+h_m)R(f_{p_l}^{(l)}+\cdots+f_{n_l}^{(l)})=0$,*

(c) *$(h_sJ/h_sJ^2)f_j^{(l)}=\overline{0}$ for any $j\geq 2$,*

we let α_l be a positive integer such that $f_1^{(l)}R/f_1^{(l)}J^j$ is injective for any j $(\geq \alpha_l+1)$ but $f_1^{(l)}R/f_1^{(l)}J^{\alpha_l}$ is not injective (see Lemma 2.1(3) below as for the existence of α_l) and put $H:=\sum_{s=1}^m h_s+\sum_{l=1,j=1}^{k\quad\alpha_l} f_j^{(l)}$, then

(d) *HRH is a hereditary ring.*

Lemma 2.1. *Let R be a ring satisfying (a) and the first condition of (b), i.e., $S_lR(1-S_l)=0$, in Theorem B(iii). Then the following hold.*

(1) *$\{f_1^{(l)}R, f_2^{(l)}R, \cdots, f_{n_l}^{(l)}R\}$ is a Kupisch series of right R-modules with $f_{n_l}^{(l)}R_R$ simple for any $l=1,\cdots,k$.*

(2) *$f_1^{(l)}R/f_1^{(l)}J^j$ is injective for any l and j $(\leq p_l)$ if $(h_1+\cdots+h_m)Rf_j^{(l)}=0$.*

(3) *Moreover, if R satisfies the whole conditions of (b), then $f_1^{(l)}R$ is injective and α_l is defined for any l.*

Proof. (1). Clear.

(2). First we show that, if $(h_1+\cdots+h_m)Rf_j^{(l)}=0$, then $f_1^{(l)}R/f_1^{(l)}J^j$ is injective as a right R-module. By (a) $f_1^{(l)}RS_l/f_1^{(l)}J^iS_l$ is an injective right S_lRS_l-module for any $i=1,\cdots,p_l$. So especially we obtain that $f_1^{(l)}RS_l/f_1^{(l)}J^jS_l$ is an injective right S_lRS_l-module. Therefore, for any $i=1,\cdots,n_l$, a right S_lRS_l-module $f_1^{(l)}RS_l/f_1^{(l)}J^jS_l$ is $f_i^{(l)}RS_l$-injective. Hence a right R-module $f_1^{(l)}R/f_1^{(l)}J^j$ is $f_i^{(l)}R$-injective because $(f_1^{(l)}R/f_1^{(l)}J^j)S_l$ $=f_1^{(l)}R/f_1^{(l)}J^j$ and $f_i^{(l)}RS_l=f_i^{(l)}R$ from $S_lR(1-S_l)=0$. Further $f_1^{(l)}R/f_1^{(l)}J^j$ is $f_i^{(t)}R$-injective for any t $(\neq l)$ and $i=1,\cdots,n_t$ because $\mathrm{Hom}_R(I, f_1^{(l)}R/f_1^{(l)}J^j)=0$ for any right submodule I of $f_i^{(t)}R$ from $S_lR(1-S_l)=0$. Furthermore we claim that $f_1^{(l)}R/f_1^{(l)}J^j$ is h_sR-injective for any s. Let I be a submodule of h_sR and $\phi\in\mathrm{Hom}_R(I, f_1^{(l)}R/f_1^{(l)}J^j)$. Assume that $\phi\neq 0$. Then $0\neq\phi^{-1}(S(f_1^{(l)}R/f_1^{(l)}J^j))\subseteq h_sRf_j^{(l)}$ since $S(f_1^{(l)}R/f_1^{(l)}J^j)\cong f_j^{(l)}R/f_j^{(l)}J$ by (1). This contradicts with the assumption that $(h_1+\cdots+h_m)Rf_j^{(l)}=0$. Hence $f_1^{(l)}R/f_1^{(l)}J(R)^j$ is R-injective by

Azumaya's Theorem (see, for instance, [1, 16.13. Proposition (2)]), i.e., $f_1^{(l)}R/f_1^{(l)}J^j$ is injective.

(3). $f_1^{(l)}R$ $(= f_1^{(l)}R/f_1^{(l)}J^{p_l})$ is injective by (2) because $(h_1 + \cdots + h_m)Rf_{p_l}^{(l)} = 0$ from (b). Further there is $0 \neq x \in (h_1 + \cdots + h_m)Rf_j^{(l)}$ for some $j \in \{1, \cdots, p_l - 1\}$ by (b). Then we have $0 \neq \phi \in \mathrm{Hom}_R(xR, S(f_1^{(l)}R/f_1^{(l)}J^j))$ because $\mathrm{S}(f_1^{(l)}R/f_1^{(l)}J^j) \cong f_j^{(l)}R/f_j^{(l)}J$ by (1). But ϕ can not be extended to a map in $\mathrm{Hom}_R((h_1 + \cdots + h_m)R, f_1^{(l)}R/f_1^{(l)}J^j)$ since $f_1^{(l)}R(h_1 + \cdots + h_m) = 0$ by (b). So $f_1^{(l)}R/f_1^{(l)}J^j$ is not injective. On the other hand, $f_1^{(l)}R$ is injective by (2). Therefore we can define a positive integer α_l. □

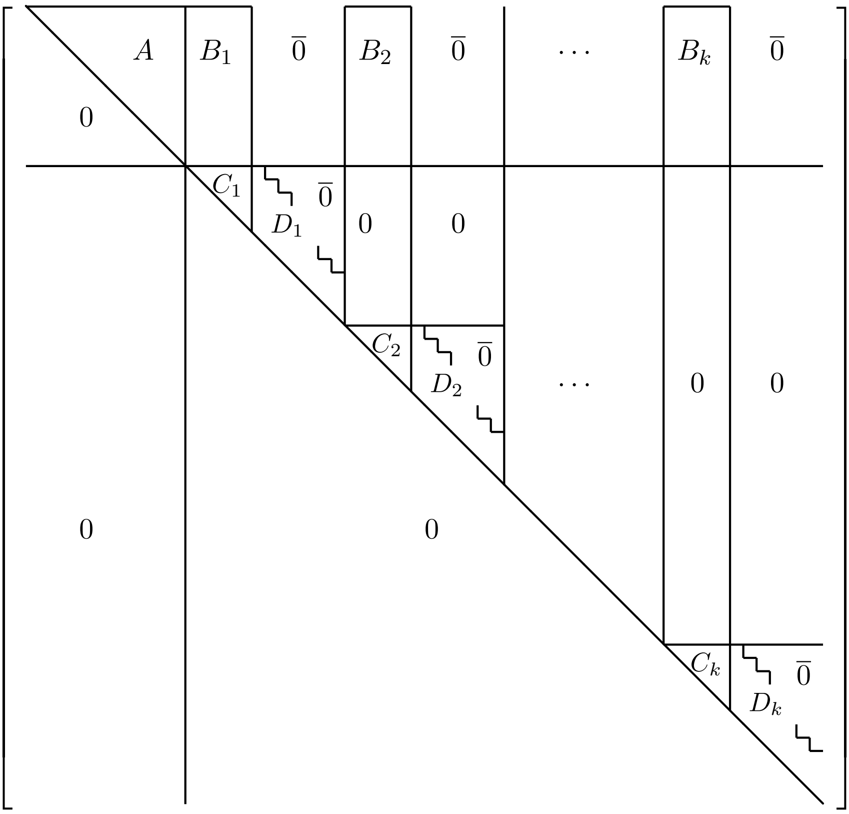
Remark 2.2. By [5] we know that a hereditary ring is represented as

$$\begin{bmatrix} D_1 & M_{1,2} & M_{1,3} & \cdots & \cdots & M_{1,n} \\ 0 & D_2 & M_{2,3} & \cdots & \cdots & M_{2,n} \\ \vdots & 0 & D_3 & \cdots & \cdots & \vdots \\ & & \ddots & \ddots & \cdots & \\ & & & 0 & D_{n-1} & M_{n-1,n} \\ 0 & & \cdots & & 0 & D_n \end{bmatrix},$$

where $D_1, D_2, \cdots, D_n$ are division rings and M_{ij} is a left D_i-right D_j-bimodule for any i, j. Further by [12] a serial ring in the first category is represented as the following factor ring:

$$\begin{bmatrix}
D & D & \cdots & \cdots & \cdots & D & \bar{0} & & & & \cdots & & & & & \bar{0} \\
\ddots & \ddots & \ddots & \cdots & \cdots & \vdots & \vdots & & & & \cdots & & & & & \bar{0} \\
 & 0 & D & D & \cdots & D & \bar{0} & & & & \cdots & & & & & \bar{0} \\
 & & 0 & D & D & \cdots & & D & \bar{0} & & & \cdots & & & & \bar{0} \\
 & & & 0 & D & \cdots & & \vdots & \vdots & & & \cdots & & & & \bar{0} \\
\vdots & & & & 0 & \ddots & & D & \bar{0} & \cdots & & & & & & \bar{0} \\
\vdots & & & & & \ddots & & \ddots & & \cdots & \cdots & & D & \bar{0} & \cdots & \bar{0} \\
\vdots & & & & & & & \ddots & & \ddots & \cdots & & \vdots & \vdots & \cdots & \bar{0} \\
\vdots & & & & & & & & & 0 & D & \cdots & D & \bar{0} & \cdots & \bar{0} \\
 & & & & & & & & & & 0 & D & \cdots & & \cdots & D \\
 & & & & & & & & & & & 0 & D & & & \vdots \\
 & & & & & & & & & & & & 0 & & \ddots & \vdots \\
0 & & & & \cdots & \cdots & & \cdots & & \cdots & & & & & 0 & D
\end{bmatrix},$$

where D is a division ring. So a ring R in Theorem B(iii) is represented as the following factor ring:



,

where $1_A = \sum_{l=1}^{m} h_l$, $1_{C_l} = \sum_{j=1}^{\alpha_l} f_j^{(l)}$ and $1_{C_l+D_l} = \sum_{j=1}^{n_l} f_j^{(l)}$ for each l. Further $HRH = A \cup (\cup_{l=1}^{k}(B_l \cup C_l))$ and $S_l R S_l = C_l \cup D_l$.

3. Characterization of a ring in Theorem B(iii)

In Theorem B a right almost hereditary ring is characterized by right ideals. The purpose of this section is to characterize a ring in Theorem B(iii) by left ideals.

First we characterize α_l in Theorem B(iii) not using the right module structure.

Lemma 3.1. *Let R be a ring satisfying $(a), (b)$ in Theorem B(iii) and α_l as in Theorem B(iii). Define an integer α_l' to satisfy $(h_1 + \cdots + h_m)Rf_j^{(l)} = 0$ for any $j = \alpha_l' + 1, \cdots, n_l$ but $(h_1 + \cdots + h_m)Rf_{\alpha_l'}^{(l)} \neq 0$. Then $\alpha_l = \alpha_l'$.*

Proof. $j \geq \alpha_l + 1$ iff $f_1^{(l)}R/f_1^{(l)}J^j$ is injective by the definition of α_l. And $j \geq \alpha_l' + 1$ iff $(h_1 + \cdots + h_m)Rf_j^{(l)} = 0$ by the definition of α_l'. Moreover, $p_l \geq \alpha_l + 1$ and $p_l \geq \alpha_l' + 1$ by Lemma 2.1(3) and (b), respectively. Hence we have only to show that $f_1^{(l)}R/f_1^{(l)}J^j$ is injective iff $(h_1 + \cdots + h_m)Rf_j^{(l)} = 0$ for any j $(\leq p_l)$.

($\Rightarrow$). Assume that $f_1^{(l)}R/f_1^{(l)}J^j$ is injective and there is h_s with $h_sRf_j^{(l)} \neq 0$. Then we have submodules $N \subset M$ of h_sR with an isomorphism $\phi_1 : M/N \to f_j^{(l)}R/f_j^{(l)}J$. Further there exists an isomorphism $\phi_2 : f_j^{(l)}R/f_j^{(l)}J \to S(f_1^{(l)}R/f_1^{(l)}J^j)$ since $j \leq p_l$ and $\{f_1^{(l)}R, f_2^{(l)}R, \cdots, f_{n_l}^{(l)}R\}$ is a Kupisch series with $f_{n_l}^{(l)}R_R$ simple by Lemma 2.1(1). So there exists an extension $\phi : h_sR/N \to f_1^{(l)}R/f_1^{(l)}J^j$ of $\phi_2\phi_1$ because $f_1^{(l)}R/f_1^{(l)}J^j$ is injective. Then $0 \neq \phi(h_s + N) \in (f_1^{(l)}R/f_1^{(l)}J^j)h_s$, i.e., $f_1^{(l)}Rh_s \neq 0$. This contradicts with (b).

($\Leftarrow$). By Lemma 2.1(2). □

Using Lemma 3.1 we have a lemma.

Lemma 3.2.

(1) *Let R be a ring in Theorem B(iii). We may assume that $h_sRh_t = 0$ for any $s > t$ by the representation form of a hereditary ring (see Remark 2.2). Then the following condition (e) holds:*

(e) *$h_sJ \cong (\oplus_{i=s+1}^m (h_iR)^{u_i}) \oplus (\oplus_{l=1}^k (f_1^{(l)}R/\ f_1^{(l)}J^{\alpha_l})^{v_l})$ as right R-modules for some non-negative integers $u_{s+1}, \cdots, u_m, v_1, \cdots, v_k$.*

(2) *Suppose that a ring R satisfies (a), (b), (e), then (c) and (d) hold. Hence (a), (b), (c), (d) in Theorem B(iii) can be replaced by (a), (b), (e).*

Proof. (1). h_sJH is projective as a right HRH-module by (d). So $h_sJH \cong (\oplus_{i=s+1}^m (h_iRH)^{u_i}) \oplus (\oplus_{l=1}^k (f_1^{(l)}RH)^{v_l})$ for some non-negative integers $u_{s+1}, \cdots, u_m, v_1, \cdots, v_k$ by (c) and the assumption that $h_sRh_t = 0$ for any $s > t$. Therefore (e) holds since $h_iR = h_iRH$ for any $i = 1, \cdots, m$ and $f_1^{(l)}R/f_1^{(l)}J^{\alpha_l}$ is a right HRH-module with $f_1^{(l)}R/f_1^{(l)}J^{\alpha_l} \cong f_1^{(l)}RH$ by Lemma 3.1 and Lemma 2.1(1), respectively.

(2). Assume that R satisfies (a), (b), (e). Clearly (c) holds. To show (d) we only show that gJH is projective as a right HRH-module for any $g \in \{h_s\}_{s=1}^m \cup \{f_j^{(l)}\}_{l=1,j=1}^{k \quad \alpha_l}$ because we always assume that rings are artinian in this paper. h_sJH $(= h_sJ)$ is a projective right HRH-module for any s by (e) since $f_1^{(l)}R/f_1^{(l)}J^{\alpha_l} \cong f_1^{(l)}RH$. Further $f_j^{(l)}JH \cong f_{j+1}^{(l)}RH$

for any $j = 1, \cdots, \alpha_l - 1$ and $f^{(l)}_{\alpha_l} JH = 0$ by Lemma 2.1(1). Therefore (d) holds. □

The following gives a characterization of a ring in Theorem B(iii) using left ideals.

Theorem 3.3. *Let R be a ring with $P(R) = \{h_1, \cdots, h_m, f^{(1)}_1, f^{(1)}_2, \cdots, f^{(1)}_{n_1}, f^{(2)}_1, \cdots, f^{(2)}_{n_2}, f^{(3)}_1, \cdots, f^{(k)}_{n_k}\}$. $P(R)$ satisfies (a),(b),(c),(d) in Theorem B(iii) if and only if the following five conditions hold for any $l = 1, \cdots, k$, we put $S_l := \sum_{j=1}^{n_l} f^{(l)}_j$,*

- (a') *S_lRS_l is a serial ring in the first category with $\{S_lRf^{(l)}_{n_l}, S_lRf^{(l)}_{n_l-1}, \cdots, S_lRf^{(l)}_1\}$ a Kupisch series of left S_lRS_l-modules,*
- (b') *$S_lR(1 - S_l) = 0$ and $(h_1 + \cdots + h_m)RS_l \neq 0$,*
- (c') *$Jf^{(l)}_j / J^2 f^{(l)}_j$ is simple as a left R-module for any $j = 2, \cdots, n_l$,*

we let α'_l be the same integer as in Lemma 3.1 and put $H' := \sum_{s=1}^{m} h_s + \sum_{l=1,j=1}^{k,\ \alpha'_l} f^{(l)}_j$, then

- (d') *$H'RH'$ is a hereditary ring , and*
- (f) *$E({}_RRf^{(l)}_1 / Jf^{(l)}_1)$ is projective as a left R-module for any $l = 1, \cdots, k$.*

Then we note that $\alpha'_l = \alpha_l$, and so $H' = H$ and (d') coincides with (d), where H and (d) are as in Theorem B(iii).

Before to show Theorem 3.3 we give a lemma.

Lemma 3.4. *Let R be a ring with $P(R) = \{h_1, \cdots, h_m, f^{(1)}_1, \cdots, f^{(1)}_{n_1}, f^{(2)}_1, \cdots, f^{(2)}_{n_2}, f^{(3)}_1, \cdots, f^{(k)}_{n_k}\}$.*

1. *Suppose that $P(R) = \{h_1, \cdots, h_m, f^{(1)}_1, \cdots, f^{(k)}_{n_k}\}$ satisfies (a'), (b'), (c') in Theorem 3.3. Then $\{Rf^{(l)}_{n_l}, Rf^{(l)}_{n_l-1}, \cdots, Rf^{(l)}_1\}$ is a Kupisch series of left R-modules for any $l = 1, \cdots, k$.*
2. *Suppose that $P(R) = \{h_1, \cdots, h_m, f^{(1)}_1, \cdots, f^{(k)}_{n_k}\}$ satisfies $(a'), (b')$, (c'), (f) in Theorem 3.3. Then $S(Rf^{(l)}_{\alpha'_l+1}) \cong Rf^{(l)}_1 / Jf^{(l)}_1$ for any $l = 1, \cdots, k$.*

Proof of Lemma 3.4. (1). $Jf^{(l)}_j / J^2 f^{(l)}_j \cong Rf^{(l)}_{j-1} / Jf^{(l)}_{j-1}$ or $\cong Rh_s / Jh_s$ for some s by $(a'), (b'), (c')$. Assume that $Jf^{(l)}_j / J^2 f^{(l)}_j \cong Rh_s / Jh_s$ for some s. Then $Jf^{(l)}_j = Rx$ for some $x \in h_s Jf^{(l)}_j$. On the other hand, there exists $0 \neq y \in f^{(l)}_{j-1} Jf^{(l)}_j$ $(\subseteq Jf^{(l)}_j = Rx)$ since $S_lJf^{(l)}_j / (S_lJS_l)^2 f^{(l)}_j \cong S_lRf^{(l)}_{j-1} / S_lJf^{(l)}_{j-1}$ by (a'). Therefore we have $0 \neq r \in f^{(l)}_{j-1} Rh_s$ with $rx = y$. This contradicts with (b).

(2). There exists an integer t with $E(Rf_1^{(l)}/Jf_1^{(l)}) \cong Rf_t^{(l)}$ by $(b'), (f)$. Then we claim that $t \geq \alpha'_l + 1$. $(h_1 + \cdots + h_m)Rf_{\alpha'_l}^{(l)} \neq 0$ by the definition of α'_l. So $(h_1 + \cdots + h_m)Rf_j^{(l)} \neq 0$ for any $j = 1, \cdots, \alpha'_l$ since $Rf_j^{(l)}$ is a projective cover of $J^{\alpha'_l - j} f_{\alpha'_l}^{(l)}$ by (1). Therefore $(h_1 + \cdots + h_m)S(Rf_j^{(l)}) \neq 0$ by (b'). Hence $t \geq \alpha'_l + 1$. On the other hand, $S_l Rf_j^{(l)} = Rf_j^{(l)}$ for any $j = \alpha'_l + 1, \cdots, n_l$ by the definition of α'_l and (b'). Therefore $S(Rf_i^{(l)}) \cong S(Rf_t^{(l)})$ $(\cong Rf_1^{(l)}/Jf_1^{(l)})$ for any $i = \alpha'_l + 1, \cdots, t$ by (1). Hence, in particular, $S(Rf_{\alpha'_l+1}^{(l)}) \cong Rf_1^{(l)}/Jf_1^{(l)}$. □

Proof of Theorem 3.3. We note that (a) and (a') are equivalent each other by, for instance, [1, 32.5. Lemma].

Assume that R is a ring in Theorem B(iii). It is obvious that (b') holds. And $\alpha'_l = \alpha_l$ holds for any l by Lemma 3.1. So $H' = H$. And (d') also holds. Further $f_1^{(l)}R$ is injective for any l by Lemma 2.1(3). And it is well known that $f_1^{(l)}R$ is injective iff $E(Rf_1^{(l)}/Jf_1^{(l)})$ is projective by Fuller's theorem (see, for instance, [1, 31.3. Theorem]). Therefore (f) holds. Hence we show that (c') holds. We obtain that $\{Rf_{n_l}^{(l)}, Rf_{n_l-1}^{(l)}, \cdots, Rf_{\alpha_l}^{(l)}\}$ is a Kupisch series of left R-modules by (a') $(\Leftrightarrow (a)), (b)$ and Lemma 3.1, i.e., $Jf_j^{(l)}/J^2 f_j^{(l)}$ is simple for any $j = \alpha_l + 1, \cdots, n_l$. So we only show that $Jf_j^{(l)}/J^2 f_j^{(l)}$ is simple for any $j = 2, \cdots, \alpha_l$. We may assume that $h_s R h_t = 0$ for any $s > t$ and (e) holds by Lemma 3.2(1). Put $f^{(l)} := \sum_{j=1}^{\alpha_l} f_j^{(l)}$. Then $h_m R f^{(l)} = h_m J f^{(l)} \cong (f_1^{(l)}R/f_1^{(l)}J^{\alpha_l})^{w_m} f^{(l)} = (f_1^{(l)}Rf^{(l)})^{w_m}$ as right $f^{(l)}Rf^{(l)}$-modules for any l, where w_m is a non-negative integer, the isomorphism is induced from $(b), (e)$ and the second equation holds since $\{f_1^{(l)}R, f_2^{(l)}R, \cdots, f_{n_l}^{(l)}R\}$ is a Kupisch series with $f_{n_l}^{(l)}R$ simple by Lemma 2.1(1). Therefore $h_{m-1}Rf^{(l)} \cong (f_1^{(l)}Rf^{(l)})^{w_{m-1}}$ for some non-negative integer w_{m-1} by $(b), (e)$. Inductively we have a right $f^{(l)}Rf^{(l)}$-isomorphism $\psi_s : h_s R f^{(l)} \to (f_1^{(l)}Rf^{(l)})^{w_s}$ for each $s = 1, \cdots, m$, where w_s is a non-negative integer. If $\alpha_l = 1$, then (c') holds for the l. So assume that $\alpha_l \geq 2$. Then $Jf_2^{(l)} = HJf_2^{(l)}$ by (a') $(\Leftrightarrow (a)), (b)$. And it is a projective left HRH-module by (d). Further $f_1^{(l)}Jf_2^{(l)}/f_1^{(l)}J^2 f_2^{(l)} \neq \bar{0}$ since $\{f_1^{(l)}R, f_2^{(l)}R, \cdots, f_{n_l}^{(l)}R\}$ is a Kupisch series. So $Jf_2^{(l)}$ contains a direct summand isomorphic to $Rf_1^{(l)}$ because $Rf_1^{(l)} = HRf_1^{(l)}$ by $(a'), (b)$. Therefore there exists a left R-monomorphism $\phi_2 : Rf_1^{(l)} \to Jf_2^{(l)}$. Then we claim that ϕ_2 is an isomorphism, i.e., $Jf_2^{(l)}/J^2 f_2^{(l)}$ is simple as a left R-module.

Concretely we show that $\phi_2|_{gRf_1^{(l)}} : gRf_1^{(l)} \to gJf_2^{(l)}$ is a bijection for any $g \in P(R)$. $f_1^{(l)}Rf_1^{(l)}$ is a division ring and ${}_{f_1^{(l)}Rf_1^{(l)}}f_1^{(l)}Jf_2^{(l)}$ is simple from (a). So $\phi_2|_{f_1^{(l)}Rf_1^{(l)}} : f_1^{(l)}Rf_1^{(l)} \to f_1^{(l)}Rf_2^{(l)}$ $(= f_1^{(l)}Jf_2^{(l)})$ is a left $f_1^{(l)}Rf_1^{(l)}$-isomorphism. Put $x_2 := \phi_2(f_1^{(l)})$. The right multiplication by x_2 induces a bijection $(x_2)_R : (f_1^{(l)}Rf_1^{(l)})^{w_s} \to (f_1^{(l)}Rf_2^{(l)})^{w_s}$ since $\phi_2|_{f_1^{(l)}Rf_1^{(l)}}$ is a bijection. For any $s = 1, \cdots, m$, let $(x_2)_R^s : h_sRf_1^{(l)} \to h_sRf_2^{(l)}$ be the right multiplication map by x_2. Then $(x_2)_R^s = (\psi_s|_{h_sRf_2^{(l)}})^{-1}(x_2)_R(\psi_s|_{h_sRf_1^{(l)}})$ holds because ψ_s is a right $f^{(l)}Rf^{(l)}$-isomorphism. Therefore $(x_2)_R^s$ is also a bijection, i.e., $\phi_2|_{h_sRf_1^{(l)}} : h_sRf_1^{(l)} \to h_sRf_2^{(l)} = h_sJf_2^{(l)}$ is a bijection for any $s = 1, \cdots, m$. Moreover, $(f_2^{(l)} + \cdots + f_{n_l}^{(l)})Rf_1^{(l)} = 0$ by (a'). And so $(f_2^{(l)} + \cdots + f_{n_l}^{(l)})Jf_2^{(l)} = 0$ because there exists a left S_lRS_l-epimorphism: $S_lRf_1^{(l)} \to S_lJf_2^{(l)}$ by (a'), i.e., $\phi_2|_{f_j^{(l)}Rf_1^{(l)}} : f_j^{(l)}Rf_1^{(l)} \to f_j^{(l)}Jf_2^{(l)}$ is a bijection for any $j = 2, \cdots, n_l$. Furthermore $f_j^{(l')}Rf_1^{(l)} = 0 = f_j^{(l')}Jf_2^{(l)}$ for any l' $(\neq l)$ and $j = 1, \cdots, n_{l'}$ by (b), i.e., $\phi_2|_{f_j^{(l')}Rf_1^{(l)}} : f_j^{(l')}Rf_1^{(l)} \to f_j^{(l')}Jf_2^{(l)}$ is a bijection. In consequence, ϕ_2 is an isomorphism, i.e., $Jf_2^{(l)}/J^2f_2^{(l)}$ is simple as a left R-module. Similarly, if $\alpha_l \geq 3$, we have a left R-monomorphism $\phi_3 : Rf_2^{(l)} \to Jf_3^{(l)}$. And $\phi_3|_{f_j^{(l)}Rf_2^{(l)}} : f_j^{(l)}Rf_2^{(l)} \to f_j^{(l)}Jf_3^{(l)}$ is a left $f_j^{(l)}Rf_j^{(l)}$-isomorphism for $j = 1, 2$. Put $x_3 := \phi_3(f_2^{(l)})$. Then the right multiplication by x_3 induces a bijection: $h_sRf_2^{(l)} \to h_sJf_3^{(l)}$ for any $s = 1, \cdots, m$. And $(\sum_{j=3}^{n_l} f_j^{(l)} + \sum_{l' \neq l, j=1}^{n_{l'}} f_j^{(l')})Rf_2^{(l)} = 0 = (\sum_{j=3}^{n_l} f_j^{(l)} + \sum_{l' \neq l, j=1}^{n_{l'}} f_j^{(l')})Jf_3^{(l)}$. Therefore ϕ_3 is an isomorphism, i.e., $Jf_3^{(l)}/J^2f_3^{(l)}$ is simple as a left R-module. Inductively $Jf_j^{(l)}/J^2f_j^{(l)}$ is simple as a left R-module for any $j = 2, \cdots, \alpha_l$.

Conversely assume that $P(R) = \{h_1, \cdots, h_m, f_1^{(1)}, \cdots, f_{n_k}^{(k)}\}$ satisfies $(a'), (b'), (c'), (d'), (f)$. To show that $P(R) = \{h_1, \cdots, h_m, f_1^{(1)}, \cdots, f_{n_k}^{(k)}\}$ satisfies (b), we only show that $(h_1 + \cdots + h_m)R(f_{p_l}^{(l)} + \cdots + f_{n_l}^{(l)}) = 0$, where $p_l := |f_1^{(l)}R|$. $Rf_{p_l}^{(l)} = E(Rf_1^{(l)}/Jf_1^{(l)})$ by (f) and Fuller's theorem because $\{f_1^{(l)}R, f_2^{(l)}R, \cdots, f_{n_l}^{(l)}\}$ is a Kupisch series by Lemma 2.1(1). So $\alpha_l' + 1 \leq p_l$ by the definition of α_l' since $\{Rf_{n_l}^{(l)}, Rf_{n_l-1}^{(l)}, \cdots, Rf_1^{(l)}\}$ is a Kupisch series by Lemma 3.4(1), i.e., $(h_1 + \cdots + h_m)R(f_{p_l}^{(l)} + \cdots + f_{n_l}^{(l)}) = 0$ holds. Then $P(R) = \{h_1, \cdots, h_m, f_1^{(1)}, \cdots, f_{n_k}^{(k)}\}$ satisfies (d) by Lemma 3.1. Last we show that $P(R) = \{h_1, \cdots, h_m, f_1^{(1)}, \cdots, f_{n_k}^{(k)}\}$ satisfies (c). Concretely we only show that $h_s J f_j^{(l)} \subseteq h_s J^2 f_j^{(l)}$ for any s, l and $j = 2, \cdots, \alpha_l$ because $h_s R f_j^{(l)} = 0$ for any s, l and $j = \alpha_l + 1, \cdots, n_l$ by Lemma 3.1. $J f_j^{(l)} / J^2 f_j^{(l)} \cong R f_{j-1}^{(l)} / J f_{j-1}^{(l)}$ for any l and $j = 2, \cdots, \alpha_l$ since $\{R f_{n_l}^{(l)}, R f_{n_l-1}^{(l)}, \cdots, R f_1^{(l)}\}$ is a Kupisch series of left R-modules. So there is a left R-epimorphism $\phi_j : R f_{j-1}^{(l)} \to J f_j^{(l)}$. Now $R f_{j-1}^{(l)} = H R f_{j-1}^{(l)}$ and $J f_j^{(l)} = H J f_j^{(l)}$ hold by (b) because $\{R f_{n_l}^{(l)}, R f_{n_l-1}^{(l)}, \cdots, R f_1^{(l)}\}$ is a Kupisch series and $j \leq \alpha_l$. So ϕ_j is considered as a left HRH-epimorphism. Therefore it is a bijection since $J f_j^{(l)}$ is projective as a left HRH-module by (d) ($\Leftrightarrow$ (d')), i.e., ϕ_j is a left R-isomorphism. Put $x_j := \phi_j(f_{j-1}^{(l)})$. Then the right multiplication by x_j induces a bijection: $h_s R f_{j-1}^{(l)} \to h_s J f_j^{(l)}$ for any s. Therefore $h_s J f_j^{(l)} = h_s R f_{j-1}^{(l)} x_j = h_s J f_{j-1}^{(l)} x_j \subseteq h_s J^2 f_j^{(l)}$ because $x_j \in f_{j-1}^{(l)} J f_j^{(l)}$, i.e., $P(R) = \{h_1, \cdots, h_m, f_1^{(1)}, \cdots, f_{n_k}^{(k)}\}$ satisfies (c). □

4. A dual ring to an almost hereditary ring

The purpose of this section is to show the following Theorem 4.1.

Theorem 4.1. *R satisfies $(\#)_l$ if and only if R is a right almost hereditary ring.*

Before giving a proof of Theorem 4.1 we recall a well known useful lemma.

Lemma C. *Put $\overline{R} = R/B$, where B is a two-sided ideal of R.*

(1) *Suppose that E is an injective left R-module. Then $r_E(B) = \{x \in E \mid Bx = 0\}$ is injective as a left $\overline{R}$-module.*

(2) *Suppose that E' is an injective left $\overline{R}$-module. Consider E' as a left R-module naturally. Then $E' = r_{E({}_RE')}(B)$.*

Now we give a proof of "if" part of Theorem 4.1. A proof of "only if" part is given in the next section.

Proof for "if" part of Theorem 4.1. We may assume that R is an indecomposable ring.

Suppose that R is a hereditary ring, then clearly the condition $(\#)_l$ holds.

Next suppose that R is a serial ring. Assume that there is an indecomposable injective left R-module E with $E/S(E)$ not injective. Then $E' := E(E/S(E))$ is a uniserial module since R is a serial ring. So we have a positive integer k with $J^k E' = E/S(E)$ and a projective cover

$\phi_i : Rg_i \to J^iE'$ for each $i = 0, \cdots, k-1$, where $g_i \in P(R)$. Then we claim that $\mathrm{Ker}\phi_i = 0$ for any $i = 0, \cdots, k-1$. Assume that there exists t with $\mathrm{Ker}\phi_t \neq 0$. Then we can naturally induce an epimorphism $\psi : J^{k-t}g_t \to E$ from ϕ_t since $\phi_t(J^{k-t}g_t) = E/S(E)$ and $S(E)$ is simple. On the other hand $J^{k-t}g_t$ is a proper submodule of Rg_t because $t \leq k-1$. This contradicts with the assumption that E is injective. Therefore J^iE' is projective for any $i = 0, \cdots, k-1$. Hence $E/S(E)$ ($\cong J^kE'$) is (cyclic) almost injective by Lemma A, i.e., the condition $(\#)_l$ holds.

Last suppose that R is a ring in Theorem B(iii). Let $P(R) = \{h_1, \cdots, h_m, f_1^{(1)}, f_2^{(1)}, \cdots, f_{n_1}^{(1)}, f_1^{(2)}, \cdots, f_{n_2}^{(2)}, f_1^{(3)}, \cdots, f_{n_k}^{(k)}\}$, α_l, H and S_l be the same notations as in Theorem B(iii). We put $E_s := E(Rh_s/Jh_s)$, $E_j^{(l)} := E(Rf_j^{(l)}/Jf_j^{(l)})$, $A_l := (1-S_l)R$ and $B := \sum_{l=1, j=\alpha_l+1}^{k,\ \ n_l} Rf_j^{(l)}$ for any s, l, j. Then we note that $A_1, \cdots, A_k$ and B are two-sided ideals, $R/A_l \cong S_lRS_l$ (which is a serial ring in the first category) and $R/B \cong HRH$ (which is a hereditary ring) by Theorem B(iii)$(a), (b), (d)$. We show that $E_s/S(E_s)$ and $E_j^{(l)}/S(E_j^{(l)})$ are either injective or finitely generated almost injective for each s, l, j.

For any l, j, $\mathrm{Hom}(R(1-S_l), E_j^{(l)}) = 0$ by Theorem B(iii)(b), i.e., $(1-S_l)RE_j^{(l)} = 0$. Therefore $r_{E_j^{(l)}}(A_l) = E_j^{(l)}$. Hence $E_j^{(l)}$ is an injective left R/A_l-module by Lemma C(1), i.e.,

$$(*) \quad (E_j^{(l)} =)\ E({}_RRf_j^{(l)}/Jf_j^{(l)}) = E({}_{R/A_l}Rf_j^{(l)}/Jf_j^{(l)}) \text{ for any } l, j.$$

So we claim that

(**) $E_j^{(l)} \cong Rf_{j'}^{(l)}/J^uf_{j'}^{(l)}$ for some j' ($\geq \alpha_l + 1$) and a positiove integer u and they are uniserial left R-modules.

Since R/A_l is a serial ring and $\{Rf_{n_l}^{(l)}, Rf_{n_l-1}^{(l)}, \cdots, Rf_1^{(l)}\}$ is a Kupisch series of left R-modules by Lemma 3.4(1), we have an isomorphism in (**) for some j' ($\geq j$) and u and they are uniserial left R-modules. $j' \geq \alpha_l + 1$ by Lemma 3.4(2) because $\{Rf_{n_l}^{(l)}, Rf_{n_l-1}^{(l)}, \cdots, Rf_1^{(l)}\}$ is a Kupisch series and $\alpha_l = \alpha'_l$ by Lemma 3.1. And we already show that serial rings satisfy the condition $(\#)_l$. So $E_j^{(l)}/S(E_j^{(l)})$ ($\cong Rf_{j'}^{(l)}/J^{u-1}f_{j'}^{(l)}$) is (cyclic) almost injective as a left R/A_l-module. If $E_j^{(l)}/S(E_j^{(l)})$ is injective as a left R/A_l-module, $E_j^{(l)}/S(E_j^{(l)}) \cong E({}_{R/A_l}Rf_{j+1}^{(l)}/Jf_{j+1}^{(l)})$ since $\{Rf_{n_l}^{(l)}, Rf_{n_l-1}^{(l)}, \cdots, Rf_1^{(l)}\}$ is a Kupisch series. So $E_j^{(l)}/S(E_j^{(l)})$ is injective also as a left R-module by (*). Assume that $E_j^{(l)}/S(E_j^{(l)})$ is (almost injective but) not injective as a left R/A_l-module. $E({}_{R/A_l}E_j^{(l)}/S(E_j^{(l)}))$

$\cong E({}_{R/A_l}Rf^{(l)}_{j+1}/Jf^{(l)}_{j+1}) \cong E^{(l)}_{j+1}$, where the second isomorphism is given by (*). There is a positive integer w such that $E^{(l)}_j/S(E^{(l)}_j) \cong J^w E^{(l)}_{j+1}$ and $J^i E^{(l)}_{j+1}$ is projective as a left R/A_l-module for any $i = 0, \cdots, w-1$ by Lemma A. Therefore to show that $E^{(l)}_j/S(E^{(l)}_j)$ is (cyclic) almost injective also as a left R-module, it is enough to show that $J^i E^{(l)}_{j+1}$ is projective also as a left R-module for any $i = 0, \cdots, w-1$ by Lemma A. There are integers j', j'' $(\geq \alpha_l + 1)$, u, v such that $E^{(l)}_j \cong Rf^{(l)}_{j'}/J^u f^{(l)}_{j'}$ and $E^{(l)}_{j+1} \cong Rf^{(l)}_{j''}/J^v f^{(l)}_{j''}$ by (**). $S_l Rf^{(l)}_{j''} = Rf^{(l)}_{j''}$ by Lemma 3.1 and Theorem B(iii)(b) since $j'' \geq \alpha_l + 1$. So $j'' \geq |{}_{S_lRS_l}S_lRf^{(l)}_{j''}| = |{}_RRf^{(l)}_{j''}|$ by Theorem 3.3(a'). And $|{}_RRf^{(l)}_{j''}| - w = |{}_RJ^w f^{(l)}_{j''}| \geq 1$ because $0 \neq E^{(l)}_j/S(E^{(l)}_j) \cong J^w E^{(l)}_{j+1} \cong J^w f^{(l)}_{j''}/J^v f^{(l)}_{j''}$. So $j'' \geq |{}_RRf^{(l)}_{j''}| \geq w+1$, i.e., $j''-w \geq 1$. Therefore, for each $p = 0, \cdots, w$, $Rf^{(l)}_{j''-p}$ is a projective cover of $J^p f^{(l)}_{j''}/J^v f^{(l)}_{j''}$ because $\{Rf^{(l)}_{n_l}, Rf^{(l)}_{n_l-1}, \cdots, Rf^{(l)}_1\}$ is a Kupisch series. Hence $J^p f^{(l)}_{j''}/J^v f^{(l)}_{j''} \cong Rf^{(l)}_{j''-p}/J^{v-p} f^{(l)}_{j''-p}$ for $Rf^{(l)}_{j''-p}$ is uniserial. So we obtain $j'' - w = j'$ $(\geq \alpha_l + 1)$ since $Rf^{(l)}_{j''-w}/J^{v-w} f^{(l)}_{j''-w} \cong J^w f^{(l)}_{j''}/J^v f^{(l)}_{j''} \cong J^w E^{(l)}_{j+1} \cong E^{(l)}_j/S(E^{(l)}_j) \cong Rf^{(l)}_{j'}/J^{u-1} f^{(l)}_{j'}$. Therefore $j'' - w \geq \alpha_l + 1$, i.e., $j'' - i \geq \alpha_l + 1$ for any $i = 0, \cdots, w-1$. So $S_l Rf^{(l)}_{j''-i} = Rf^{(l)}_{j''-i}$ by Lemma 3.1 and Theorem B(iii)(b). Hence $Rf^{(l)}_{j''-i}$ is a left R/A_l-module. Therefore we can consider a natural left R/A_l-epimorphism: $Rf^{(l)}_{j''-i} \to Rf^{(l)}_{j''-i}/J^{v-i} f^{(l)}_{j''-i} \cong J^i f^{(l)}_{j''}/J^v f^{(l)}_{j''} \cong J^i E^{(l)}_{j+1}$ and it splits because $J^i E^{(l)}_{j+1}$ is projective as a left R/A_l-module, i.e., it is an isomorphism. Therefore $J^i E^{(l)}_{j+1}$ $(\cong Rf^{(l)}_{j''-i})$ is projective as a left R-module.

By the definition of α'_l $(= \alpha_l)$, $h_s Rf^{(l)}_j = 0$ for any s, l, and j $(\geq \alpha_l + 1)$. So $\mathrm{Hom}(Rf^{(l)}_j, E_s) = 0$, i.e., $f^{(l)}_j E_s = 0$. Therefore $BE_s = 0$, i.e., $r_{E_s}(B) = E_s$. Hence E_s is injective as a left R/B-module by Lemma C(1), i.e.,

(***) $\quad E_s = E({}_{R/B}Rh_s/Jh_s)$ for any s.

So $E_s/S(E_s)$ is injective as a left R/B-module since R/B is a hereditary ring. Let E' be an indecomposable direct summand of $E_s/S(E_s)$. And consider E' as a left R-module. We show that E' is injective or finitely generated almost injective. If $S(E') \cong Rh_{s'}/Jh_{s'}$ for some s', then $E' \cong E({}_{R/B}Rh_{s'}/Jh_{s'}) = E({}_RRh_{s'}/Jh_{s'})$ by (***), i.e., E' is injective also as a left R-module. Assume that $S(E') \cong Rf^{(l)}_j/Jf^{(l)}_j$ for some j and

l. Then we claim that $j = 1$. There exists $x \in E_s$ with $Rx/S(E_s) = S(E')$ because E' is an indecomposable direct summand of $E_s/S(E_s)$. Then $Rf_j^{(l)}$ is a projective cover of Rx since $Rx/S(E_s) = S(E') \cong Rf_j^{(l)}/Jf_j^{(l)}$. Therefore $Jf_j^{(l)}/J^2f_j^{(l)}$ contains a direct summand isomorphic to Jx ($= S(E_s) \cong Rh_s/Jh_s$). But, if $j \geq 2$, then $Jf_j^{(l)}/J^2f_j^{(l)} \cong Rf_{j-1}^{(l)}/Jf_{j-1}^{(l)}$ since $\{Rf_{n_l}^{(l)}, Rf_{n_l-1}^{(l)}, \cdots, Rf_1^{(l)}\}$ is a Kupisch series by Lemma 3.4(1). This is a contradiction. Hence $j = 1$. Therefore $E(_RE') \cong E_1^{(l)}$. Now there are integers j' ($\geq \alpha_l + 1$) and u such that $E_1^{(l)} \cong Rf_{j'}^{(l)}/J^uf_{j'}^{(l)}$ and they are uniserial left R-modules by (**). Then we claim that $E_1^{(l)} \cong Rf_{j'}^{(l)}$. It is enough to show that $J^uf_{j'}^{(l)} = 0$. $J^{u-1}f_{j'}^{(l)}/J^uf_{j'}^{(l)} \cong S(_RE_1^{(l)}) \cong Rf_1^{(l)}/Jf_1^{(l)}$. So $S_lJ^uf_{j'}^{(l)} = 0$ by Theorem 3.3(a'). Further $(h_1 + \cdots + h_m)J^uf_{j'}^{(l)} = 0$ because $j' \geq \alpha_l + 1 = \alpha_l' + 1$. Therefore $J^uf_{j'}^{(l)} = 0$ by Theorem 3.3(b'). Moreover we claim that

$$(****) \quad J^if_{j'}^{(l)} \cong Rf_{j'-i}^{(l)} \text{ for any } i = 0, \cdots, j' - \alpha_l - 1.$$

$S(Rf_{j'}^{(l)}) \cong S(E_1^{(l)}) \cong Rf_1^{(l)}/Jf_1^{(l)} \cong S(Rf_{\alpha_l+1}^{(l)})$ by Lemma 3.4(2) and Lemma 3.1. So, for any $i = 0, \cdots, j' - \alpha_l - 1$, $S(Rf_{j'}^{(l)}) \cong S(Rf_{j'-i}^{(l)})$. Therefore $J^if_{j'}^{(l)} \cong Rf_{j'-i}^{(l)}$. Now to show that E' is cyclic almost injective as a left R-module, we have only to show

(1) $J^{j'-\alpha_l}E_1^{(l)} \cong E'$, and
(2) $J^iE_1^{(l)}$ is projective as a left R-module for any $i = 0, \cdots, j' - \alpha_l - 1$

by Lemma A since $E(_RE') \cong E_1^{(l)}$ and $E_1^{(l)}$ is a uniserial left R-module.

(1). $E' = r_{E(_RE')}(B)$ by Lemma C(2) since E' is injective as a R/B-module. On the other hand, $E(_RE') \cong E_1^{(l)} \cong Rf_{j'}^{(l)}$. Therefore $E' \cong r_{Rf_{j'}^{(l)}}(B)$. So we only show that $r_{Rf_{j'}^{(l)}}(B) = J^{j'-\alpha_l}E_1^{(l)}$. For any $j = \alpha_l + 1, \cdots, n_l$, $f_j^{(l)}J^{j'-\alpha_l}E_1^{(l)} \cong f_j^{(l)}J^{j'-\alpha_l}f_{j'}^{(l)}$. On the other hand, $Rf_{\alpha_l}^{(l)}$ is a projective cover of $J^{j'-\alpha_l}f_{j'}^{(l)}$ and $f_j^{(l)}Rf_{\alpha_l}^{(l)} = 0$ since $\{Rf_{n_l}^{(l)}, Rf_{n_l-1}^{(l)}, \cdots, Rf_1^{(l)}\}$ is a Kupisch series and $j \geq \alpha_l + 1$. So there is a left $f_j^{(l)}Rf_j^{(l)}$-epimorphism: $(0 =)\ f_j^{(l)}Rf_{\alpha_l}^{(l)} \to f_j^{(l)}J^{j'-\alpha_l}f_{j'}^{(l)}$. Therefore $f_j^{(l)}J^{j'-\alpha_l}E_1^{(l)} = 0$. Hence $BJ^{j'-\alpha_l}E_1^{(l)} = 0$ by Theorem B(iii)(b). Further $f_{\alpha_l+1}^{(l)}J^{j'-\alpha_l-1}E_1^{(l)} \cong f_{\alpha_l+1}^{(l)}J^{j'-\alpha_l-1}f_{j'}^{(l)} \cong f_{\alpha_l+1}^{(l)}Rf_{\alpha_l+1}^{(l)} \neq 0$, where we obtain the last isomorphism from (****). Hence $r_{Rf_{j'}^{(l)}}(B) = J^{j'-\alpha_l}E_1^{(l)}$.

(2). $J^iE_1^{(l)} \cong Rf_{j'-i}^{(l)}$ for any $i = 0, \cdots, j' - \alpha_l - 1$ by (****) since $E_1^{(l)} \cong Rf_{j'}^{(l)}$. Hence each $J^iE_1^{(l)}$ is projective as a left R-module. □

5. A proof for "only if" part of Theorem 3.1

The purpose of this section is to give a proof for "only if" part of Theorem 4.1. Throughout this section, we let R be a ring satisfying $(\#)_l$.

First we consider a special case.

Lemma 5.1. *(cf.* [9, Lemma 6]*). Suppose that Rg is not injective for any $g \in P(R)$. Then R is a hereditary ring.*

Proof. Any finitely generated almost injective left R-module is injective by assumption and Lemma A. Therefore R is hereditary by Lemma A since R satisfies $(\#)_l$. □

So we may assume that there is $f_1 \in P(R)$ with Rf_1 injective. Then $Rf_1/S_{w-1}(Rf_1)$ is injective for any $w = 1, \cdots, |_RRf_1|$ or there exists $\gamma_1 \in \{1, \cdots, |_RRf_1| - 1\}$ such that $Rf_1/S_{w-1}(Rf_1)$ is injective for any $w = 1, \cdots, \gamma_1$ and $Rf_1/S_{\gamma_1}(Rf_1)$ is not injective but almost injective since R satisfies the condition $(\#)_l$. If there exists γ_1, then we have $f_2 \in P(R)$ with Rf_2 injective and a positive integer β_2 such that $J^{\beta_2}f_2 \cong Rf_1/S_{\gamma_1}(Rf_1)$ and $J^{j-1}f_2$ is projective for any $j = 1, \cdots, \beta_2$ by Lemma A. For each $j = 1, \cdots, \beta_2$, let $f_{2,j} \in P(R)$ such that $Rf_{2,j} \cong J^{j-1}f_2$. (So $f_{2,1} = f_2$.) Moreover, $Rf_{2,1}/S_{w-1}(Rf_{2,1})$ is injective for any $w = 1, \cdots, |_RRf_{2,1}|$ or there exists $\gamma_2 \in \{1, \cdots, |_RRf_{2,1}| - 1\}$ such that $Rf_{2,1}/S_{w-1}(Rf_{2,1})$ is injective for any $w = 1, \cdots, \gamma_2$ and $Rf_{2,1}/S_{\gamma_2}(Rf_{2,1})$ is not injective but almost injective. Continuing this procedure and put $f_{1,1} := f_1$, it terminates when either the following (I) or (II) holds.

(I) $f_{n,1} = f_{1,1}$ for some n (≥ 2), i.e., $\{Rf_{n,1}, Rf_{n,2}, \cdots, Rf_{n,\beta_n}, Rf_{n-1,1}, \cdots, Rf_{2,1}, \cdots, Rf_{2,\beta_2}\}$ is a cyclic Kupisch series.

(II) There exists n (≥ 1) such that $Rf_{n,1}/S_{w-1}(Rf_{n,1})$ are injective for any $w = 1, \cdots, |Rf_{n,1}|$. (Then $\{Rf_{n,1}, Rf_{n,2}, \cdots, Rf_{n,\beta_n}, Rf_{n-1,1}, \cdots, Rf_{2,\beta_2}, Rf_{1,1}\}$ is a Kupisch series.)

Then we claim that the following (†) holds in both cases (I),(II).

(†) $Rf_{i,1}$ is uniserial for any $i = 1, \cdots, n$.

First assume that (II) holds. Then $Rf_{n,1}$ is uniserial since $Rf_{n,1}/S_{w-1}(Rf_{n,1})$ is injective for any $w = 1, \cdots, |_RRf_{n,1}|$. Further $Rf_{n-1,1}$ is also uniserial since $Rf_{n-1,1}/S_{\gamma_{n-1}}(Rf_{n-1,1}) \cong J^{\beta_n}f_{n,1}$ and $S_{\gamma_{n-1}}(Rf_{n-1,1})$ is uniserial. So we obtain (†) inductively. Next assume that (I) holds. $S_{\gamma_n}(Rf_{n,1})$ is uniserial since $Rf_{n,1}/S_{w-1}(Rf_{n,1})$ is indecomposable injective for any $w = 1, \cdots, \gamma_n$. So $S_{\gamma_n+\gamma_{n-1}}(Rf_{n-1,1})$ is uniserial since $J^{\beta_n}f_{n,1} \cong$

$Rf_{n-1,1}/S_{\gamma_{n-1}}(Rf_{n-1,1})$ and $S_{\gamma_{n-1}}(Rf_{n-1,1})$ is uniserial by the same reason as $f_{n,1}$. Further we obtain that $S_{\gamma_n+\gamma_{n-1}+\gamma_{n-2}}(Rf_{n-2,1})$ is also uniserial. Continue this argument, we see that (†) holds because $\{Rf_{n,1}, \cdots, Rf_{2,\beta_2}\}$ is a cyclic Kupisch series.

Now, when (II) holds, put $\beta_1 := |_R Rf_{1,1}|$ and we have $f_{1,j} \in P(R)$ with $Rf_{1,j}/Jf_{1,j} \cong J^{j-1}f_{1,1}/J^j f_{1,1}$ for each $j = 2, \cdots, \beta_1$ by (†). Then the following (††) holds in both cases (I),(II) by the definition of $\{f_{n,1}, f_{n,2}, \cdots\}$.

(††) For each i, j, there exist integers p, q such that $E(Rf_{i,j}/Jf_{i,j}) \cong Rf_{p,1}/S_q(Rf_{p,1})$.

Therefore, when (II) holds, $\{f_{i,j}\}_{i=1,j=1}^{n \ \ \beta_i}$ is a set of distinct elements in $P(R)$.

Put $S := \sum_{i=2,j=1}^{n \ \ \beta_i} f_{i,j}$ if (I) holds and $S := \sum_{i=1,j=1}^{n \ \ \beta_i} f_{i,j}$ if (II) holds. Then $S \cdot E(Rf_{i,j}/Jf_{i,j}) = E(Rf_{i,j}/Jf_{i,j})$ holds for any i, j by (††) and the definition of $\{f_{n,1}, f_{n,2}, \cdots\}$, i.e., $E(Rf_{i,j}/Jf_{i,j})$ is considered as a left SRS-module. And further we claim that the following (†††) holds in both cases (I),(II).

(†††) Suppose that $SRf_{i,j} = Rf_{i,j}$ holds for any i, j. Then $E(_R Rf_{i,j}/Jf_{i,j}) = E(_{SRS} SRf_{i,j}/SJf_{i,j})$.

A left SRS-module $E(_R Rf_{i,j}/Jf_{i,j})$ is $SRf_{s,t}$-injective for any s, t since it is $Rf_{s,t}$-injective as a left R-module and $SRf_{s,t} = Rf_{s,t}$ by assumption. So (†††) holds by Azumaya's Theorem (see, for instance, [1, 16.13. Proposition (2)]).

Lemma 5.2 (cf. [9, Lemmas 7 and 8]). *Suppose that (I) holds. Then*

(1) *SRS is a serial ring in the second category, and*
(2) *$R = SRS \oplus (1-S)R(1-S)$ as rings.*

Proof. (1). SRS is a left serial ring by (†) and the definition of $\{f_{i,j}\}_{i=2,j=1}^{n \ \ \beta_i}$. Further $SRf_{i,j} = Rf_{i,j}$ for any i, j because $\{Rf_{n,1}, \cdots, Rf_{2,\beta_2}\}$ is a cyclic Kupisch series of left R-modules. Therefore SRS is a left co-serial ring by (†),(††),(†††). Hence SRS is a serial ring by, for instance, [1, 32.3. Theorem]. Moreover SRS is in the second category since $\{Rf_{n,1}, \cdots, Rf_{2,\beta_2}\}$ is a cyclic Kupisch series of left R-modules and $SRf_{i,j} = Rf_{i,j}$ for any i, j.

(2). Since $SRf_{i,j} = Rf_{i,j}$ for any i, j, it is clear that $(1-S)RS = 0$. So it suffices to prove $SR(1-S) = 0$. Assume that there are u, v with $f_{u,v}R(1-S) \neq 0$. Then there exist left R-submodules $X \supset Y$ of $R(1-S)$ with a left R-isomorphism $\phi : X/Y \to Rf_{u,v}/Jf_{u,v}$. Further we have an isomorphism $\phi' : E(Rf_{u,v}/Jf_{u,v}) \to Rf_{w,1}/J^m f_{w,1}$ for some w and m by the definition of $\{f_{n,1}, \cdots, f_{2,\beta_2}\}$. So there exists a nonzero homomorphism $\tilde{\phi} : R(1-S)/Y \to Rf_{w,1}/J^m f_{w,1}$ with $\tilde{\phi}|_{X/Y} = \phi'\phi$. Therefore since $R(1-$

S) is a projective left R-module, there exists a nonzero homomorphism: $R(1-S) \to Rf_{w,1}$, i.e., $(1-S)Rf_{w,1} \neq 0$, a contradiction. $\square$

By Lemmas 5.1 and 5.2 we only show the following Lemma 5.3 to complete a proof of "only if part" of Theorem 4.1.

Lemma 5.3. *Suppose that R is an indecomposable ring, there is $g \in P(R)$ with Rg injective and R does not have a cyclic Kupisch series. Then R is a serial ring in the first category or a ring in Theorem B(iii).*

In the remainder of this section we show Lemma 5.3.

Let $f_{1,1} \in P(R)$ with $Rf_{1,1}$ injective. By the same way as in just before Lemma 5.2, we define primitive idempotents $f_{2,1}, f_{2,2}, \cdots, f_{2,\beta_2}, f_{3,1}, \cdots$ inductively. Then (II) holds since R does not have a cyclic Kupisch series, i.e., we obtain a Kupisch series $\{Rf_{n,1}, Rf_{n,2}, \cdots, Rf_{2,\beta_2}, Rf_{1,1}\}$.

Assume that there exists another $f'_{1,1} \in P(R)$ with $Rf'_{1,1}$ injective. We obtain a Kupisch series $\{Rf'_{m,1}, \cdots, Rf'_{m,\beta'_m}, Rf'_{m-1,1}, \cdots, Rf'_{2,\beta'_2}, Rf'_{1,1}\}$ by the same way as $\{Rf_{n,1}, \cdots, Rf_{n,\beta_n}, Rf_{n-1,1}, \cdots, Rf_{2,\beta_2}, Rf_{1,1}\}$. We claim that, if $f_{i,j} = f'_{k,l}$ for some i, j, k, l, then either $\{Rf_{n,1}, \cdots, Rf_{1,1}\} \subseteq \{Rf'_{m,1}, \cdots, Rf'_{1,1}\}$ or $\{Rf_{n,1}, \cdots, Rf_{1,1}\} \supseteq \{Rf'_{m,1}, \cdots, Rf'_{1,1}\}$ holds. $S(Rf_{i,1}) \cong S(Rf_{i,j}) = S(Rf'_{k,l}) \cong S(Rf'_{k,1})$. Hence $Rf_{i,1} = Rf'_{k,1}$ since $Rf_{i,1}$ and $Rf'_{k,1}$ are injective, i.e., $f_{i,1} = f'_{k,1}$ holds. Then we note that $\{Rf_{n,1}, \cdots, Rf_{i,1}\} = \{Rf'_{m,1}, \cdots, Rf'_{k,1}\}$ by the definition of $\{f_{n,1}, \cdots, f_{i,1}\}$ and $\{f'_{m,1}, \cdots, f'_{k,1}\}$. So, if $i = 1$ (resp. $k = 1$), then $\{Rf_{n,1}, \cdots, Rf_{1,1}\} \subseteq \{Rf'_{m,1}, \cdots, Rf'_{1,1}\}$ (resp. $\{Rf_{n,1}, \cdots, Rf_{1,1}\} \supseteq \{Rf'_{m,1}, \cdots, Rf'_{1,1}\}$) holds. Therefore we assume that $i > 1$ and $k > 1$. Then $\beta_i = \beta'_k$ holds since $f_{i,1} = f'_{k,1}$ and β_i (resp. β'_k) is the smallest positive integer t such that $J^t f_{i,1}$ (resp. $J^t f_{k,1}$) is not projective. So $Rf_{i-1,1}/S_{\gamma_{i-1}}(Rf_{i-1,1}) \cong J^{\beta_i} f_{i,1} = J^{\beta'_k} f'_{k,1} \cong Rf'_{k-1,1}/S_{\gamma'_{k-1}}(Rf'_{k-1,1})$, where γ'_{k-1} is an integer defined as γ_{i-1}. Therefore $Rf_{i-1,1} \cong Rf'_{k-1,1}$, i.e., $f_{i-1,1} = f'_{k-1,1}$. Inductively we obtain $f_{i-p,1} = f'_{k-p,1}$ for any $p = 1, 2, \cdots$. Then $i - p = 1$ or $k - p = 1$ holds for some p, i.e., the previous case holds. Hence we may let $f_{1,1}$ be a primitive idempotent with $Rf_{1,1}$ injective such that it induces the longest Kupisch series $\{Rf_{n,1}, \cdots, Rf_{n,\beta_n}, Rf_{n-1,1}, \cdots, Rf_{2,\beta_2}, Rf_{1,1}\}$.

Since (II) holds, we can further define primitive idempotents $f_{1,2}, \cdots, f_{1,\beta_1}$ by the same way as in just before Lemma 5.2. In consequence, we obtain a sequence $\{f_{n,1}, \cdots, f_{n,\beta_n}, f_{n-1,1}, \cdots, f_{2,1}, \cdots, f_{2,\beta_2}, f_{1,1}, \cdots, f_{1,\beta_1}\}$ of distinct elements in $P(R)$ such that its subsequence induces a Kupisch series $\{Rf_{n,1}, Rf_{n,2}, \cdots, Rf_{2,\beta_2}, Rf_{1,1}\}$, $Rf_{n,1}/S_{w-1}(Rf_{n,1})$ is injective for any $w = 1, \cdots, |{}_R Rf_{n,1}|$ and $Rf_{i,1}/S_{w(i)-1}(Rf_{i,1})$ is also injective for any $i = 1, \cdots, n-1$ and $w(i) = 1, \cdots, \gamma_i$.

Suppose that $\{Rf_{n,1},\ Rf_{n,2},\ \cdots,\ Rf_{2,\beta 2},\ Rf_{1,1},\ \cdots,\ Rf_{1,\beta_1}\}$ is a Kupisch series with Rf_{1,β_1} a simple left R-module. Then a ring SRS is left serial and left co-serial by $(\dagger),(\dagger\dagger),(\dagger\dagger\dagger)$ since $SRf_{i,j} = Rf_{i,j}$ holds for any i, j. So it is a serial ring in the first category (see, for instance, [1, 32.3. Theorem]). Further it is obvious that $(1-S)RS = 0$. And $SR(1-S) = 0$ also holds by the same argument as the proof of Lemma 5.2(2) using $(\dagger\dagger)$. Therefore $R = SRS \oplus (1-S)R(1-S)$. Hence $1 - S = 0$ because R is an indecomposable ring, i.e., R is a serial ring in the first category.

Therefore we may assume that $\{Rf_{n,1}, Rf_{n,2}, \cdots, Rf_{2,\beta 2}, Rf_{1,1}, \cdots, Rf_{1,\beta_1}\}$ is not a Kupisch series with Rf_{1,β_1} a simple left R-module. Put $f^{(1)}_{i,j} := f_{i,j}$, $n_1 := n$ and $\beta^{(1)}_i := \beta_i$ for any i, j.

If there is another $g \in P(R) - \{f^{(1)}_{i,1}\}_{i=1}^{n_1}$ with Rg injective, we obtain another sequence $\{f^{(2)}_{n_2,1}, \cdots, f^{(2)}_{n_2,\beta^{(2)}_{n_2}}, \cdots, f^{(2)}_{2,1}, \cdots, f^{(2)}_{2,\beta^{(2)}_2}, f^{(2)}_{1,1}, \cdots, f^{(2)}_{1,\beta^{(2)}_1}\}$ by the same way as $\{f^{(1)}_{n_1,1}, \cdots, f^{(1)}_{1,\beta^{(1)}_1}\}$. (We note that $g = f^{(2)}_{i,1}$ for some i.) Then $\{f^{(1)}_{n_1,1}, \cdots, f^{(1)}_{1,\beta^{(1)}_1}\}$ and $\{f^{(2)}_{n_2,1}, \cdots, f^{(2)}_{1,\beta^{(2)}_1}\}$ are disconnected because we assume that $\{Rf^{(1)}_{n_1,1}, \cdots, Rf^{(1)}_{1,1}\}$ is the longest Kupisch series.

Repeating this proceeding, we obtain disconnected sequences:

$$\begin{gathered}
\{f^{(1)}_{n_1,1}, \cdots, f^{(1)}_{n_1,\beta^{(1)}_{n_1}}, f^{(1)}_{n_1-1,1}, \cdots, f^{(1)}_{2,1}, \cdots, f^{(1)}_{2,\beta^{(1)}_2}, f^{(1)}_{1,1}, \cdots, f^{(1)}_{1,\beta^{(1)}_1}\},\\
\{f^{(2)}_{n_2,1}, \cdots, f^{(2)}_{n_2,\beta^{(2)}_{n_2}}, f^{(2)}_{n_2-1,1}, \cdots, f^{(2)}_{2,1}, \cdots, f^{(2)}_{2,\beta^{(2)}_2}, f^{(2)}_{1,1}, \cdots, f^{(2)}_{1,\beta^{(2)}_1}\},\\
\cdots\cdots\cdots\cdots\cdots\\
\{f^{(k)}_{n_k,1}, \cdots, f^{(k)}_{n_k,\beta^{(k)}_{n_k}}, f^{(k)}_{n_k-1,1}, \cdots, f^{(k)}_{2,1}, \cdots, f^{(k)}_{2,\beta^{(k)}_2}, f^{(k)}_{1,1}, \cdots, f^{(k)}_{1,\beta^{(k)}_1}\}
\end{gathered}$$

such that Rg is not injective for any $g \in P(R) - \{\text{all above } f^{(l)}_{i,j}\}$.

Put $\{h_1, \cdots, h_m\} := P(R) - \{\text{all above } f^{(l)}_{i,j}\}$. And we show that a complete set

$$(\star)\ \{h_1, \cdots, h_m, f^{(1)}_{1,\beta^{(1)}_1}, \cdots, f^{(1)}_{1,1}, f^{(1)}_{2,\beta^{(1)}_2}, \cdots, f^{(1)}_{2,1}, \cdots, f^{(1)}_{n_1,1}, f^{(2)}_{1,\beta^{(2)}_1}, \cdots, f^{(2)}_{n_2,1}, f^{(3)}_{1,\beta^{(3)}_1}, \cdots, f^{(k-1)}_{n_{k-1},1}, f^{(k)}_{1,\beta^{(k)}_1}, \cdots, f^{(k)}_{n_k,1}\}$$

of orthogonal primitive idempotents (we remark that the order of $\{f^{(l)}_{n_l,1}, \cdots, f^{(l)}_{1,1}, \cdots, f^{(l)}_{1,\beta^{(l)}_1}\}$ is inversed for each $l = 1, \cdots, k$) satisfies the conditions $(a'), (b'), (c'), (d'), (f)$ in Theorem 3.3 in this order to complete a proof of "only if part" of Theorem 4.1.

For each $l = 1, \cdots, k$, put $S_l := \sum_{i=1,j=1}^{n_l\ \beta^{(l)}_i} f^{(l)}_{i,j}$ and define a positive integer $\tilde{\alpha}_l$ to satisfy $Rf^{(l)}_{1,j} \cong J^{j-1}f^{(l)}_{1,1}$ for any $j = 1, \cdots, \tilde{\alpha}_l$ but $Rf^{(l)}_{1,\tilde{\alpha}_l+1} \ncong$

$J^{\tilde{\alpha}_l} f_{1,1}^{(l)}$. Then we note that $\{Rf_{n_l,1}^{(l)}, \cdots, Rf_{1,1}^{(l)}, \cdots, Rf_{1,\tilde{\alpha}_l}^{(l)}\}$ is a Kupisch series and $\tilde{\alpha}_l \leq \beta_1^{(l)} - 1$ by the assumption that $\{Rf_{n,1}^{(l)}, Rf_{n,2}^{(l)}, \cdots, Rf_{2,\beta 2}^{(l)}, Rf_{1,1}^{(l)}, \cdots, Rf_{1,\beta_1}^{(l)}\}$ is not a Kupisch series with $Rf_{1,\beta_1}^{(l)}$ a simple left R-module.

First we show that $(\star)$ satisfies $(a'), (b'), (f)$ in the following Claim 5.4(4),(5),(6).

Claim 5.4. *Then*

(1) $S_l J f_{1,j}^{(l)} / S_l J^2 f_{1,j}^{(l)} \cong S_l R f_{1,j+1}^{(l)} / S_l J f_{1,j+1}^{(l)}$ *for any l and $j = \tilde{\alpha}_l + 1, \cdots, \beta_1^{(l)} - 1$,*

(2) $S_l R f_{1,j}^{(l)} / S_l J^{\beta_1^{(l)} - j + 1} f_{1,j}^{(l)} \cong S_l J^{j-1} f_{1,1}^{(l)}$ *for any l and $j = \tilde{\alpha}_l + 1, \cdots, \beta_1^{(l)}$,*

(3) $S_l J^{\beta_1^{(l)} - j + 1} f_{1,j}^{(l)} = 0$ *for any l and $j = \tilde{\alpha}_l + 1, \cdots, \beta_1^{(l)}$,*

(4) $S_l R S_l$ *is a serial ring in the first category with $\{S_l R f_{n_l,1}^{(l)}, S_l R f_{n_l,2}^{(l)}, \cdots, S_l R f_{1,1}^{(l)}, \cdots, S_l R f_{1,\beta_1^{(l)}}^{(l)}\}$ a Kupisch series of left $S_l R S_l$-modules, i.e., $(\star)$ satisfies (a'),*

(5) *$(\star)$ satisfies (b'), and*

(6) $E({}_R R f_{1,\beta_1^{(l)}}^{(l)} / J f_{1,\beta_1^{(l)}}^{(l)})$ *is projective as a left R-module for any $l = 1, \cdots, k$, i.e., $(\star)$ satisfies (f).*

Proof of Claim 5.4. (1). Let $x \in S_l J f_{1,j}^{(l)} - S_l J^2 f_{1,j}^{(l)}$ with $f_{u,v}^{(l)} x = x$ for some u, v. Put $E := E(Rx/Jx)$ $(\cong E(Rf_{u,v}^{(l)} / J f_{u,v}^{(l)}))$. There is an epimorphism $\phi : Rx \to S(E)$. And let $\tilde{\phi} : R f_{1,j}^{(l)} \to E$ be an extension map of ϕ. Then $\tilde{\phi}(f_{1,j}^{(l)}) \in S_2(E) - S(E)$ since $x \in J f_{1,j}^{(l)} - J^2 f_{1,j}^{(l)}$ and $0 \neq \tilde{\phi}(x) \in S(E)$. So $f_{1,j}^{(l)} \cdot (S_2(E)/S(E)) \neq 0$. Therefore $S_2(E)/S(E) \cong R f_{1,j}^{(l)} / J f_{1,j}^{(l)}$ because E is uniserial by (†) and (††), i.e., $S_2(E)/S(E) \cong J^{j-1} f_{1,1}^{(l)} / J^j f_{1,1}^{(l)}$. So $S(E) \cong J^j f_{1,1}^{(l)} / J^{j+1} f_{1,1}^{(l)}$ by (††) and the definition of uniserial modules $R f_{1,1}^{(l)}, \cdots, R f_{n_l,1}^{(l)}$. Therefore $f_{u,v}^{(l)} = f_{1,j+1}^{(l)}$ because $S(E) \cong R f_{u,v}^{(l)} / J f_{u,v}^{(l)}$ and $J^j f_{1,1}^{(l)} / J^{j+1} f_{1,1}^{(l)} \cong R f_{1,j+1}^{(l)} / J f_{1,j+1}^{(l)}$. Hence $S_l J f_{1,j}^{(l)} / S_l J^2 f_{1,j}^{(l)} \cong (S_l R f_{1,j+1}^{(l)} / S_l J f_{1,j+1}^{(l)})^{m'}$ for some positive integer m'. Assume that $m' \geq 2$. Put $X := J^2 f_{1,j}^{(l)}$. Then we obtain $y_1, y_2 \in f_{1,j+1}^{(l)} J f_{1,j}^{(l)} - X$ with $S_l R y_1 + S_l R y_2$ not a local left $S_l R S_l$-module. By (††), $E(R f_{1,j+1}^{(l)} / J f_{1,j+1}^{(l)}) \cong R f_{p,1}^{(l)} / S_q(R f_{p,1}^{(l)})$ for some p, q. Put $Y_t := S_t(R f_{p,1}^{(l)})$ for any positive integer t. We have a nonzero homomorphism $\psi_i : (Ry_1 + Ry_2 + X)/X \to Y_{q+1}/Y_q$ with $\mathrm{Ker}\psi_i \ni y_{i'} + X$ for each $i = 1, 2$, where $(i \neq)\ i' \in \{1, 2\}$. Since $R f_{p,1}^{(l)} / Y_q$ is injective, we have an extension homomorphism $\tilde{\psi}_i : Rf_{1,j}^{(l)}/X \to Rf_{p,1}^{(l)}/Y_q$ of ψ_i and put $z_i + Y_q := \tilde{\psi}_i(f_{1,j}^{(l)} + X)$ for each i, where $z_i \in f_{1,j}^{(l)}Rf_{p,1}^{(l)}$. Then we claim that there exists an isomorphism $\eta : Rz_2/Y_q \to Rz_1/Y_q$ with $\eta(z_2 + Y_q) = z_1 + z' + Y_q$ for some $z' \in Jz_1$. We can define an isomorphism $\xi : Rz_2/Y_{q+1} \to Rz_1/Y_{q+1}$ by $\xi(z_2+Y_{q+1}) = z_1+Y_{q+1}$ because $Rz_i/Y_{q+1} = Y_{q+2}/Y_{q+1}$ is simple for any $i = 1, 2$. Now $Rf_{p,1}^{(l)}/Y_{q+1}$ is almost injective by $(\#)_l$. Suppose that $Rf_{p,1}^{(l)}/Y_{q+1}$ is injective. Then we have an extension homomorphism $\tilde{\xi} \in \mathrm{End}_R(Rf_{p,1}^{(l)}/Y_{q+1})$ of ξ. So there is $\zeta \in \mathrm{End}_R(Rf_{p,1}^{(l)})$ with $\pi\zeta = \tilde{\xi}\pi$, where we let $\pi : Rf_{p,1}^{(l)} \to Rf_{p,1}^{(l)}/Y_{q+1}$ be a natural epimorphism. Then $\zeta(Y_q) = Y_q$ and $\zeta(z_2) = z_1 + z'$ for some $z' \in Jz_1$ since $Rf_{p,1}^{(l)}$ is uniserial. Hence ζ induces an isomorphism η. Next suppose that $Rf_{p,1}^{(l)}$ is almost injective but not injective. Then we have an isomorphism $\iota : E(Rf_{p,1}^{(l)}/Y_{q+1}) \to Rf_{p+1,1}^{(l)}$. So there is $\xi' \in \mathrm{End}_R(Rf_{p+1,1}^{(l)})$ with $\xi'\iota = \iota\xi$. And we have $\tilde{\xi} \in \mathrm{End}_R(Rf_{p,1}^{(l)}/Y_{q+1})$ with $\iota\tilde{\xi} = \xi'\iota$ since $Rf_{p+1,1}^{(l)}$ is uniserial. Then $\tilde{\xi}$ is an extension of ξ. So we obtain an isomorphism η by the same way as the case that $Rf_{p,1}^{(l)}/Y_{q+1}$ is injective. Therefore $\tilde{\psi}_2(y_2 + X) = y_2z_2 + Y_q = \eta^{-1}(y_2(z_1 + z') + Y_q) = \eta^{-1}(y_2z_1+Y_q) = \eta^{-1}(\tilde{\psi}_1(y_2+X)) = \eta^{-1}(Y_q) = Y_q$, where the third equation is given since $y_2 \in J$ induces $y_2z' \in J^2z_1 \subseteq Y_q$ and we have the fifth equation because $y_2 + X \in \mathrm{Ker}\tilde{\psi}_1$. This contradicts with the definition of $\tilde{\psi}_2$. Hence $m' = 1$, i.e., $S_lJf_{1,j}^{(l)}/S_lJ^2f_{1,j}^{(l)} \cong S_lRf_{1,j+1}^{(l)}/S_lJf_{1,j+1}^{(l)}$ for any l and $j = \tilde{\alpha}_l + 1, \cdots, \beta_1^{(l)} - 1$.

(2). We first show that $S_lR(1-S_l) = 0$ for any $l = 1, \cdots, k$, i.e., the first half of (b') holds. Take any $f_{i,j}^{(l)}$ and assume that $f_{i,j}^{(l)}Rg \neq 0$ for some $g \in P(R)$. Then there are submodules $X \supset Y$ of Rg with an isomorphism: $X/Y \to Rf_{i,j}^{(l)}/Jf_{i,j}^{(l)}$. We have an extension homomorphism: $Rg/Y \to E(Rf_{i,j}^{(l)}/Jf_{i,j}^{(l)})$, i.e., $g \cdot E(Rf_{i,j}^{(l)}/Jf_{i,j}^{(l)}) \neq 0$. Therefore $g \in \{f_{i,j}^{(l)}\}_{i=1,j=1}^{n_l\ \ \beta_i^{(l)}}$ by $(\dagger\dagger)$ and the definition of $\{f_{i,1}^{(l)}\}_{i=1}^{n_l}$, i.e., $S_lR(1-S_l) = 0$ holds.

For any l and $j \in \{\tilde{\alpha}_l + 1, \cdots, \beta_1^{(l)} - 1\}$ there exists a left S_lRS_l-epimorphism $\phi_{j+1} : S_lRf_{1,j+1}^{(l)} \to S_lJf_{1,j}^{(l)}$ by (1). On the other hand, $S_lJ^if_{1,j}^{(l)} = S_lJS_lJ^{i-1}f_{1,j}^{(l)} + S_lJ(1 - S_l)J^{i-1}f_{1,j}^{(l)} = S_lJS_lJ^{i-1}f_{1,j}^{(l)} = \cdots = (S_lJS_l)^if_{1,j}^{(l)}$ for any $i \in \mathbf{N}$. So $\phi_{j+1}(S_lJ^{i-1}f_{1,j+1}^{(l)}) = \phi((S_lJS_l)^{i-1} \cdot S_lRf_{1,j+1}^{(l)}) = (S_lJS_l)^{i-1} \cdot S_lJf_{1,j}^{(l)} = S_lJ^if_{1,j}^{(l)}$. Hence for any $i \in \{1,$

$\cdots,\ \beta_1^{(l)}-j\}$ we have an epimorphism $\phi_{j+1}\phi_{j+2}\cdots\phi_{j+i}\ :\ S_lRf_{1,j+i}^{(l)}\rightarrow S_lJ^if_{1,j}^{(l)}$ with $\phi_{j+1}\phi_{j+2}\cdots\phi_{j+i}(S_lJf_{1,j+i}^{(l)})=S_lJ^{i+1}f_{1,j}^{(l)}$. Therefore $S_lJ^if_{1,j}^{(l)}/S_lJ^{i+1}f_{1,j}^{(l)}\cong S_lRf_{1,j+i}^{(l)}/S_lJf_{1,j+i}^{(l)}$, i.e., $S_lJ^if_{1,j}^{(l)}/S_lJ^{i+1}f_{1,j}^{(l)}$ is a simple as a left S_lRS_l-module. Therefore $S_lRf_{1,j}^{(l)}/S_lJ^{\beta_1^{(l)}-j+1}f_{1,j}^{(l)}$ is uniserial as a left S_lRS_l-module. Hence $S_lRf_{1,j}^{(l)}/\ S_lJ^{\beta_1^{(l)}-j+1}f_{1,j}^{(l)}\cong S_lJ^{j-1}f_{1,1}^{(l)}$ for any $j\in\{\tilde{\alpha}_l+1,\cdots,\beta_1^{(l)}\}$ since $\beta_1^{(l)}=|{}_RRf_{1,1}^{(l)}|$ and $S_lRf_{1,j}^{(l)}$ is a projective cover of $S_lJ^{j-1}f_{1,1}^{(l)}$ by the definition of $f_{1,j}^{(l)}$.

(3). Assume that there are l and $j'\in\{\tilde{\alpha}_l+1,\cdots,\beta_1^{(l)}\}$ with $S_lJ^{\beta_1^{(l)}-j'+1}f_{1,j'}^{(l)}\neq 0$, i.e., $f_{u,v}^{(l)}J^{\beta_1^{(l)}-j'+1}f_{1,j'}^{(l)}\neq 0$ for some u,v. Now $S_lJ^{\beta_1^{(l)}-j'}f_{1,j'}^{(l)}/S_lJ^{\beta_1^{(l)}-j'+1}f_{1,j'}^{(l)}=S({}_{S_lRS_l}S_lRf_{1,j'}^{(l)}/S_lJ^{\beta_1^{(l)}-j'+1}f_{1,j'}^{(l)})\cong S({}_{S_lRS_l}S_lJ^{j'-1}f_{1,1}^{(l)})\cong S_lRf_{1,\beta_1^{(l)}}^{(l)}/S_lJf_{1,\beta_1^{(l)}}^{(l)}$ because $S_lRf_{1,j'}^{(l)}/S_lJ^{\beta_1^{(l)}-j'+1}f_{1,j'}^{(l)}\cong S_lJ^{j'-1}f_{1,1}^{(l)}$ by (2) and $Rf_{1,\beta_1^{(l)}}^{(l)}/Jf_{1,\beta_1^{(l)}}^{(l)}\cong S({}_RRf_{1,1}^{(l)})$ by the defintion of $f_{1,\beta_1^{(l)}}^{(l)}$. So $S_lRf_{1,\beta_1^{(l)}}^{(l)}$ is a projective cover of a left S_lRS_l-module $S_lJ^{\beta_1^{(l)}-j'}f_{1,j'}^{(l)}$. Hence there exists $0\neq x\in f_{u,v}^{(l)}Jf_{1,\beta_1^{(l)}}^{(l)}$ by the assumption that $f_{u,v}^{(l)}J^{\beta_1^{(l)}-j'+1}f_{1,j'}^{(l)}\neq 0$. Therefore we have $0\neq\phi\in\mathrm{Hom}_R(Rx,Rf_{u,v}^{(l)}/Jf_{u,v}^{(l)})$. By (††), $E(Rf_{u,v}^{(l)}/Jf_{u,v}^{(l)})\cong Rf_{p,1}^{(l)}/S_q(Rf_{p,1}^{(l)})$ for some p,q. So there is $0\neq\tilde{\phi}\in\mathrm{Hom}_R(Rf_{1,\beta_1^{(l)}}^{(l)},Rf_{p,1}^{(l)}/S_q(Rf_{p,1}^{(l)}))$. Then $\tilde{\phi}\ (f_{1,\beta_1^{(l)}}^{(l)})\notin S_{q+1}(Rf_{p,1}^{(l)})/S_q(Rf_{p,1}^{(l)})$ because $x\in Jf_{1,\beta_1^{(l)}}^{(l)}$ and $0+S_q(Rf_{p,1}^{(l)})\neq\tilde{\phi}(x)\in S(Rf_{p,1}^{(l)}/S_q(Rf_{p,1}^{(l)}))=S_{q+1}(Rf_{p,1}^{(l)})/S_q(Rf_{p,1}^{(l)})$. Therefore $f_{1,\beta_1^{(l)}}^{(l)}\cdot(Rf_{p,1}^{(l)}/S_{q+1}(Rf_{p,1}^{(l)}))\neq 0$, i.e., $Rf_{1,\beta_1^{(l)}}^{(l)}/Jf_{1,\beta_1^{(l)}}^{(l)}$ is isomorphic to a subfactor of $Rf_{p,1}^{(l)}/S(Rf_{p,1}^{(l)})$. So $f_{1,\beta_1^{(l)}}^{(l)}\in\{f_{n_l,1}^{(l)},\cdots,f_{1,1}^{(l)},\cdots,f_{1,\beta_1^{(l)}-1}^{(l)}\}$. This contradicts with the fact that $\{f_{i,j}^{(l)}\}_{i=1,j=1}^{n_l\ \ \beta_i^{(l)}}$ is a set of distinct elements in $P(R)$.

(4). $Rf_{n_l,1}^{(l)},Rf_{n_l,2}^{(l)},\cdots,Rf_{1,1}^{(l)},\cdots,Rf_{1,\tilde{\alpha}_l}^{(l)}$ are uniserial for any $l=1,\cdots,k$ by (†) and the definitions of $\{f_{i,j}^{(l)}\}_{i=1,j=1}^{n_l\ \ \beta_i^{(l)}}$ and $\tilde{\alpha}_l$. So $S_lRf_{n_l,1}^{(l)}$, $S_lRf_{n_l,2}^{(l)},\ \cdots,\ S_lRf_{1,1}^{(l)},\ \cdots,S_lRf_{1,\tilde{\alpha}_l}^{(l)}$ are uniserial left S_lRS_l-modules for any l. Further $S_lRf_{1,\tilde{\alpha}_l+1}^{(l)}$, $S_lRf_{1,\tilde{\alpha}_l+2}^{(l)},\ \cdots,\ S_lRf_{1,\beta_1^{(l)}}^{(l)}$ are also uniserial left S_lRS_l-modules for any l by (2),(3) and (†). So S_lRS_l is a left serial ring.

For any $l = 1, \cdots, k$, $E({}_RRf^{(l)}_{i,j}/Jf^{(l)}_{i,j})$ is a uniserial left R-module for any i, j by (†),(††). Further $S_lRf_{i,j} = Rf_{i,j}$ holds for any i, j by the definition of $\{f^{(l)}_{i,j}\}^{n_l}_{i=1,j=1}{}^{\beta^{(l)}_i}$ and (2),(3). So $E({}_{S_lRS_l}S_lRf^{(l)}_{i,j})$ is a uniserial left S_lRS_l-module by (†††), i.e., S_lRS_l is a left co-serial ring. Therefore S_lRS_l is a serial ring (see, for instance, [1, 32.3. Theorem]). Further $\{S_lRf^{(l)}_{n_l,1}, \cdots, S_lRf^{(l)}_{1,\beta^{(l)}_1}\}$ is a Kupisch series of left S_lRS_l-modules by the definition of $\{f^{(l)}_{i,j}\}^{n_l}_{i=1,j=1}{}^{\beta^{(l)}_i}$ and (2). Hence S_lRS_l is a serial ring in the first category because $S_lJf^{(l)}_{1,\beta^{(l)}_1} = 0$ by (3).

(5). We already show the first half of (5) in the proof of (2). We show the second half.

$\tilde{\alpha}_l \leq \beta^{(l)}_1 - 1$ which we note just before Claim 5.4. So $f^{(l)}_{\tilde{\alpha}_l+1}$ exists. Therefore $(h_1 + \cdots + h_m)Rf^{(l)}_{1,\tilde{\alpha}_l+1} \neq 0$ by (a') and the definition of $\tilde{\alpha}_l$ since $S_lR(1 - S_l) = 0$ which we already show.

(6). $E(Rf^{(l)}_{1,\beta^{(l)}_1}/Jf^{(l)}_{1,\beta^{(l)}_1}) \cong Rf^{(l)}_{1,1}$ by the definition of $\{f^{(l)}_{i,j}\}^{n_l}_{i=1,j=1}{}^{\beta^{(l)}_i}$, i.e., $E(Rf^{(l)}_{1,\beta^{(l)}_1}/Jf^{(l)}_{1,\beta^{(l)}_1})$ is projective. □

By $(a'), (b')$ which we already show in Claim 5.4(4),(5) and the definitions of $\{f^{(l)}_{i,j}\}^{n_l}_{i=1,j=1}{}^{\beta^{(l)}_i}$ and $\tilde{\alpha}_l$, $(h_1 + \cdots + h_m)Rg = 0$ for any $g \in \{f^{(l)}_{n_l,1}, \cdots, f^{(l)}_{1,1}, \cdots, f^{(l)}_{1,\tilde{\alpha}_l}\}$ and $(h_1 + \cdots + h_m)Rf^{(l)}_{1,\tilde{\alpha}_l+1} \neq 0$. So put $H := \sum_{s=1}^m h_s + \sum_{l=1,j=\tilde{\alpha}_l+1}^{k}{}^{\beta^{(l)}_1} f^{(l)}_{1,j}$. And to show that $(\star)$ satisfies (d'), we have to show that a ring HRH is hereditary.

Claim 5.5. *Then*

(1) *Jg/J^2g is a simple left R-module for any l and $g \in \{f^{(l)}_{n_l,1}, \cdots, f^{(l)}_{1,1}, \cdots, f^{(l)}_{1,\beta^{(l)}_1-1}\}$, i.e., $(\star)$ satisfies (c'), and*

(2) *a ring HRH is hereditary, i.e., $(\star)$ satisfies (d').*

Proof of Claim 5.5. Put $B := \sum_{l=1}^k (Rf^{(l)}_{n_l,1} + \cdots + Rf^{(l)}_{1,1} + \cdots + Rf^{(l)}_{1,\tilde{\alpha}_l})$. Then B is a two sided ideal of R with $R/B \cong HRH$ by $(a'), (b')$. Further put $\overline{R} := R/B$, $\overline{J} := J(\overline{R})$, $\overline{f^{(l)}_{1,j}} := f^{(l)}_{1,j} + B$, $\overline{h_s} := h_s + B$, $\overline{E^{(l)}_{1,j}} := E({}_{\overline{R}}\overline{Rf^{(l)}_{1,j}}/\overline{Jf^{(l)}_{1,j}})$ and $\overline{E_s} := E({}_{\overline{R}}\overline{Rh_s}/\overline{Jh_s})$ for any $l = 1, \cdots, k$, $j = \tilde{\alpha}_l + 1, \cdots, \beta^{(l)}_1$, $s = 1, \cdots, m$.

Then first we claim that $\overline{E_s}$ is injective also as a left R-module. $h_sB = 0$ for any s by the definition of B. So $\mathrm{Hom}_R(B', Rh_s/Jh_s) = 0$

for any left R-submodule B' of B. Therefore, for any left ideal N of R and $\phi \in \mathrm{Hom}_R(N, \overline{E_s})$, where we consider $\overline{E_s}$ as a left R-module, there is $\tilde{\phi} \in \mathrm{Hom}_R(R, \overline{E_s})$ with $\tilde{\phi}|_N = \phi$ because $\overline{E_s}$ is injective as a left $\overline{R}$-module. Hence $\overline{E_s}$ is injective also as a left R-module.

(1). It is obvious that Jg/J^2g is a simple left R-module for any l and $g \in \{f^{(l)}_{n_l,1}, \cdots, f^{(l)}_{1,1}, \cdots, f^{(l)}_{1,\tilde{\alpha}_l}\}$ by (†) and the definition of $\{f^{(l)}_{i,j}\}_{i=1,j=1}^{n_l \ \ \beta_i^{(l)}}$ since $\tilde{\alpha}_l \le \beta^{(l)}_1 - 1$. Assume that there exist l and $j' \in \{\tilde{\alpha}_l + 1, \cdots, \beta^{(l)}_1 - 1\}$ such that $Jf^{(l)}_{1,j'}/J^2 f^{(l)}_{1,j'}$ is not simple. Then $Jf^{(l)}_{1,j'}/J^2 f^{(l)}_{1,j'}$ contains a simple submodule isomorphic to some Rh_s/Jh_s by (a'), (b'). Now we already show that $\overline{E_s}$ is injective also as a left R-module. So $f^{(l)}_{1,j'}(S_2(\overline{E_s})/S(\overline{E_s})) \neq 0$ and $\overline{E_s}/S(\overline{E_s})$ is a direct sum of an injective left R-module and finitely generated almost injective left R-modules by $(\#)_l$. Therefore there is a direct summand I of $\overline{E_s}/S(\overline{E_s})$ with $S(I) \cong {}_R Rf^{(l)}_{1,j'}/Jf^{(l)}_{1,j'}$ since any finitely generated indecomposable almost injective left R-module has a simple socle by Lemma A. Then a left R-module I is injective or finitely generated almost injective. Assume that I is injective. Then I contains a submodule isomorphic to $Rf^{(l)}_{1,1}/J^{j'} f^{(l)}_{1,1}$ since $S(I) \cong Rf^{(l)}_{1,j'}/Jf^{(l)}_{1,j'} \cong S(Rf^{(l)}_{1,1}/J^{j'} f^{(l)}_{1,1})$. So $f^{(l)}_{1,1} I \neq 0$. But $f^{(l)}_{1,1} I = 0$ since I can be considered as a left $\overline{R}$-module, a contradiction. So I is not injective but finitely generated almost injective. Then $E(I) \cong Rf^{(l)}_{u,1}$ and $I \cong J^v f^{(l)}_{u,1}$ for some $u \in \{1, \cdots, n_l\}$ and $v \in \{1, \cdots, \beta^{(l)}_u\}$ by Lemma A and the definition of $\{f^{(l)}_{i,j}\}_{i=1,j=1}^{n_l \ \ \beta_i^{(l)}}$. And we claim that $u = 1$, i.e., $I \cong J^v f^{(l)}_{1,1}$. Assume that $u \ge 2$. There exists a monomorphism: $Rf^{(l)}_{1,1}/J^{j'} f^{(l)}_{1,1} \to E(I)$ $(\cong Rf^{(l)}_{u,1})$ since $S(I) \cong Rf^{(l)}_{1,j'}/Jf^{(l)}_{1,j'} \cong J^{j'-1} f^{(l)}_{1,1}/J^{j'} f^{(l)}_{1,1}$. So $f^{(l)}_{1,1} R f^{(l)}_{u,1} \neq 0$. Further $J^{j-1} f^{(l)}_{u,1}/J^j f^{(l)}_{u,1} \not\cong Rf^{(l)}_{1,1}/Jf^{(l)}_{1,1}$ for any $j = 1, \cdots, \beta^{(l)}_u$ by the definition of $\{f^{(l)}_{i,j}\}_{i=1,j=1}^{n_l \ \ \beta_i^{(l)}}$ because $u \ge 2$. Therefore $(f^{(l)}_{1,1} I \cong)$ $f^{(l)}_{1,1} J^v f^{(l)}_{u,1} \neq 0$. But $f^{(l)}_{1,1} I = 0$ since I can be considered as a left $\overline{R}$-module, a contradiction. Hence $Rf^{(l)}_{1,j'}/Jf^{(l)}_{1,j'} \cong S(I) \cong S(J^v f^{(l)}_{1,1}) = S(Rf^{(l)}_{1,1}) \cong Rf^{(l)}_{1,\beta^{(l)}_1}/Jf^{(l)}_{1,\beta^{(l)}_1}$, i.e., $j' = \beta^{(l)}_1$. This contradicts with $j' \le \beta^{(l)}_1 - 1$.

(2). We show that $\overline{R}$ is a hereditary ring. Concretely we show that $\overline{E^{(l)}_{1,j}}/S(\overline{E^{(l)}_{1,j}})$ and $\overline{E_s}/S(\overline{E_s})$ are injective as a left $\overline{R}$-module for any $l = 1, \cdots, k$, $j = \tilde{\alpha}_l + 1, \cdots, \beta^{(l)}_1$, $s = 1, \cdots, m$.

Put $E^{(l)}_{1,j} := E({}_RRf^{(l)}_{1,j}/Jf^{(l)}_{1,j})$ for each l, j. $E^{(l)}_{1,j} \cong Rf^{(l)}_{p,1}/S_q(Rf^{(l)}_{p,1})$ for some p, q by (††). Then either of the following two cases holds by the definition of $\{f^{(l)}_{i,j}\}^{n_l\ \beta^{(l)}_i}_{i=1,j=1}$:

(α) $E(E^{(l)}_{1,j}/S(E^{(l)}_{1,j})) \cong Rf^{(l)}_{p,1}/S_{q+1}(Rf^{(l)}_{p,1})$;

(β) $E(E^{(l)}_{1,j}/S(E^{(l)}_{1,j})) \cong Rf^{(l)}_{p+1,1}$ and $J^{\beta^{(l)}_{p+1}}f^{(l)}_{p+1,1} \cong Rf^{(l)}_{p,1}/S_{q+1}(Rf^{(l)}_{p,1})$.

On the other hand, $BRf^{(l)}_{1,\tilde{\alpha}_l+1} = 0$ by (a') but $BRf^{(l)}_{1,\tilde{\alpha}_l} \neq 0$. So put $r := \tilde{\alpha}_l + \sum_{i=2}^{p}\beta^{(l)}_i$, then $BJ^rf^{(l)}_{p,1} = 0$ but $BJ^{r-1}f^{(l)}_{p,1} \neq 0$ by the definition of $\{f^{(l)}_{i,j}\}^{n_l\ \beta^{(l)}_i}_{i=1,j=1}$. Therefore $r_{Rf^{(l)}_{p,1}}(B) = J^rf^{(l)}_{p,1}$ by (†). Hence $J^rf^{(l)}_{p,1}/S_q(Rf^{(l)}_{p,1})$ is injective as a left $\overline{R}$-module by Lemma C(1), i.e., $\overline{E^{(l)}_{1,j}} \cong J^rf^{(l)}_{p,1}/S_q(Rf^{(l)}_{p,1})$. Therefore $\overline{E^{(l)}_{1,j}}/S(\overline{E^{(l)}_{1,j}}) \cong J^rf^{(l)}_{p,1}/S_{q+1}(Rf^{(l)}_{p,1})$. When the case ($\alpha$) holds, $\overline{E^{(l)}_{1,j}}/S(\overline{E^{(l)}_{1,j}})$ ($\cong J^rf^{(l)}_{p,1}/S_{q+1}(Rf^{(l)}_{p,1})$) is also injective as a left $\overline{R}$-module since $r_{Rf^{(l)}_{p,1}}(B) = J^rf^{(l)}_{p,1}$. When the case ($\beta$) holds, $BJ^rf^{(l)}_{p,1} = 0$ and $BJ^{r-1}f^{(l)}_{p,1} \neq 0$ induce $BJ^{r+\beta^{(l)}_{p+1}}f^{(l)}_{p+1,1} = 0$ and $BJ^{r+\beta^{(l)}_{p+1}-1}f^{(l)}_{p+1,1} \neq 0$ since $r \geq 1$ and $J^{\beta^{(l)}_{p+1}}f^{(l)}_{p+1,1} \cong Rf^{(l)}_{p,1}/S_{q+1}(Rf^{(l)}_{p,1})$. Therefore $r_{Rf^{(l)}_{p+1,1}}(B) = J^{r+\beta^{(l)}_{p+1}}f^{(l)}_{p+1,1}$ and it is injective as a left $\overline{R}$-module by Lemma C(1). Hence $\overline{E^{(l)}_{1,j}}/S(\overline{E^{(l)}_{1,j}})e$ ($\cong J^rf^{(l)}_{p,1}/S_{q+1}(Rf^{(l)}_{p,1}) \cong J^{r+\beta^{(l)}_{p+1}}f^{(l)}_{p+1,1}$) is also injective as a left $\overline{R}$-module.

We already show that $\overline{E_s}$ is injective also as a left R-module. So $\overline{E_s}/S(\overline{E_s})$ is a direct sum of an injective left R-module and finitely generated almost injective left R-modules by $(\#)_l$. Let I be an indecomposable direct summand of ${}_{\overline{R}}\overline{E_s}/S(\overline{E_s})$. If I is injective as a left R-module, it is injective also as a left $\overline{R}$-module by Lemma C(1). So we may assume that I is not injective but finitely generated almost injective as a left R-module. Then there exist integers l, u, v such that $I \cong J^vf^{(l)}_{u,1}$ and $J^{j-1}f^{(l)}_{u,1}$ is projective for any $j = 1, \cdots, v$ by Lemma A because Rg is injective iff $g \in \{f^{(l)}_{i,1}\}^{k\ \ n_l}_{l=1,i=1}$ for any $g \in P(R)$. Then we claim $u = 1$. Assume that $u \geq 2$. $v \leq \beta^{(l)}_u$ since $J^{j-1}f^{(l)}_{u,1}$ is projective for any $j = 1, \cdots, v$. So $f^{(l)}_{u-1,1}I \cong f^{(l)}_{u-1,1}J^vf^{(l)}_{u,1} \neq 0$ because $J^{\beta^{(l)}_u}f^{(l)}_{u,1} \cong Rf^{(l)}_{u-1,1}/S_\gamma(Rf^{(l)}_{u-1,1})$ for some γ by the definition of $\{f^{(l)}_{i,j}\}^{n_l\ \beta^{(l)}_i}_{i=1,j=1}$. But $f^{(l)}_{u-1,1}I = 0$ since I is considered as a $\overline{R}$-module, a contradiction. Then $J^{j-1}f^{(l)}_{1,1} \cong Rf^{(l)}_{1,j}$ for any

$j = 1, \cdots, v$ since $J^{j-1} f_{1,1}^{(l)}$ is projective for the j. So $v \leq \tilde{\alpha}_l$ by the definition of $\tilde{\alpha}_l$. On the other hand, $BJ^{\tilde{\alpha}_l} f_{1,1}^{(l)} = 0$ but $BJ^{\tilde{\alpha}_l - 1} f_{1,1}^{(l)} \neq 0$ by the definitions of B and $\{f_{i,j}^{(l)}\}_{i=1,j=1}^{n_l \ \beta_i^{(l)}}$. So $r_{Rf_{1,1}^{(l)}}(B) = J^{\tilde{\alpha}_l} f_{1,1}^{(l)}$ since $Rf_{1,1}^{(l)}$ is uniserial. Now I $(\cong J^v f_{1,1}^{(l)})$ is a left $\overline{R}$-module. So $BJ^v f_{1,1}^{(l)} = 0$, i.e., $J^v f_{1,1}^{(l)} \subseteq r_{Rf_{1,1}^{(l)}}(B)$ $(= J^{\tilde{\alpha}_l} f_{1,1}^{(l)})$. Therefore $v \geq \tilde{\alpha}_l$. In consequence, we obtain $v = \tilde{\alpha}_l$, i.e., $I \cong J^{\tilde{\alpha}_l} f_{1,1}^{(l)} = r_{Rf_{1,1}^{(l)}}(B)$. Hence I is injective as a left $\overline{R}$-module by Lemma C(1). □

ACKNOWLEDGEMENT

The condition $(\#)_r$ in this paper is first considered by M. Harada. He called a ring satisfying this condition a right co-almost hereditary ring and gave the structure theorem which is a dual form of Theorem B in an unpublished paper. The authors thank M. Harada for using results in this unpublished paper.

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(*Received November 13, 2000*)