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THE 19 AND 20-TH HOMOTOPY GROUPS OF THE ROTATION GROUPS R_n

HIDEYUKI KACHI AND JUNO MUKAI

Abstract. We determine the group structures of the homotopy groups of the rotation groups $\pi_k(R_n)$ for $k = 19$ and 20. The main tool of the computaion is the homotopy exact sequence associated with the fibration $R_{n+1}/R_n = S^n$.

Introduction

The present paper is the continuation of [6] and is devoted to the computation of the homtopy groups $\pi_k(R_n)$ of the rotation groups R_n for $k = 19$ and 20. According to [8] and [11], the group structures of $\pi_k(R_n)$ for $k \leq 22$ and $n \leq 9$ are known. For $-1 \leq r \leq 4$, the group structure of $\pi_{m+r}(R_m)$ are determined in [7]. On [1], Barratt-Mahowald shows that for $n \geq 13$ and $k \leq 2n-1$

$$\pi_k(R_n) \cong \pi_k(R_\infty) \oplus \pi_{k+1}(V_{2n,n}),$$

where $V_{2n,n} = R_{2n}/R_n$ is the Stiefel manifold. By use of this splitting, [13] and [14], we can obtain $\pi_k(R_n)$ for $k = 19$ and 20 with $n \geq 13$. So our main task is to determine $\pi_k(R_n)$ for $k = 19$ and 20 in the case $10 \leq n \leq 12$. Summarizing these results, we have the following table:

n	3	4	5	6	7
$\pi_{19}(R_n)$	$Z_6 \oplus Z_2$	$(Z_6)^2 \oplus (Z_2)^2$	$(Z_2)^2$	$Z_{12} \oplus Z_2$	Z_2
$\pi_{20}(R_n)$	$Z_{12} \oplus (Z_2)^2$	$(Z_{12})^2 \oplus (Z_2)^4$	$(Z_2)^2$	$Z_{60} \oplus Z_2$	$(Z_2)^2$

8	9	10	11	12	13	14
Z_2	Z_2	Z_{12}	$Z \oplus (Z_2)^2$	$Z \oplus (Z_2)^4$	$Z \oplus (Z_2)^2$	$Z \oplus Z_4$
$Z_6 \oplus (Z_2)^2$	Z_2	$Z_8 \oplus Z_2$	$(Z_2)^3$	$(Z_2)^5$	$(Z_2)^3$	Z_{240}

15	16	17	18	19	20	21	$22 \leq n$
$Z \oplus Z_2$	$Z \oplus Z_2$	$Z \oplus Z_2$	Z	Z	$Z \oplus Z$	Z	Z
0	0	0	Z_{12}	Z_2	$(Z_2)^2$	Z_2	0

Here $(Z_k)^n$ indicates the direct sum of n-copies of a cyclic group Z_k of order k. Odd primary componets of $\pi_k(R_n)$ for $k = 19, 20$ are easily obtained from [8] and its method. We denote by $\pi_k(R_n : 2)$ a suitablly chosen subgroup of the homotopy group $\pi_k(R_n)$ which consists of the 2-primary components and a free part such that the index $|\pi_k(R_n) : \pi_k(R_n : 2)|$ is odd. This group $\pi_k(R_n : 2)$ is roughly called as the 2-primary components of $\pi_k(R_n)$. The results of $\pi_k(R_n : 2)$ for $k = 19$ and 20 are stated in Theorems 2.1 and 3.2, in which generators of the 2-primary components $\pi_k(R_n : 2)$ are given.

The main tool of the computation is the following homotopy exact sequence associated with the fibration $R_{n+1}/R_n = S^n$;

$$\to \pi_{k+1}(S^n) \xrightarrow{\Delta} \pi_k(R_n) \xrightarrow{i_*} \pi_k(R_{n+1}) \xrightarrow{p_*} \pi_k(S^n) \xrightarrow{\Delta} \pi_{k-1}(R_n) \to,$$

where $i : R_n \to R_{n+1}$ is the inclusion, $p : R_{n+1} \to S^n$ is the projection and $\Delta : \pi_{k+1}(S^n) \to \pi_k(R_n)$ is the connecting map. We can form the following exact sequence:

$$(k)_n \quad \pi_{k+1}(S^n : 2) \xrightarrow{\Delta} \pi_k(R_n : 2) \xrightarrow{i_*} \pi_k(R_{n+1} : 2) \xrightarrow{p_*} \pi_k(S^n : 2) \xrightarrow{\Delta} \pi_{k-1}(R_n : 2).$$

The computation will be done by use of the exact sequence $(k)_n$, the results [18], [10] on the homotopy groups of sphere S^n and the J-homomorphism $J : \pi_k(R_n : 2) \to \pi_{k+n}(S^n : 2)$.

We use the notations and the results of [5], [6], [18] and [10], freely. For an element $\alpha \in \pi_k(S^n)$, we denote by $[\alpha] \in \pi_k(R_n)$ an element satisfying $p_*[\alpha] = \alpha$. Though $[\alpha]$ is only determined modulo Im $i_* = i_*(\pi_k(R_n))$, we will sometimes give restriction on $[\alpha]$ to fix it more concretely. We set $[\alpha]_m = j_*[\alpha] \in \pi_k(R_m)$, where $j : R_n \to R_m$ for $n+1 \leq m$ is the inclusion. Hereafter we only deal with the 2-primary components $\pi_k(X : 2)$ of the homotopy group $\pi_k(X)$ and its is denoted by $\pi_k(X)$ for simplicity.

1. Preriminaries

We recall the elements $[\eta_2] \in \pi_3(R_3)$, $[\iota_3] \in \pi_3(R_4)$ and $[\iota_7] \in \pi_7(R_7)$ given in [5]. It is well know that the homomorphism

$$(1) \qquad [\eta_2]_* : \pi_k(S^3) \to \pi_k(R_3)$$

is an isomorphism for all $k \geq 3$ and $p_* \circ [\eta_2]_* = \eta_{2*}$. For $n = 4$ or 8,

$$(2) \qquad i_* + [\iota_{n-1}]_* : \pi_k(R_{n-1}) \oplus \pi_k(S^{n-1}) \to \pi_k(R_n)$$

is an isomorphism for all k.

For the J-homomorphism $J : \pi_k(R_n) \to \pi_{k+n}(S^n)$, we have the diagram

(3)

$$\begin{array}{ccccccc}
\pi_k(R_{n-1}) & \xrightarrow{i_*} & \pi_k(R_n) & \xrightarrow{p_*} & \pi_k(S^{n-1}) & \xrightarrow{\Delta} & \pi_{k-1}(R_{n-1}) \\
\downarrow J & & \downarrow J & & \downarrow \Sigma^n & & \downarrow J \\
\pi_{k+n-1}(S^{n-1}) & \xrightarrow{\Sigma} & \pi_{k+n}(S^n) & \xrightarrow{H} & \pi_{k+n}(S^{2n-1}) & \xrightarrow{P} & \pi_{k+n-2}(S^{n-1})
\end{array}$$

which is commutative up to sign and its lower sequence is the EHP exact sequence ([18]). Moreover, we have a formula

(4) $$J(\alpha \circ \beta) = J(\alpha) \circ \Sigma^n \beta$$

for $\alpha \in \pi_j(R_n)$ and $\beta \in \pi_k(S^j)$.

The connecting homomorphism $\Delta : \pi_k(S^n) \to \pi_{k-1}(R_n)$ satisfies the following relation

(5) $$\Delta(\alpha \circ \Sigma\beta) = \Delta(\alpha) \circ \beta$$

for $\alpha \in \pi_j(S^n)$ and $\beta \in \pi_{k-1}(S^{j-1})$.

As to the Toda bracket, from Theorem 5.2 of [12], we have following

Proposition 1.1. *Assume that $\alpha \circ \Sigma\beta = \beta \circ \gamma = 0$ for $\alpha \in \pi_{j+1}(S^n), \beta \in \pi_m(S^j)$ and $\gamma \in \pi_k(S^m)$. Then we have*

$$\Delta\{\alpha, \Sigma\beta, \Sigma\gamma\}_1 \subset \{\Delta\alpha, \beta, \gamma\}.$$

On $\pi_{12}(R_n)$, from [5], we have the following.

Lemma 1.2. (1) *There exists an element $[\sigma'''] \in \pi_{12}(R_6) \cong Z_4$ which satisfies the relations $p_*([\sigma''']) = \sigma''', 2[\sigma'''] = [\iota_3]_6\mu_3$ and $\Delta(\sigma'') = [\sigma''']$, where $\Delta : \pi_{13}(S^6) \to \pi_{12}(R_6)$.*

(2) *There exists an element $[\eta^2_{10}] \in \pi_{12}(R_{11})$ of order 2 such that*

$$J([\eta^2_{10}]) = \theta'.$$

(3) *There exists an element $[\eta_{11}] \in \pi_{12}(R_{12})$ of order 2 such that*

$$J([\eta_{11}]) = \theta.$$

From Table 2 of [5], we have $\pi_8(R_6) \cong Z_8\{[\nu_5]\}$ and $\pi_9(R_6) \cong Z_2\{[\nu_5]\eta_8\}$. Since $8[\nu_5] = 8\nu_8 = 0$, a Toda bracket $\{[\nu_5], 8\iota_8, \nu_8\}$ is defined and it is a coset of $[\nu_5] \circ \pi_{12}(S^8) + \pi_9(R_6) \circ \nu_9 = 0$. From Lemma 5.13 of [18], $\sigma''' = \{\nu_5, 8\iota_8, \mu_8\}$. Then we have

$$p_*(\{[\nu_5], 8\iota_8, \nu_8\}) = \{\nu_5, 8\iota_8, \nu_8\} = \sigma'''.$$

Consequently we have

Lemma 1.3.

$$[\sigma'''] = \{[\nu_5], 8\iota_8, \nu_8\}.$$

From Table 2 of [5], we have

$$\pi_{14}(R_4) \cong Z_4\{[\eta_2]_4\epsilon'\} \oplus Z_2\{[\eta_2]_4\eta_3\mu_4\} \oplus Z_4\{[\iota_3]\epsilon'\} \oplus Z_2\{[\iota_3]\eta_3\mu_4\}.$$

By Theorem 7.3 of [18], we have

$$\pi_{14}(S^4) \cong Z_8\{\nu_4\sigma'\} \oplus Z_4\{\Sigma\epsilon'\} \oplus \{\eta_4\mu_5\}.$$

For the homomorphism $\Delta : \pi_{14}(S^4) \to \pi_{13}(R_4)$, we prove

Lemma 1.4.

$$\Delta(\nu_4\sigma') = 2[\iota_3]\epsilon' + 2c[\eta_2]_4\epsilon'$$

where c is an integer.

Proof. Set $\Delta(\nu_4\sigma') = a[\iota_3]\epsilon' + b[\iota_3]\eta_3\mu_4 + c[\eta_2]_4\epsilon' + d[\eta_2]_4\eta_3\mu_4$, for some integers a, b, c and d. Apply the J-homomorphism $J : \pi_{13}(R_4) \to \pi_{17}(S^4)$ to the both sides of this equation, where $\pi_{17}(S^4) \cong Z_8\{\nu_4^2\sigma_{10}\} \oplus Z_2\{\nu_4\eta_7\} \oplus Z_2\{(\Sigma\nu')\eta_7\mu_8\}$ by Theorem 7.7 of [18]. From the diagram (3) and the relation in p.76 of [18], we have

$$J\Delta(\nu_4\sigma') = P(4\sigma_9\nu_{16}) = 4x'\nu_4^2\sigma_{10},$$

where x' is an odd integer.

From [5], we have $J([\iota_3]) = \nu_4$ and $J([\eta_2]_4) = \Sigma\nu'$. Then, by (7.10) of [18], we have

$$J([\iota_3]\epsilon') = \nu_4\Sigma^4\epsilon' = \pm 2\nu_4^2\sigma_{10},$$

$$J([\iota_3]\eta_3\mu_4) = \nu_4\eta_7\mu_8,$$

$$J([\eta_2]_4\eta_3\mu_4) = (\Sigma\nu')\eta_7\mu_8$$

and

$$J([\eta_2]_4\epsilon') = \Sigma\nu'\Sigma^4\epsilon' = 0,$$

since $\nu'\Sigma^3\epsilon' = 2(\nu'\nu_6\sigma_9) = 0$ by (7.10) of [18] and $\nu'\nu_6 \in \pi_9(S^3) = 0$. So we obtain that $a = 2$ and

$$\Delta(\nu_4\sigma') = 2[\iota_3]\epsilon' + c[\eta_2]_4\epsilon'.$$

On the other hand, we have

$$\Delta(2(\nu_4\sigma')) = \Delta(\nu_4 \circ 2\sigma') = \Delta(\nu_4 \circ \Sigma\sigma'') = \Delta(\nu_4)\sigma'' = ([\iota_3] + a[\eta_2]_4)\nu'\sigma''$$

by Lemma 1.1 of [6]. Here we have

$$\Sigma^3(\nu'\sigma'') = 8\nu_6\sigma_9 = 0.$$

Since $\Sigma^3 : \pi_{13}(S^3) \to \pi_{16}(S^6)$ is a monomorphism by (7.15) of [18], we have $\nu'\sigma'' = 0$. It follows that $2c[\eta_2]_4\epsilon' = 0$. Therefore we obtain that c is an even integer. □

2. Determination of $\pi_{19}(R_n : 2)$

In this section, we shall determine the generators of the 2-primary components $\pi_{19}(R_n : 2)$. The results for $\pi_{19}(R_n : 2)$ are stated as follow:

Theorem 2.1.

$$
\begin{array}{lll}
\pi_{19}(R_3:2) & = & \{[\eta_2]\mu_3\sigma_{12}\} \oplus \{[\eta_2]\eta_3\bar{\epsilon}_4\} \cong Z_2 \oplus Z_2, \\
\pi_{19}(R_4:2) & = & \{[\eta_2]_4\mu_3\sigma_{12}\} \oplus \{[\eta_2]_4\eta_3\bar{\epsilon}_4\} \oplus \{[\iota_3]\mu_3\sigma_{12}\} \oplus \{[\iota_3]\eta_3\bar{\epsilon}_4\} \\
 & \cong & Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2, \\
\pi_{19}(R_5:2) & = & \{[\iota_3]_5\mu_3\sigma_{12}\} \oplus \{[\iota_3]_5\eta_3\bar{\epsilon}_4\} \cong Z_2 \oplus Z_2, \\
\pi_{19}(R_6:2) & = & \{[\sigma''']\sigma_{12}\} \oplus \{[\nu_5]\bar{\nu}_8\nu_{16}\} \cong Z_4 \oplus Z_2, \\
\pi_{19}(R_n:2) & = & \{[\nu_5]_n\bar{\nu}_8\nu_{16}\} \cong Z_2 \textit{ for } n = 7, 8 \textit{ and } 9, \\
\pi_{19}(R_{10}:2) & = & \{[2\iota_9]\sigma_9\nu_{16}\} \cong Z_4, \\
\pi_{19}(R_{11}:2) & = & \{[32[\iota_{10},\iota_{10}]]\} \oplus \{[\epsilon_{10}]\eta_{18}\} \oplus \{[\eta_{10}^2]\sigma_{12}\} \\
 & \cong & Z \oplus Z_2 \oplus Z_2, \\
\pi_{19}(R_{12}:2) & = & \{[32[\iota_{10},\iota_{10}]]_{12}\} \oplus \{[\epsilon_{10}]_{12}\eta_{18}\} \oplus \{[\eta_{10}^2]_{12}\sigma_{12}\} \\
 & & \oplus\{[\eta_{11}]\sigma_{12}\} \oplus \{[\epsilon_{11}]\} \\
 & \cong & Z \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2, \\
\pi_{19}(R_{13}:2) & = & \{[32[\iota_{10},\iota_{10}]]_{13}\} \oplus \{[\eta_{11}]_{13}\sigma_{12}\} \oplus \{[\epsilon_{11}]_{13}\} \\
 & \cong & Z \oplus Z_2 \oplus Z_2 \\
\pi_{19}(R_{14}:2) & = & \{[32[\iota_{10},\iota_{10}]]_{14}\} \oplus \{[\nu_{13}^2]\} \cong Z \oplus Z_4 \\
\pi_{19}(R_n:2) & = & \{[32[\iota_{10},\iota_{10}]]_n\} \oplus \{[\nu_{13}^2]_n\} \cong Z \oplus Z_2 \\
 & & \textit{for } n = 15, 16 \textit{ and } 17 \\
\pi_{19}(R_n:2) & = & \{[32[\iota_{10},\iota_{10}]]_n\} \cong Z \textit{ for } n = 18 \textit{ and } 19, \\
\pi_{19}(R_{20}:2) & = & \{[32[\iota_{10},\iota_{10}]]_{20}\} \oplus \{[2\iota_{19}]\} \cong Z \oplus Z \\
\pi_{19}(R_n:2) & = & \{[32[\iota_{10},\iota_{10}]]_n\} \cong Z \textit{ for } n \geq 21.
\end{array}
$$

The relations hold: $2[\sigma''']\sigma_{12} = [\iota_3]_5\mu_3\sigma_{12}$ *and* $2[\nu_{13}^2] = [\epsilon_{11}]_{14}$.

Since $\pi_{19}(S^3) \cong Z_2\{\mu_3\sigma_{12}\} \oplus Z_2\{\eta_3\bar{\epsilon}_4\}$ by Theorem 12.6 of [18], we have the results for $\pi_{19}(R_3)$ and $\pi_{19}(R_4)$ from (1) and (2), respectively.

For $n \geq 4$, we will determine the group $\pi_{19}(R_{n+1})$ by applying the exact sequence

$$(19)_n \qquad \pi_{20}(S^n) \xrightarrow{\Delta} \pi_{19}(R_n) \xrightarrow{i_*} \pi_{19}(R_{n+1}) \xrightarrow{p_*} \mathrm{Ker}\Delta \to 0,$$

where Im $p_* =$ Ker Δ for $\Delta : \pi_{19}(S^n) \to \pi_{18}(R_n)$. For the results on this Ker Δ, we have the following lemma as the results of the computation of $\pi_{18}(R_n)$ in [6].

Lemma 2.2. *The homomrphism* $\Delta : \pi_{19}(S^n) \to \pi_{18}(R_n)$ *are monomorphisms for* $n = 4, 6, 7, 8, 12, 14, 15, 16, 17$ *and* 18. *For the other values of n, we have the following table of the kernel of* Δ*:*

n	5	9	10	11	13	19
$\mathrm{Ker}\Delta \cong$	$Z_2 \oplus Z_2$	Z_4	$Z \oplus Z_2 \oplus Z_2$	$Z_2 \oplus Z_2$	Z_2	Z
generators	$\nu_5\zeta_8,\ \nu_5\overline{\nu}_8\nu_{16}$	$2\sigma_9\nu_{16}$	$32[\iota_{10},\iota_{10}],\ \eta_{10}\overline{\nu}_{11},\ \eta_{10}\epsilon_{11}$	$\overline{\nu}_{11},\ \epsilon_{11}$	ν_{13}^2	$2\iota_{19}$

From the exact sequence $(19)_4$ and Lemma 2.2, we have the exact sequence

$$\pi_{20}(S^4) \xrightarrow{\Delta} \pi_{19}(R_4) \xrightarrow{i_*} \pi_{19}(R_5) \to 0,$$

where $\pi_{20}(S^4) \cong Z_2\{\nu_4^2\sigma_{10}\nu_{17}\} \oplus Z_2\{\mu_4\sigma_{13}\} \oplus Z_2\{\eta_4\overline{\epsilon}_5\}$ by Theorem 12.6 of [18].

Since $\Delta(\nu_4^2) = 0$ and $\Delta(\iota_4) = 2[\iota_3] - [\eta_2]_4$ by Table 3 of [5], we have

$\Delta(\nu_4^2\sigma_{10}\nu_{17}) = \Delta(\nu_4^2)\sigma_9\nu_{16} = 0,$

$\Delta(\mu_4\sigma_{13}) = \Delta(\iota_4)\mu_3\sigma_{13} = (2[\iota_3] - [\eta_2]_4)\mu_3\sigma_{13} = [\eta_2]_4\mu_3\sigma_{13}$

and

$\Delta(\eta_4\overline{\epsilon}_5) = \Delta(\iota_4)\eta_3\overline{\epsilon}_4 = (2[\iota_3] - [\eta_2]_4)\eta_3\overline{\epsilon}_4 = [\eta_2]\eta_3\overline{\epsilon}_4.$

It follows from the above exact sequence that

$$\pi_{19}(R_5) = \{[\iota_3]_5\mu_3\sigma_{12}\} \oplus \{[\iota_3]_5\eta_3\overline{\epsilon}_4\} \cong Z_2 \oplus Z_2.$$

Consider the homomorphism $\Delta : \pi_{20}(S^5) \to \pi_{20}(R_5)$, where $\pi_{20}(S^5) \cong Z_2\{\overline{\epsilon}_5\} \oplus Z_2\{\rho^{IV}\}$ by Theorem 10.5 of [18]. Since $\Delta(\iota_5) = [\iota_3]_5\eta_3$ by Table 3 of [5], we have

$$\Delta(\overline{\epsilon}_5) = [\iota_3]_5\eta_3\overline{\epsilon}_4.$$

By the definition of ρ^{IV} (p.103 of [18]),

$$\rho^{IV} \in \{\sigma''', 2\iota_{12}, 8\sigma_{12}\}_1 \tag{6}$$

where the Toda bracket $\{\sigma''', 2\iota_{12}, 8\sigma_{12}\}_1$ consists of a single element.

By Proposition 1.1 and Lemma 1.2 (1), we have

$$\Delta(\rho^{IV}) \in \{\Delta(\sigma'''), 2\iota_{12}, 8\sigma_{12}\} = \{0, 2\iota_{12}, 8\sigma_{12}\} \equiv 0$$

and its coset is $\pi_{12}(R_5) \circ 8\sigma_{12} = 0$ because $\pi_{12}(R_5) \cong Z_2 \oplus Z_2$ by Table 2 of [5]. Therefore we have

$$\Delta(\rho^{IV}) = 0.$$

It follows from the exact sequence $(19)_5$ and Lemma 2.2 that the sequence

$$0 \to Z_2\{[\iota_3]_5\mu_3\sigma_{13}\} \xrightarrow{i_*} \pi_{19}(R_6) \xrightarrow{p_*} Z_2\{\nu_5\zeta_8\} \oplus Z_2\{\nu_5\overline{\nu}_8\nu_{16}\} \to 0$$

is exact.

The element $[\nu_5]\overline{\nu}_8\nu_{16}$ is of order 2 and $p_*([\nu_5]\overline{\nu}_8\nu_{16}) = \nu_5\overline{\nu}_8\nu_{16}$.

By Lemma 1.2 (1) , we have

(7) $$[\iota_3]_6\mu_3\sigma_{12} = 2[\sigma'''] \sigma_{12}.$$

Hence we obtain that $[\sigma''']\sigma_{12}$ is of order 4.

We need the following.

Lemma 2.3. (N. Oda) $\sigma'''\sigma_{12} = \nu_5\zeta_8$

Proof. From (10.7) and Lemma 5.14 of [18], we have

$$\Sigma(\sigma'''\sigma_{12}) = 2\sigma''\sigma_{13} = \nu_6\zeta_9 = \Sigma(\nu_5\zeta_8).$$

By Theorem 10.3 of [18], we have

$$\sigma'''\sigma_{12} = \nu_5\zeta_8 + a\nu_5\overline{\nu}_8\nu_{16},$$

where $a = 0$ or 1.

By (7.20) of [18],

$$\sigma'''\sigma_{12}\nu_{19} = 0.$$

By Proposition 2.4 (2) of [15] and p.145 of [18],

$$\nu_5\zeta_8\nu_{19} = \nu_5^2\zeta_{11} = 0.$$

By (15.1) of [10] and p.145 of [18] and Theorem 12.7 of [18],

$$\nu_5\overline{\nu}_8\nu_{16}^2 = \nu_5(2\kappa_8) = 2(\nu_5\kappa_8) \neq 0.$$

It follows that $a = 0$. This completes the proof. □

From Lemma 2.3, we have

(8) $$[\sigma''']\sigma_{12} = [\nu_5]\zeta_8.$$

By the exactness of the above sequence, we have the result

$$\pi_{19}(R_6) = \{[\sigma''']\sigma_{12}\} \oplus \{[\nu_5]\overline{\nu}_8\nu_{16}\} \cong Z_4 \oplus Z_2.$$

From the exact sequence $(19)_6$ and Lemma 2.2, the following sequence

$$\pi_{20}(S^6) \xrightarrow{\Delta} \pi_{19}(R_6) \xrightarrow{i_*} \pi_{19}(R_7) \to 0$$

is exact, where $\pi_{20}(S^6) \cong Z_4\{\sigma''\sigma_{13}\} \oplus Z_2\{\overline{\nu}_6\nu_{14}^2\}$ by Theorem 10.3 of [18].

From Lemma 1.2 (1) and the formula (5), we have

(9) $$\Delta(\sigma''\sigma_{13}) = \Delta(\sigma'')\sigma_{12} = [\sigma''']\sigma_{12}.$$

Since $\Delta(\overline{\nu}_6 + \epsilon_6) = 0$ by Table 3 of [5] and $\epsilon_6\nu_{14} = 0$ by (7.13) of [18], we have

$$\Delta(\overline{\nu}_6\nu_{14}^2) = \Delta((\overline{\nu}_6 + \epsilon_6)\nu_{14})\nu_{16} = 0.$$

It follows from the above exact sequence that

$$\pi_{19}(R_7) = \{[\nu_5]_7\overline{\nu}_8\nu_{16}\} \cong Z_2.$$

From (2) and $\pi_{19}(S^7) = 0$ by Theorem 7.6 of [18], we have

$$\pi_{19}(R_8) = \{[\nu_5]_8\overline{\nu}_8\nu_{16}\} \cong Z_2.$$

From the exact sequence $(19)_8$ and Lemma 2.2, we obtain that

$$\pi_{19}(R_9) = \{[\nu_5]_9\overline{\nu}_8\nu_{16}\} \cong Z_2$$

since $\pi_{20}(S^8) = 0$ by Theorem 7.4 of [18].

Consider the homomorphism $\Delta : \pi_{20}(S^9) \to \pi_{19}(R_9)$, where $\pi_{20}(S^9) \cong Z_8\{\zeta_9\} \oplus Z_2\{\overline{\nu}_9\nu_{17}\}$ by Theorem 7.4 of [18]. By Lemma 1.1 of [6], $\Delta(\iota_9) = [\nu_5]_9 + [\iota_7]_9\eta_7$. So by the fact $\eta_7\zeta_8 \in \pi_{19}(S^7) = 0$ of Theorem 7.6 of [18], we have

$$\begin{aligned}\Delta(\zeta_9) &= \Delta(\iota_9)\zeta_8 \\ &= ([\nu_5]_9 + [\iota_7]_9\eta_7)\zeta_8 \\ &= [\nu_5]_9\zeta_8 + [\iota_7]_9\eta_7\zeta_8 \\ &= 0 \qquad\qquad \text{by (8) and (9)}\end{aligned}$$

and

$$\begin{aligned}\Delta(\overline{\nu}_9\nu_{17}) &= \Delta(\iota_9)\overline{\nu}_8\nu_{16} \\ &= [\nu_5]_9\overline{\nu}_8\nu_{16} + [\iota_7]_9\eta_7\overline{\nu}_8\nu_{16} \\ &= [\nu_5]_9\overline{\nu}_8\nu_{16} + [\iota_7]_9\overline{\nu}_7\eta_{15}\nu_{16} \\ &= [\nu_5]_9\overline{\nu}_8\nu_{16}.\end{aligned}$$

Hence $\Delta : \pi_{20}(S^9) \to \pi_{19}(R_9)$ is an epimorphism. It follows from the exact sequence $(19)_9$ that $p_* : \pi_{19}(R_{10}) \to \mathrm{Ker}\,\Delta$ is an isomorphism, where Ker $\Delta \cong Z_4\{2\sigma_9\nu_{16}\}$ by Lemma 2.2. Then we have

$$\pi_{19}(R_{10}) = \{[2\iota_9]\sigma_9\nu_{16}\} \cong Z_4.$$

Consider the homomorphism $\Delta : \pi_{20}(S^{10}) \to \pi_{19}(R_{10})$, where $\pi_{20}(S^{10}) = Z_4\{\sigma_{10}\nu_{17}\} \oplus Z_2\{\eta_{10}\mu_{11}\}$ by Theorem 7.7 of [18]. We know $\Delta(\iota_{10}) = [2\iota_9]$ by [16] and $\Delta(\eta_{10}) = 2[\iota_7]_{10}\nu_7$ by Theorem 1 of [7] and Table 2 of [5]. So we have

$$\Delta(\sigma_{10}\nu_{17}) = \Delta(\iota_{10})\sigma_9\nu_{16} = [2\iota_9]\sigma_9\nu_{16}$$

and

$$\Delta(\eta_{10}\mu_{11}) = 2[\iota_7]_{10}\nu_7\mu_{10} = 0.$$

Therefore $\Delta : \pi_{20}(S^{10}) \to \pi_{19}(R_{10})$ is an epimorphism. It follows from the exact sequence $(19)_{10}$ that $p_* : \pi_{19}(R_{11}) \to \mathrm{Ker}\,\Delta$ is an isomorphism. From Lemma 2.2, we have

$$\pi_{19}(R_{11}) = \{[32[\iota_{10}, \iota_{10}]]\} \oplus \{[\epsilon_{10}]\eta_{18}\} \oplus \{[\eta_{10}^2]\sigma_{12}\} \cong Z \oplus Z_2 \oplus Z_2$$

where $p_*([\epsilon_{10}]\eta_{18}) = \epsilon_{10}\eta_{18} = \eta_{10}\epsilon_{11}$ and $p_*([\eta_{10}^2]\sigma_{12}) = \eta_{10}^2\sigma_{12} = \eta_{10}\overline{\nu}_{11} + \eta_{10}\epsilon_{11}$.

To determine the group structures of $\pi_{19}(R_n)$ for $n = 12$ and 13, we need some lemmas. By Lemma 3.10 (ii) of [6], we have $J([\epsilon_{10}]) \equiv \lambda' \mod \Sigma\pi_{28}(S^{10})$. We recall that the stable J-image is trivial in $\pi_{18}^S(S^0) \cong Z_8\{\nu^*\} \oplus Z_2\{\eta\bar{\mu}\}$. We also recall that $\pi_{28}(S^{10}) \cong (Z_8 \oplus Z_2 \oplus Z_2)\{\lambda'', \xi'', \eta_{10}\bar{\mu}_{11}\}$, $\Sigma\lambda'' = 2\lambda'$, $\Sigma\xi'' = 2\xi'$, $\Sigma^\infty\lambda' = 4\nu^*$, $\Sigma^\infty\xi' = -2\nu^*$. So we obtain

$$(*) \quad J[\varepsilon_{10}] = a\lambda' + 2b\xi' \ (a, b : odd)$$

We show the following.

Lemma 2.4. *For the J-homomorphism $J : \pi_{19}(R_{11}) \to \pi_{30}(S^{11})$, we have*

(1) $J([32[\iota_{10}, \iota_{10}]]) \in \Sigma\pi_{29}(S^{10}) \cong Z_2\{\overline{\sigma}_{11}\} \oplus Z_8\{\overline{\zeta}_{11}\}$,
(2) $J([\epsilon_{10}]\eta_{18}) = \lambda'\eta_{29}$,
(3) $J([\eta_{10}^2]\sigma_{12}) = \xi'\eta_{29}$.

Proof. From the diagram (3), we have

$$HJ([32[\iota_{10}, \iota_{10}]]) = \Sigma^{11}p_*([32[\iota_{10}, \iota_{10}]]) = \Sigma^{11}(32[\iota_{10}, \iota_{10}]) = 0.$$

By the exactness of the EHP sequence, we have the first relation.

By use of the formula (4) and $(*)$, we have $J([\epsilon_{10}]\eta_{18}) = \lambda'\eta_{29}$.

By (2.6) of [9], $\theta'\sigma_{23} = \xi'\eta_{29}$. By Lemma 1.2 (2), $J([\eta_{10}^2]) = \theta'$. Hence we obtain $J([\eta_{10}^2]\sigma_{12}) = \theta'\sigma_{23} = \xi'\eta_{29}$. This completes the proof. □

Consider the homomorphism $\Delta : \pi_{20}(S^{11}) \to \pi_{19}(R_{11})$, where $\pi_{20}(S^{11}) \cong Z_2\{\eta_{11}\overline{\nu}_{12}\} \oplus Z_2\{\eta_{11}\epsilon_{12}\} \oplus Z_2\{\mu_{11}\}$ by Theorem 7.2 of [18]. Since $\Delta(\iota_{11}) = [\iota_7]_{11}\nu_7$ by Lemma 1.1 (i) of [6], we have

$$\Delta(\eta_{11}\overline{\nu}_{12}) = [\iota_7]_{11}\nu_7\eta_{10}\overline{\nu}_{11} = 0,$$
$$\Delta(\eta_{11}\epsilon_{12}) = [\iota_7]_{11}\nu_7\eta_{10}\epsilon_{11} = 0$$

and

$$\Delta(\mu_{11}) = [\iota_7]_{11}\nu_7\mu_{10} = 0.$$

Thus we obtained that $\Delta : \pi_{20}(S^{11}) \to \pi_{19}(R_{11})$ is a trivial homomorphism. It follows from the exact sequence $(19)_{11}$ that the sequence

$$0 \to \pi_{19}(R_{11}) \xrightarrow{i_*} \pi_{19}(R_{12}) \xrightarrow{p_*} \mathrm{Ker}\,\Delta \to 0$$

is exact, where $\mathrm{Ker}\,\Delta \cong Z_2\{\overline{\nu}_{11}\} \oplus Z_2\{\epsilon_{11}\} \cong Z_2\{\eta_{11}\sigma_{12}\} \oplus Z_2\{\epsilon_{11}\}$ by Lemma 6.4 of [18] and Lemma 2.2.

Consider the elements $[32[\iota_{10}, \iota_{10}]]_{12}, [\epsilon_{10}]_{12}\eta_{18}, [\eta_{10}^2]_{12}\sigma_{12}, [\eta_{11}]\sigma_{12}$ and $[\epsilon_{11}]$ of $\pi_{19}(R_{12})$ and the J-homomorphism $J : \pi_{19}(R_{12}) \to \pi_{31}(S^{12})$. We recall from Theorem 12.23 of [18] that

$$\pi_{31}(S^{12}) \cong Z_2\{\Sigma\lambda' \circ \eta_{30}\} \oplus Z_2\{\Sigma\xi' \circ \eta_{30}\} \oplus Z_2\{\overline{\sigma}_{12}\} \oplus Z_8\{\overline{\zeta}_{12}\} \oplus Z_2\{\omega'\} \oplus Z_2\{\xi_{12}\eta_{30}\}.$$

Then we have

Lemma 2.5. (1) $J([32[\iota_{10}, \iota_{10}]]_{12}) \in \Sigma^2\pi_{29}(S^{10}) \cong Z_2\{\overline{\sigma}_{12}\} \oplus Z_8\{\overline{\zeta}_{12}\}$,
(2) $J([\epsilon_{10}]_{12}\eta_{18}) = (\Sigma\lambda')\eta_{30}$,
(3) $J([\eta_{10}^2]_{12}\sigma_{12}) = (\Sigma\xi')\eta_{30}$,
(4) $J([\eta_{11}]\sigma_{12}) = \theta\sigma_{24} \equiv \xi_{12}\eta_{30} \mod \Sigma\pi_{30}(\mathrm{S}^{11})$,
(5) $J([\epsilon_{11}]) \equiv \omega' \mod \Sigma\pi_{30}(\mathrm{S}^{11}) + \xi_{12}\eta_{30}$.

Proof. From Lemma 2.4 and the diagram (3), we obtain (1), (2) and (3). By Lemma 12.14 of [18], the element ξ_{12} of $\pi_{30}(S^{12})$ satisfies $H(\xi_{12}) \equiv \sigma_{23} \mod 2\sigma_{23}$. From Lemma 7.5 of [18], the element $\theta \in \pi_{24}(S^{12})$ satisfies $H(\theta) = \eta_{23}$. We have

$$H(\theta\sigma_{24}) = H(\theta)\sigma_{24} = \eta_{23}\sigma_{24} = \sigma_{23}\eta_{30} = H(\xi_{12}\eta_{30}).$$

Hence it follows from the EHP sequence that

$$\theta\sigma_{24} \equiv \xi_{12}\eta_{30} \mod \Sigma\pi_{30}(\mathrm{S}^{11}).$$

Since $J([\eta_{11}]) = \theta$ by Lemma 1.1 (3), we obtain (4).

From Lemma 12.21 of [18], the element ω' of $\pi_{31}(S^{12})$ satisfies $H(\omega') \equiv \epsilon_{23} \mod \overline{\nu}_{23} + \epsilon_{23}$. Since $HJ([\epsilon_{11}]) = \epsilon_{23}$, we have

$$H(\omega') \equiv HJ([\epsilon_{11}]) \mod \overline{\nu}_{23} + \epsilon_{23}.$$

Hence we obtain that

$$J([\epsilon_{11}]) \equiv \omega' \mod \Sigma\pi_{30}(\mathrm{S}^{11}) + \xi_{12}\eta_{30}.$$

□

Here we have

$p_*([\eta_{11}]\sigma_{12}) = \eta_{11}\sigma_{12} = \overline{\nu}_{11} + \epsilon_{11}$ and $[\eta_{11}]\sigma_{12}$ is of order 2.

From Lemma 2.5, the J-image of five elements $[32[\iota_{10}, \iota_{10}]]_{12}$, $[\epsilon_{10}]_{12}\eta_{18}$, $[\eta_{10}^2]_{12}\sigma_{12}$, $[\eta_{11}]\sigma_{12}$ and $[\epsilon_{11}]$ of $\pi_{19}(R_{12})$ are independent on $\pi_{31}(S^{12})$. It follows that

$$\begin{aligned}\pi_{19}(R_{12}) &= \{[32[\iota_{10}, \iota_{10}]]_{12}\} \oplus \{[\epsilon_{10}]_{12}\eta_{18}\} \oplus \{[\eta_{10}^2]_{12}\sigma_{12}\} \oplus \{[\eta_{11}]\sigma_{12}\} \\ &\quad \oplus \{[\epsilon_{11}]\} \\ &\cong Z \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.\end{aligned}$$

We note that $[\epsilon_{11}] \in \{[\eta_{11}], 2\iota_{12}, \nu_{12}^2\}_1$. We show

Lemma 2.6. $[\epsilon_{11}] \in \{[\eta_{11}], \nu_{12}^2, 2\iota_{18}\}_1$

Proof. From Theorem 2.1 of [5], we have $[\eta_{11}]\nu_{12} \in \pi_{15}(R_{12}) \cong Z\{[8\sigma_8]_{12}\} \oplus Z_2\{[\nu_5]_{12}\sigma_8\}$. Hence

$$[\eta_{11}]\nu_{12} = x[\nu_5]_{12}\sigma_8.$$

for some integer x. Then, from Lemma 2.3 (ii) of [6],

$$\begin{aligned} [\eta_{11}]\nu_{12}^2 &= x[\nu_5]_{12}\sigma_8\nu_{15} \\ &= x[\overline{\nu}_6 + \epsilon_6]_{12}\eta_{14}\nu_{15} \\ &= 0 \end{aligned}$$

By the fact $2\nu_{12}^2 = 0$, a Toda bracket $\{[\eta_{11}], \nu_{12}^2, 2\iota_{18}\}_1$ is defined and it is a coset of $[\eta_{11}] \circ \pi_{19}(S^{12}) + \pi_{19}(R_{12}) \circ 2\iota_{19} = \{[\eta_{11}]\sigma_{12}\} + \{2[32[\iota_{10}, \iota_{10}]]_{12}\}$.

From (6.1) of [18], $\epsilon_{11} \in \{\eta_{11}, \nu_{12}^2, 2\iota_{18}\}_1$ and it is a coset of $\{\eta_{11}\sigma_{12}\}$. Since

$$p_*(\{[\eta_{11}], \nu_{12}^2, 2\iota_{18}\}_1) \subset \{\eta_{11}, \nu_{12}^2, 2\iota_{18}\}_1,$$

we can take $[\epsilon_{11}] \in \{[\eta_{11}], \nu_{12}^2, 2\iota_{18}\}_1$. □

From the exact sequence $(19)_{12}$ and Lemma 2.2, the following sequence

$$\pi_{20}(S^{12}) \xrightarrow{\Delta} \pi_{19}(R_{12}) \xrightarrow{i_*} \pi_{19}(R_{13}) \to 0$$

is exact, where $\pi_{20}(S^{12}) \cong Z_2\{\overline{\nu}_{12}\} \oplus Z_2\{\epsilon_{12}\}$ by Theorem 7.1 of [18].

By (12.26) of [18], $J\Delta(\pi_{20}(S^{12})) = P(\pi_{33}(S^{25})) = \{(\Sigma\lambda')\eta_{30}\} + \{(\Sigma\xi')\eta_{30}\}$ has four elements. From Lemma 2.5,

$$J(\{[\epsilon_{10}]_{12}\eta_{18}\} \oplus \{[\eta_{10}^2]_{12}\sigma_{12}\}) = \{(\Sigma\lambda')\eta_{30}\} + \{(\Sigma\xi')\eta_{30}\}.$$

It follows that $\Delta(\pi_{20}(S^{12}))$ has four elements and

$$\{[\epsilon_{10}]_{12}\eta_{18}\} \oplus \{[\eta_{10}^2]_{12}\sigma_{12}\} \subset \Delta(\pi_{20}(S^{12})).$$

Hence we obtain that

$$\pi_{19}(R_{13}) = \{[32[\iota_{10}, \iota_{10}]]_{13}\} \oplus \{[\eta_{11}]_{13}\sigma_{12}\} \oplus \{[\epsilon_{11}]_{13}\} \cong Z \oplus Z_2 \oplus Z_2.$$

Consider the homomorphism $\Delta : \pi_{20}(S^{13}) \to \pi_{19}(R_{13})$, where $\pi_{20}(S^{13}) \cong Z_{16}\{\sigma_{13}\}$ by Propsition 5.15 of [18]. Since $\Delta(\iota_{13}) = [\eta_{11}]_{13}$ by Table 3 of [5], we have

$$\Delta(\sigma_{13}) = \Delta(\iota_{13})\sigma_{12} = [\eta_{11}]_{13}\sigma_{12}. \tag{10}$$

It follows from the exact sequence $(19)_{13}$ that the sequence

$$0 \to Z_2\{[\epsilon_{11}]_{13}\} \oplus Z\{[32[\iota_{10}, \iota_{10}]]_{13}\} \xrightarrow{i_*} \pi_{19}(R_{14}) \xrightarrow{p_*} \text{Ker }\Delta \to 0$$

is exact, where Ker $\Delta \cong Z_2\{\nu_{13}^2\}$ by Lemma 2.2.

Since $\Delta(\iota_{13})\nu_{12}^2 = \Delta(\nu_{13}^2) = 0$ and $2\nu_{12}^2 = 0$, a Toda bracket $\{\Delta(\iota_{13}), \nu_{12}^2, 2\iota_{18}\}$ is defined. Then, from Lemma 2.6, we have

$$\begin{array}{rcl} [\epsilon_{11}]_{13} = i_*([\epsilon_{11}]) & \in & i_*(\{[\eta_{11}], \nu_{12}^2, 2\iota_{18}\}) \\ & \subset & \{[\eta_{11}]_{13}, \nu_{12}^2, 2\iota_{18}\} \\ & = & \{\Delta(\iota_{13}), \nu_{12}^2, 2\iota_{18}\} \end{array}$$

by $\Delta(\iota_{13}) = [\eta_{11}]_{13}$ ([7]), where $i_* : \pi_{19}(R_{12}) \to \pi_{19}(R_{13})$. According to Theorem 2.1 of [11], for an element $[\epsilon_{11}]_{13}$, there exists an element $[\nu_{13}^2] \in \pi_{19}(R_{14})$ such that

$$p_*([\nu_{13}^2]) = \nu_{13}^2 \quad \text{and} \quad [\epsilon_{11}]_{14} = 2[\nu_{13}^2].$$

It follows from the above exact sequence $(19)_{13}$ that

$$\pi_{19}(R_{14}) = \{[32[\iota_{10}, \iota_{10}]]_{14}\} \oplus \{[\nu_{13}^2]\} \cong Z \oplus Z_4.$$

From the exact sequence $(19)_{14}$ and Lemma 2.2, the following sequence

$$\pi_{20}(S^{14}) \xrightarrow{\Delta} \pi_{19}(R_{14}) \xrightarrow{i_*} \pi_{19}(R_{15}) \to 0$$

is exact, where $\pi_{20}(S^{14}) \cong Z_2\{\nu_{14}^2\}$ by Propsition 5.11 of [18].

By (12.27) of [18], we have $J\Delta(\nu_{14}^2) = P(\nu_{29}^2) = \Sigma^2\omega' = 2\omega_{14}\nu_{30}$. It follows that $\Delta(\nu_{14}^2)$ is of order 2. Since $\pi_{19}(R_{14}) \cong Z \oplus Z_4$, we have

(11) $$\Delta(\nu_{14}^2) = 2[\nu_{13}^2].$$

From the above exact sequence we have

$$\pi_{19}(R_{15}) = \{[32[\iota_{10}, \iota_{10}]]_{15}\} \oplus \{[\nu_{13}^2]_{15}\} \cong Z \oplus Z_2.$$

In the exact sequence $(19)_{15}$, Ker $\Delta = 0$ and $\pi_{20}(S^{15}) = 0$ and in the exact sequence $(19)_{16}$, Ker $\Delta = 0$ and $\pi_{20}(S^{16}) = 0$. Hence we have

$$\pi_{19}(R_n) = \{[32[\iota_{10}, \iota_{10}]]_n\} \oplus \{[\nu_{13}^2]_n\} \cong Z \oplus Z_2 \quad \text{for } n = 16, 17.$$

Consider the homomorphism $\Delta : \pi_{20}(S^{17}) \to \pi_{19}(R_{17})$, where $\pi_{20}(S^{17}) \cong Z_8\{\nu_{17}\}$ by Proposition 5.6 of [18]. By Theorem 3 (v) of [7], $\Delta(\nu_{17}) \neq 0$ and $\Delta(2\nu_{17}) = 0$. Therefore we have

(12) $$\Delta(\nu_{17}) = [\nu_{13}^2]_{17}.$$

From the exact sequence $(19)_{17}$, we have

$$\pi_{19}(R_{18}) = \{[32[\iota_{10}, \iota_{10}]]_{18}\} \cong Z.$$

In the exact sequence $(19)_{18}$, Ker $\Delta = 0$ by Lemma 2.2. According to [7], $\Delta(\eta_{18}^2) = 0$. Hence we have

$$\pi_{19}(R_{19}) = \{[32[\iota_{10}, \iota_{10}]]_{19}\} \cong Z.$$

From the exact sequence $(19)_{19}$, we have the following exact sequence;

$$0 \to \pi_{19}(R_{19}) \to \pi_{19}(R_{20}) \to Z\{2\iota_{19}\} \to 0$$

Hence we have

$$\pi_{19}(R_{20}) = \{[32[\iota_{10}, \iota_{10}]]_{20}\} \oplus \{[2\iota_{19}]\} \cong Z \oplus Z$$

where $\Delta(\iota_{20}) = [2\iota_{19}]$ by [16].

From the stability of $\pi_k(R_n)$ ([2]), we have

$$\pi_{19}(R_n) = \{[32[\iota_{10}, \iota_{10}]]_n\} \cong Z \quad \text{for} \quad n \geq 21.$$

3. Determination of $\pi_{20}(R_n : 2)$

First, we list the kernel of the homomorphism $\Delta : \pi_{20}(S^n) \to \pi_{19}(R_n)$ from the computations of $\pi_{19}(R_n)$.

Lemma 3.1. *The homomrphism $\Delta : \pi_{20}(S^n) \to \pi_{19}(R_n)$ are monomorphisms for $n = 7, 8, 12, 14, 15, 16$ and 20. For the other values of n, we have the following table of the kernel of Δ:*

n	4	5	6	9	10
Ker $\Delta \cong$	Z_2	Z_2	Z_2	Z_8	Z_2
generators	$\nu_4^2\sigma_{10}\nu_{17}$	ρ^{IV}	$\overline{\nu}_6\nu_{14}^2 = (\overline{\nu}_6 + \epsilon_6)\nu_{14}^2$	ζ_9	$\eta_{10}\mu_{11}$

11	13	17	18	19
$Z_2 \oplus Z_2 \oplus Z_2$	Z_8	Z_4	Z_2	Z_2
$\eta_{11}\overline{\nu}_{12}, \mu_{11}, \eta_{11}\epsilon_{12}$	$2\sigma_{13}$	$2\nu_{17}$	η_{18}^2	η_{19}

In this section, we shall determine the generators of the 2-primary components $\pi_{20}(R_n : 2)$. The results for $\pi_{20}(R_n : 2)$ are stated as follow:

Theorem 3.2.

$$\begin{aligned}
\pi_{20}(R_3 : 2) &= \{[\eta_2]\overline{\epsilon}'\} \oplus \{[\eta_2]\overline{\mu}_3\} \oplus \{[\eta_2]\eta_3\mu_4\sigma_{13}\} \cong Z_4 \oplus Z_2 \oplus Z_2,\\
\pi_{20}(R_4 : 2) &= \{[\eta_2]_4\overline{\epsilon}'\} \oplus \{[\eta_2]_4\overline{\mu}_3\} \oplus \{[\eta_2]_4\eta_3\mu_4\sigma_{13}\} \oplus \{[\iota_3]\overline{\epsilon}'\}\\
&\quad \oplus \{[\iota_3]\overline{\mu}_3\} \oplus \{[\iota_3]\eta_3\mu_4\sigma_{13}\}\\
&\cong Z_4 \oplus Z_2 \oplus Z_2 \oplus Z_4 \oplus Z_2 \oplus Z_2,\\
\pi_{20}(R_5 : 2) &= \{[\nu_4^2]\sigma_{10}\nu_{17}\} \oplus \{[\iota_3]_5\overline{\mu}_3\} \oplus \{[\iota_3]_5\eta_3\mu_4\sigma_{13}\}\\
&\cong Z_2 \oplus Z_2 \oplus Z_2,\\
\pi_{20}(R_6 : 2) &= \{[\rho^{IV}]\} \oplus \{[\nu_4^2]_6\sigma_{10}\nu_{17}\} \cong Z_4 \oplus Z_2,\\
\pi_{20}(R_7 : 2) &= \{[\nu_4^2]_7\sigma_{10}\nu_{17}\} \oplus \{[\overline{\nu}_6 + \epsilon_6]\nu_{14}^2\} \cong Z_2 \oplus Z_2,\\
\pi_{20}(R_8 : 2) &= \{[\nu_4^2]_8\sigma_{10}\nu_{17}\} \oplus \{[\overline{\nu}_6 + \epsilon_6]_8\nu_{14}^2\} \oplus \{[\iota_7]\nu_7\sigma_{10}\nu_{17}\}\\
&\cong Z_2 \oplus Z_2 \oplus Z_2,\\
\pi_{20}(R_9 : 2) &= \{[\overline{\nu}_6 + \epsilon_6]_9\nu_{14}^2\} \cong Z_2,\\
\pi_{20}(R_{10} : 2) &= \{[\overline{\nu}_6 + \epsilon_6]_{10}\nu_{14}^2\} \oplus \{[\zeta_9]\} \cong Z_2 \oplus Z_8,\\
\pi_{20}(R_{11} : 2) &= \{[\overline{\nu}_6 + \epsilon_6]_{11}\nu_{14}^2\} \oplus \{[\zeta_9]_{11}\} \oplus \{[\eta_{10}\mu_{11}]\}\\
&\cong Z_2 \oplus Z_2 \oplus Z_2,
\end{aligned}$$

$$\begin{aligned}
\pi_{20}(R_{12}:2) &= \{[\zeta_9]_{12}\} \oplus \{[\eta_{10}\mu_{11}]_{12}\} \oplus \{[\eta_{11}]\overline{\nu}_{12}\} \oplus \{[\eta_{11}]\epsilon_{12}\} \\
&\quad \oplus\{[\mu_{11}]\} \\
&\cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2, \\
\pi_{20}(R_{13}:2) &= \{[\eta_{11}]_{13}\overline{\nu}_{12}\} \oplus \{[\eta_{11}]_{13}\epsilon_{12}\} \oplus \{[\mu_{11}]_{13}\} \cong Z_2 \oplus Z_2 \oplus Z_2, \\
\pi_{20}(R_{14}:2) &= \{[2\iota_{13}]\sigma_{13}\} \cong Z_{16}, \\
\pi_{20}(R_n:2) &= 0 \quad \text{for } n = 15, 16 \text{ and } 17, \\
\pi_{20}(R_{18}:2) &= \{[2\nu_{17}]\} \cong Z_4, \\
\pi_{20}(R_{19}:2) &= \{[\eta^2_{18}]\} \cong Z_2, \\
\pi_{20}(R_{20}:2) &= \{[\eta^2_{18}]_{20}\} \oplus \{[\eta_{19}]\} \cong Z_2 \oplus Z_2, \\
\pi_{20}(R_{21}:2) &= \{[\eta_{19}]_{21}\} \cong Z_2, \\
\pi_{20}(R_n:2) &= 0 \quad forn \geq 22,
\end{aligned}$$

We have the relations $2[\rho^{IV}] = [\iota_3]_6\overline{\mu}_3$ *and* $8[2\iota_{13}]\sigma_{13} = [\mu_{14}]_{14}$.

Since $\pi_{20}(S^3) \cong Z_4\{\overline{\epsilon}'\} \oplus Z_2\{\overline{\mu}_3\} \oplus Z_2\{\eta_3\mu_4\sigma_{13}\}$ by Theorem 12.7 of [18], we have the results for $\pi_{20}(R_3)$ and $\pi_{20}(R_4)$ from (1) and (2), respectively.

Consider the J-homomorphism $J : \pi_{20}(R_4) \to \pi_{24}(S^4)$, where $\pi_{24}(S^4) \cong Z_2\{(\Sigma\nu')\overline{\mu}_7\} \oplus Z_2\{(\Sigma\nu')\eta_7\mu_8\sigma_{17}\} \oplus Z_2\{\nu_4\sigma'\eta_{14}\mu_{15}\} \oplus Z_2\{\nu_4^2\kappa_{10}\} \oplus Z_2\{\nu_4\overline{\mu}_7\} \oplus Z_2\{\nu_4\eta_7\mu_8\sigma_{17}\}$ by Theorem of [10].

Lemma 3.3. (1) $J([\eta_2]_4\overline{\epsilon}') = 0$,
(2) $J([\eta_2]_4\overline{\mu}_3) = (\Sigma\nu')\overline{\mu}_7$,
(3) $J([\eta_2]_4\eta_3\mu_4\sigma_{13}) = (\Sigma\nu')\eta_7\mu_8\sigma_{17}$,
(4) $J([\iota_3]\overline{\epsilon}') = 0$,
(5) $J([\iota_3]\overline{\mu}_3) = \nu_4\overline{\mu}_7$,
(6) $J([\iota_3]\eta_3\mu_4\sigma_{13}) = \nu_4\eta_7\mu_8\sigma_{17}$.

Proof. From Lemma 12.3 and Theorem 12.7 of [18], we have

$$\Sigma^4\overline{\epsilon}' = \Sigma^3(\Sigma\nu' \circ \kappa_7) = 2\nu_7\kappa_{10} = 0.$$

By use of the formula (4), $J([\eta_2]_4) = \Sigma\nu'$ and $J([\iota_3]) = \nu_4$, we obtain the lemma. □

For $n \geq 4$, we will determine the group $\pi_{20}(R_{n+1})$ by applying the exact sequence

$$(20)_n \qquad \pi_{21}(S^n) \xrightarrow{\Delta} \pi_{20}(R_n) \xrightarrow{i_*} \pi_{20}(R_{n+1}) \xrightarrow{p_*} \text{Ker}\ \Delta \to 0$$

where $\text{Im}\ p_* = \text{Ker}\ \Delta$ for $\Delta : \pi_{20}(S^n) \to \pi_{19}(R_n)$.

From the exact sequence $(20)_4$, we obtain the following exact sequence;

$$0 \to \text{Coker}\ \Delta \xrightarrow{i_*} \pi_{20}(R_5) \xrightarrow{p_*} \text{Ker}\ \Delta \to 0$$

where $\text{Ker}\ \Delta \cong Z_2\{\nu_4^2\sigma_{10}\nu_{17}\}$ by Lemma 3.1.

From Table 2 of [5], $\pi_{10}(R_5) \cong Z_8\{[\nu_4^2]\}$. Obviously, $p_*([\nu_4^2]\sigma_{10}\nu_{17}) = \nu_4^2\sigma_{10}\nu_{17}$. $\Sigma(2\sigma_{10}\nu_{17}) = 2\sigma_{11}\nu_{18} = 0$ by (7.20) of [18]. Then it follows from Lemma 2.2 of [11]

$$2([\nu_4^2]\sigma_{10}\nu_{17}) = [\nu_4^2] \circ 2(\sigma_{10}\nu_{17}) = [\nu_4^2] \circ 0 = 0.$$

This shows that

$$\pi_{20}(R_5) \cong \text{Coker } \Delta \oplus \mathrm{Z}_2\{[\nu_4^2]\sigma_{10}\nu_{17}\}. \tag{13}$$

Consider the homomorphism $\Delta : \pi_{21}(S^4) \to \pi_{20}(R_4)$, where $\pi_{21}(S^4) \cong Z_8\{\nu_4\sigma'\sigma_{14}\} \oplus Z_4\{\nu_4\kappa_7\} \oplus Z_4\{E\overline{\epsilon}'\} \oplus Z_2\{\overline{\mu}_4\} \oplus Z_2\{\eta_4\mu_5\sigma_{14}\}$ by Theorem 12.7 of [18]. By use of $\Delta(\iota_4) = 2[\iota_3] - [\eta_2]_4$ and the formula (5), we obtain that

$$\Delta(\Sigma\overline{\epsilon}') = \Delta(\iota_4)\overline{\epsilon}' = (2[\iota_3] - [\eta_2]_4)\overline{\epsilon}' = 2[\iota_3]\overline{\epsilon}' - [\eta_2]_4\overline{\epsilon}',$$
$$\Delta(\overline{\mu}_4) = \Delta(\iota_4)\overline{\mu}_3 = [\eta_2]_4\overline{\mu}_3,$$
$$\Delta(\eta_4\mu_5\sigma_{14}) = \Delta(\iota_4)\eta_3\mu_4\sigma_{13} = [\eta_2]_4\eta_3\mu_4\sigma_{13}.$$

By Lemma 1.4 and by Lemma 6.6 of [18], we have

$$\begin{aligned}\Delta(\nu_4\sigma'\sigma_{14}) &= \Delta(\nu_4\sigma')\sigma_{13}\\ &= 2[\iota_3]\epsilon'\sigma_{13} + 2c[\eta_2]_4\epsilon'\sigma_{13}\\ &= [\iota_3]\eta_3^2\epsilon_5\sigma_{13} + c[\eta_2]_4\eta_3^2\epsilon_5\sigma_{13}\\ &= 0,\end{aligned}$$

since $\epsilon_5\sigma_{13} = 0$ by Lemma 10.7 of [18].

We need the following

Lemma 3.4. $\nu'\overline{\nu}_6\nu_{14}^2 = 2\overline{\epsilon'}$.

Proof. By by (7.12) and Lemma 6.3 of [18], we have

$$\nu'\overline{\nu}_6\nu_{14}^2 = \epsilon_3\nu_{11}^3 = \epsilon_3\eta_{11}\overline{\nu}_{12}.$$

By Lemma 12.3 and Lemma 12.10 of [18], we have

$$2\overline{\epsilon'} = \eta_3^2\overline{\epsilon}_5 = \eta_3\epsilon_4\overline{\nu}_{12}.$$

Since $\eta_3\epsilon_4 = \epsilon_3\eta_{11}$ by (7.5) of [18], we obtain that

$$\nu'\overline{\nu}_6\nu_{14}^2 = 2\overline{\epsilon'}.$$

□

Next we prove

Lemma 3.5. $\Delta(\nu_4\kappa_7) = d'[\iota_3]\overline{\epsilon}' + a'[\eta_2]_4\overline{\epsilon}'$,
where d' is odd integer and $a' \equiv a$ mod 2. Here $\Delta(\nu_4) = [\iota_3]\nu' + a[\eta_2]_4\nu'$ ([6]).

Proof. Set $\Delta(\nu_4\kappa_7)=a'[\eta_2]_4\overline{\epsilon}'+b'[\eta_2]_4\overline{\mu}_3+c'[\eta_2]_4\eta_3\mu_4\sigma_{13}+d'[\iota_3]\overline{\epsilon}'+e'[\iota_3]\overline{\mu}_3+f'[\iota_3]\eta_3\mu_4\sigma_{13}$. Apply the J-homomorphism to the both sides. By the diagram (3) and p.46 of [10], we have

$$J\Delta(\nu_4\kappa_7)=P(\nu_9\kappa_{12})=0.$$

Hence, from Lemma 3.3, we obtain that

$$\Delta(\nu_4\kappa_7)=a'[\eta_2]_4\overline{\epsilon}'+d'[\iota_3]\overline{\epsilon}'.$$

where a' and d' are integers such that $0\le a',d'\le 3$.

From Lemma 10.1 of [18], we have

$$2\kappa_7=\overline{\nu}_7\nu_{15}^2+x\nu_7\zeta_{10}=\Sigma(\overline{\nu}_6\nu_{14}^2+x\nu_6\zeta_9)$$

for some integer x. Then we have

$$\begin{aligned}\Delta(2(\nu_4\kappa_7))=\Delta(\nu_4\circ 2\kappa_7) &= \Delta(\nu_4)(\overline{\nu}_6\nu_{14}^2+x\nu_6\zeta_9)\\ &= ([\iota_3]+a[\eta_2]_4)(\nu'\overline{\nu}_6\nu_{14}^2+x\nu'\nu_6\zeta_9)\\ &= ([\iota_3]+a[\eta_2]_4)\nu'\overline{\nu}_6\nu_{14}^2,\end{aligned}$$

since $\nu'\nu_6\in\pi_9(S^3)=0$ by Proposition 5.11 of [18].

From Lemma 3.4, we obtain that

$$\Delta(2(\nu_4\kappa_7))=2[\iota_3]\overline{\epsilon}'+2a[\eta_2]_4\overline{\epsilon}'.$$

This completes the proof. □

It follows from (13) that

$$\pi_{20}(R_5:2)=\{[\nu_4^2]\sigma_{10}\nu_{17}\}\oplus\{[\iota_3]_5\overline{\mu}_3\}\oplus\{[\iota_3]_5\eta_3\mu_4\sigma_{13}\}\cong Z_2\oplus Z_2\oplus Z_2$$

and

$$[\iota_3]_5\overline{\epsilon}'=[\eta_2]_5\overline{\epsilon}'=0. \tag{14}$$

Consider the homomorphism $\Delta:\pi_{21}(S^5)\to\pi_{20}(R_5)$, where $\pi_{21}(S^5)\cong Z_2\{\mu_5\sigma_{14}\}\oplus Z_2\{\eta_5\overline{\epsilon}_6\}$ by Theorem 12.6 of [18]. By the formula (5) and the relation $\Delta(\iota_5)=[\iota_3]_5\eta_3$, we have

$$\Delta(\mu_5\sigma_{14})=\Delta(\iota_5)\mu_4\sigma_{13}=[\iota_3]_5\eta_3\mu_4\sigma_{13}.$$

By (14) and the relation $\eta_3^2\overline{\epsilon}_5=2\overline{\epsilon}'$ (Lemma 12.3 of [18]), we have

$$\Delta(\eta_5\overline{\epsilon}_6)=[\iota_3]_5\eta_3^2\overline{\epsilon}_5=2[\iota_3]_5\overline{\epsilon}'=0.$$

Then, from the exact sequence $(20)_5$ and Lemma 3.1, we obtain that the following sequence

$$0\to Z_2\{[\nu_4^2]\sigma_{10}\nu_{17}\}\oplus Z_2\{[\iota_3]_5\overline{\mu}_3\}\xrightarrow{i_*}\pi_{20}(R_6)\xrightarrow{p_*}Z_2\{\rho^{IV}\}\to 0.$$

is exact.

Lemma 3.6. $[\rho^{IV}]=\{[\sigma'''],4\iota_{12},4\sigma_{12}\}_1$ *and* $2[\rho^{IV}]=[\iota_3]_6\overline{\mu}_3$.

Proof. By Lemma 1.2 (1) and the fact $16\sigma_{12} = 0$, the Toda bracket $\{[\sigma'''], 4\iota_{12}, 4\sigma_{12}\}$ is defined and it is a coset of $[\sigma''']\circ\pi_{20}(S^{12})+\pi_{13}(R_6)\circ 4\sigma_{13}$. Since $\pi_{13}(R_6) \cong Z_4$ by Table 2 of [5], $\pi_{13}(R_6)\circ 4\sigma_{13} = 0$. By Lemma 1.3,

$$[\sigma''']\epsilon_{12} = \{[\nu_5], 8\iota_8, \nu_8\}_1 \circ \epsilon_{12} = -[\nu_5]\circ \Sigma\{8\iota_7, \nu_7, \epsilon_{10}\} = 0,$$

since $\Sigma\{8\iota_7, \nu_7, \epsilon_{10}\} \subset \pi_{20}(S^8) = 0$ by Theorem 7.6 of [18].

$$[\sigma''']\eta_{12} = \{[\nu_5], 8\iota_8, \nu_8\}_1 \circ \eta_{12} = -[\nu_5]\circ \Sigma\{8\iota_7, \nu_7, \eta_{10}\} = 0,$$

since $\Sigma\{8\iota_7, \nu_7, \eta_{10}\} \subset \pi_{13}(S^8) = 0$ by Proposition 5.9 of [18]. Then we have

$$[\sigma''']\overline{\nu}_{12} = [\sigma'''](\epsilon_{12} + \overline{\nu}_{12}) = [\sigma''']\eta_{12}\sigma_{13} = 0.$$

Hence we have $[\sigma''']\circ\pi_{20}(S^{12}) = 0$. It follows that the Toda bracket $\{[\sigma'''], 4\iota_{12}, 4\sigma_{12}\}_1$ consists of a single element. By (6), we have

$$p_*(\{[\sigma'''], 4\iota_{12}, 4\sigma_{12}\}_1) = \{\sigma''', 4\iota_{12}, 4\sigma_{12}\}_1 = \rho^{IV}.$$

Denote by ι_{R_5} the homotopy class of the identity map of R_5. By Lemma 1.2 (1) and the definition of $\overline{\mu}_3$ (p.136 of [18]), we have

$$\begin{aligned} 2[\rho^{IV}] &= 2\iota_{R_5}\circ[\rho^{IV}] \\ &\subset \{2[\sigma'''], 4\iota_{12}, 4\sigma_{12}\}_1 \\ &= \{[\iota_3]_6\mu_3, 4\iota_{12}, 4\sigma_{12}\}_1 \\ &\supset [\iota_3]_6\circ\{\mu_3, 4\iota_{12}, 4\sigma_{12}\}_1 \\ &\supset [\iota_3]_6\circ\{\mu_3, 2\iota_{12}, 8\sigma_{12}\}_1 \\ &\ni [\iota_3]_6\overline{\mu}_3 \end{aligned}$$

The indeterminacy of $\{2[\sigma'''], 4\iota_{12}, 4\sigma_{12}\}_1$ is $2[\sigma''']\circ\pi_{20}(S^{12}) + \pi_{13}(R_6)\circ 4\sigma_{13} = 0$. Thus we have

$$2[\rho^{IV}] = [\iota_3]_6\overline{\mu}_3.$$

□

From the above exact sequence and Lemma 3.6, we obtain that

$$\pi_{20}(R_6) = \{[\rho^{IV}]\}\oplus\{[\nu_4^2]_6\sigma_{10}\nu_{17}\} \cong Z_4\oplus Z_2.$$

From the exact sequence $(20)_6$ and Lemma 3.1, we obtain the following exact sequence;

$$0 \to \text{Coker}\ \Delta \to \pi_{20}(R_7) \to Z_2\{(\overline{\nu}_6+\epsilon_6)\nu_{14}^2\} \to 0.$$

We recall the element $[\overline{\nu}_6+\epsilon_6] \in \pi_{14}(R_7)$ given in [5]. The element $[\overline{\nu}_6+\epsilon_6]\nu_{14}^2 \in \pi_{20}(R_7)$ is of order two and satisfies $p_*([\overline{\nu}_6+\epsilon_6]\nu_{14}^2) = (\overline{\nu}_6+\epsilon_6)\nu_{14}^2$. It follows from the above sequence that

$$\pi_{20}(R_7) \cong \text{Coker}\ \Delta \oplus \mathrm{Z}_2\{[\overline{\nu}_6+\epsilon_6]\nu_{14}^2\}. \tag{15}$$

Consider the homomorphism $\Delta : \pi_{21}(S^6) \to \pi_{20}(R_6)$, where $\pi_{21}(S^6) \cong Z_4\{\rho'''\}\oplus Z_2\{\overline{\epsilon}_6\}$, by Theorem 10.5 of [18].

By the definition of ρ''' (p.103 of [18]),

$$\rho''' \in \{\sigma'', 4\iota_{13}, 4\sigma_{13}\}_1.$$

By Proposition 1.1, Lemmas 1.2 and 3.4, we have

$$\begin{array}{rcl} \Delta(\rho''') & \in & \Delta\{\sigma'', 4\iota_{13}, 4\sigma_{13}\}_1 \\ & \subset & \{\Delta(\sigma''), 4\iota_{13}, 4\sigma_{13}\} \\ & = & \{[\sigma'''], 4\iota_{13}, 4\sigma_{13}\} = [\rho^{IV}]. \end{array}$$

Consider the element $[\eta_6]\kappa_7$ of $\pi_{21}(R_7)$. From (15.3) of [10], we have $\eta_6\kappa_7 = \bar{\epsilon}_6$. Then we obtain that

$$\Delta(\bar{\epsilon}_6) = \Delta(\eta_6\kappa_7) = \Delta(p_*([\eta_6]\kappa_7) = 0.$$

That is Coker $\Delta \cong Z_2\{[\nu_4^2]_6\sigma_{10}\nu_{17}\}$.

It follows from (15) that

$$\pi_{20}(R_7) = \{[\nu_4^2]_7\sigma_{10}\nu_{17}\} \oplus \{[\overline{\nu}_6 + \epsilon_6]\nu_{14}^2\} \cong Z_2 \oplus Z_2.$$

From (2) and $\pi_{20}(S^7) \cong Z_2\{\nu_7\sigma_{10}\nu_{17}\}$ by Theorem 7.7 of [18], we have

$$\pi_{20}(R_8) = \{[\nu_4^2]_8\sigma_{10}\nu_{17}\} \oplus \{[\overline{\nu}_6 + \epsilon_6]_8\nu_{14}^2\} \oplus \{[\iota_7]\nu_7\sigma_{10}\nu_{17}\} \cong Z_2 \oplus Z_2 \oplus Z_2.$$

Conside the homomorphism $\Delta : \pi_{21}(S^8) \to \pi_{20}(R_8)$, where $\pi_{21}(S^8) \cong Z_2\{\sigma_8\nu_{15}^2\} \oplus Z_2\{\nu_8\sigma_{11}\nu_{18}\}$ by Theorem 7.7 of [18]. By Lemma 1.2 (ii) of [6] and (7.19) of [18], we have

$$\begin{array}{rcl} \Delta(\sigma_8\nu_{15}^2) & \equiv & [\iota_7]\sigma'\nu_{14}^2 + c[\overline{\nu}_6 + \epsilon_6]_8\nu_{14}^2 \mod \{[\nu_5]\nu_8^4, [\eta_5\epsilon_6]_8\nu_{14}^2\} \\ & \equiv & x[\iota_7]\nu_7\sigma_{10}\nu_{17} + c[\overline{\nu}_6 + \epsilon_6]_8\nu_{14}^2 \mod \{[\nu_5]\nu_8^4, [\eta_5\epsilon_6]_8\nu_{14}^2\} \end{array}$$

where x, c are odd integers. Since $\Delta(\iota_8) = 2[\iota_7] - [\eta_6]_8$ by Table 3 of [5], we have

$$\begin{array}{rcll} \Delta(\nu_8\sigma_{11}\nu_{18}) & = & (2[\iota_7] - [\eta_6]_8)\nu_7\sigma_{10}\nu_{17} & \\ & = & [\eta_6]_8\nu_7\sigma_{10}\nu_{17} & \\ & = & b[\nu_4^2]_8\sigma_{10}\nu_{17} & \text{by Lemma 1.1 (iv) of [6]} \end{array}$$

where b is an odd integer. That is Coker $\Delta \cong Z_2\{[\overline{\nu}_6 + \epsilon_6]_8\nu_{14}^2\}$.

In the exact sequence $(20)_8$, Ker $\Delta = 0$ by Lemma 3.1. It follows that

$$i_* : \text{Coker } \Delta \to \pi_{20}(R_9)$$

is an isomorphism where $\Delta : \pi_{21}(S^8) \to \pi_{20}(R_8)$. So we have

$$\pi_{20}(R_9) = \{[\overline{\nu}_6 + \epsilon_6]_9\nu_{14}^2\} \cong Z_2.$$

In the exact sequence $(20)_9$, Ker $\Delta \cong Z_8\{\zeta_9\}$ by Lemma 3.1 and $\pi_{21}(S^9) = 0$ by Theorem 7.6 of [18]. It follows that the sequence

$$(16) \qquad 0 \to \pi_{20}(R_9) \xrightarrow{i_*} \pi_{20}(R_{10}) \xrightarrow{p_*} Z_8\{\zeta_9\} \to 0$$

is exact. By Lemma 1 of [17], $\Sigma^{13}p_*\Delta(\zeta_{10}) = \Sigma^{13}p_*(2[\zeta_9])$ for $\Delta : \pi_{21}(S^{10}) \to \pi_{20}(R_{10})$. By Theorem 7.4 of [18], Ker $\{\Sigma^{13} : \pi_{20}(S^9) \to \pi_{33}(S^{22})\} \cong Z_2\{\bar{\nu}_9\nu_{17}\}$. When we determined the group $\pi_{19}(R_{10})$, we used the relation $\Delta(\bar{\nu}_9\nu_{17}) = [\nu_5]_9\bar{\nu}_8\nu_{16} \neq 0$. This implies the relation $p_*(\Delta(\zeta_{10}) - 2[\zeta_9]) = 0$. From the exactness of the sequence (16), we have

$$\Delta(\zeta_{10}) \equiv 2[\zeta_9] \quad \bmod\ i_*\pi_{20}(R_9). \tag{17}$$

Consider the element $[\eta_{10}^2]\mu_{12}$ of $\pi_{21}(R_{11})$. By (7.14) of [18], we have

$$p_*([\eta_{10}^2]\mu_{12}) = \eta_{10}^2\mu_{12} = 4\zeta_{10}.$$

Therefore we have

$$8[\zeta_9] = \Delta(4\zeta_{10}) = \Delta p_*([\eta_{10}^2]\mu_{12}) = 0.$$

Since the element $[\zeta_9]$ is of order 8, the above exact sequence (16) is split. So we have

$$\pi_{20}(R_{10}) = \{[\bar{\nu}_6 + \epsilon_6]_{10}\nu_{14}^2\} \oplus \{[\zeta_9]\} \cong Z_2 \oplus Z_8.$$

From the exact sequence $(20)_{10}$, (17) and Lemma 3.1, we obtain that the following sequence

$$0 \to Z_2\{[\zeta_9]_{11}\} \oplus Z_2\{[\bar{\nu}_6 + \epsilon_6]_{11}\nu_{14}^2\} \xrightarrow{i} \pi_{20}(R_{11}) \xrightarrow{p_*} Z_2\{\eta_{10}\mu_{11}\} \to 0$$

is exact.

A Toda bracket $\{[\eta_{10}^2], 2\iota_{12}, 8\sigma_{12}\}_1$ is defined by Lemma 1.2 (2) and the fact $16\sigma_{12} = 0$ and it is a coset of $[\eta_{10}^2] \circ \pi_{20}(S^{12}) + \pi_{13}(R_{11}) \circ 8\sigma_{13} = \{[\eta_{10}^2]\bar{\nu}_{12}, [\eta_{10}^2]\epsilon_{12}\}$ by Table 2 of [5]. Since $\mu_{11} \in \{\eta_{11}, 2\iota_{12}, 8\sigma_{12}\}_1 + \{\nu_{11}^3\}$ by Lemma 6.5 of [18], we have

$$\eta_{10}\mu_{11} \in \{\eta_{10}^2, 2\iota_{12}, 8\sigma_{12}\}_1,$$

where its indeterminacy is $\eta_{10}^2 \circ \pi_{20}(S^{12}) + \pi_{13}(S^{10}) \circ 8\sigma_{13} = \{\eta_{10}^2\bar{\nu}_{12}, \eta_{10}^2\epsilon_{12}\} = 0$. Then we have

$$p_*\{[\eta_{10}^2], 2\iota_{12}, 8\sigma_{12}\}_1 \subset \{\eta_{10}^2, 2\iota_{12}, 8\sigma_{12}\}_1 = \eta_{10}\mu_{11}.$$

Therefore we can take the element $[\eta_{10}\mu_{11}] \in \pi_{20}(R_{11})$ such that $[\eta_{10}\mu_{11}] \in \{[\eta_{10}^2]_{12}, 2\iota_{12}, 8\sigma_{12}\}_1$. By Corollary 3.7 of [18], we have

$$2[\eta_{10}\mu_{11}] = \{[\eta_{10}^2], 2\iota_{12}, 8\sigma_{12}\}_1 \circ 2\iota_{19} = [\eta_{10}^2] \circ \{2\iota_{12}, 8\sigma_{12}, 2\iota_{19}\}_1 \equiv 0$$

where the indeterminacy of $\{2\iota_{12}, 8\sigma_{12}, 2\iota_{19}\}$ is $2\pi_{20}(S^{12}) = 0$. It follows from the above exact sequence that

$$\pi_{20}(R_{11}) \cong Z_2\{[\bar{\nu}_6 + \epsilon_6]_{11}\nu_{14}^2\} \oplus Z_2\{[\zeta_9]_{11}\} \oplus Z_2\{[\eta_{10}\mu_{11}]\}.$$

Lemma 3.7. *For the J-homomorphism $J : \pi_{20}(R_{11}) \to \pi_{31}(S^{11})$, we have*

$$J([\overline{\nu}_6+\epsilon_6]_{11}\nu_{14}^2)=0,$$
$$J([\zeta_9]_{11})=\Sigma\beta'=\theta'\epsilon_{23},$$
$$J([\eta_{10}\mu_{11}])\equiv\beta'' \mod \{\Sigma\beta'\}$$

Proof. By the formula (4) and $J([\overline{\nu}_6+\epsilon_6])=\sigma'\sigma_{14}$, we have

$$\begin{aligned}J([\overline{\nu}_6+\epsilon_6]_{11}\nu_{14}^2) &= \Sigma^4 J([\overline{\nu}_6+\epsilon_6])\Sigma^7\nu_{14}^2\\ &= \Sigma^4(\sigma'\sigma_{14})\nu_{21}^2\\ &= 2\sigma_{11}^2\nu_{21}^2=0,\end{aligned}$$

since $2\nu_{21}^2=0$.

From Lemma 16.1 of [10], the element β' of $\pi_{30}(S^{10})$ satisfies the relations $\Sigma\beta'=\theta'\epsilon_{23}$ and $H(\beta')=\zeta_{19}$. From the diagram (3), we have

$$HJ([\zeta_9])=\zeta_{19}=H(\beta').$$

By the exactness of the EHP sequence, we have

$$J([\zeta_9])\equiv\beta' \mod \Sigma\pi_{29}(S^9)=\{\overline{\kappa}_{10}\}.$$

Since $\overline{\kappa}_{10}$ is not in the J-image, we have $J([\zeta_9])=\beta'$. By the diagram (3), we have $J([\zeta_9]_{11})=\Sigma\beta'$.

From Lemma 16.2 of [10], the element β'' of $\pi_{31}(S^{11})$ satisfies $\Sigma\beta''=P(\mu_{25})$ and $H(\beta'')=\eta_{21}\mu_{22}$. Since $HJ([\eta_{10}\mu_{11}])=\eta_{21}\mu_{22}$ and $\overline{\kappa}_{11}$ is not in the J-image, we have

$$J([\eta_{10}\mu_{11}])\equiv\beta'' \mod \Sigma\beta'.$$

□

Consider the homomorphism $\Delta:\pi_{21}(S^{11})\to\pi_{20}(R_{11})$, where $\pi_{21}(S^{11})\cong Z_2\{\sigma_{11}\nu_{18}\}\oplus Z_2\{\eta_{11}\mu_{12}\}$ by Theorem 7.3 of [18]. By use of $\Delta(\iota_{11})=[\iota_7]_{11}\nu_7$ (Lemma 1.1 of [6]) and Lemma 2.3 (v) of [6], we have

$$\Delta(\sigma_{11}\nu_{18})=\Delta(\iota_{11})\sigma_{10}\nu_{17}=[\iota_7]_{11}\nu_7\sigma_{10}\nu_{17}=d[\overline{\nu}_6+\epsilon_6]_{11}\nu_{14}^2,$$

where d is an odd integer and

$$\Delta(\eta_{11}\mu_{12})=[\iota_7]_{11}\nu_7\eta_{10}\mu_{11}=0.$$

From the exact sequence $(20)_{11}$ and Lemma 3.1, we obtain that the following sequence

$$\begin{aligned}0\to Z_2\{[\zeta_9]_{11}\}\oplus Z_2\{[\eta_{10}\mu_{11}]\}\to\pi_{20}(R_{12})\to&\\ Z_2\{\eta_{11}\overline{\nu}_{12}\}\oplus Z_2\{\eta_{11}\epsilon_{12}\}\oplus Z_2\{\mu_{11}\}\to 0&\end{aligned}$$

is exact.

Now consider the elements $[\zeta_9]_{12}, [\eta_{10}\mu_{11}]_{12}, [\eta_{11}]\overline{\nu}_{12}, [\eta_{11}]\epsilon_{12}$ and $[\mu_{11}]$ of $\pi_{20}(R_{12})$ and the J-homomorphism $J : \pi_{20}(R_{12}) \to \pi_{32}(S^{12})$. Here, by Theorem of [10],

$$\begin{aligned}\pi_{32}(S^{12}) \;\cong\; & Z_8\{\overline{\kappa}_{12}\} \oplus Z_2\{(\Sigma\theta')\epsilon_{24}\} \oplus Z_2\{\Sigma\beta''\} \oplus Z_2\{\theta\epsilon_{24}\} \oplus \\ & Z_2\{\theta\overline{\nu}_{24}\} \oplus Z_2\{\beta'''\}\end{aligned}$$

with the relations ;

$$\Sigma^2\beta''' = 8P(\sigma_{29}), \quad H(\beta''') = \mu_{23}. \tag{18}$$

by Lemma 16.3 of [10]. Then we have

Lemma 3.8. (1) $J([\zeta_9]_{12}) = (\Sigma\theta')\epsilon_{24}$,
(2) $J([\eta_{10}\mu_{11}]_{12}) \equiv \Sigma\beta'' \mod (\Sigma\theta')\epsilon_{24}$,
(3) $J([\eta_{11}]\overline{\nu}_{12}) = \theta\overline{\nu}_{24}$,
(4) $J([\eta_{11}]\epsilon_{12}) = \theta\epsilon_{24}$,
(5) $J([\mu_{11}]) \equiv \beta''' \mod \{(\Sigma\theta')\epsilon_{24}, \Sigma\beta''\}$.

Proof. From the diagram (3) and Lemma 3.7, we have (1) and (2). By Lemma 1.2 (3) and the formula (4), we have (3) and (4). By the diagram (3) and (18), we have $HJ([\mu_{11}]) = \mu_{23} = H(\beta''')$. From the exactness of the EHP sequence, we obtain that

$$J([\mu_{11}]) \equiv \beta''' \mod \Sigma\pi_{31}(S^{11}).$$

By [10], $\Sigma\pi_{31}(S^{11}) \cong Z_8\{\overline{\kappa}_{12}\} \oplus Z_2\{(\Sigma\theta')\epsilon_{24}\} \oplus Z_2\{\Sigma\beta''\}$. Since $\overline{\kappa}_{12}$ is not in the J-image, we have (5). □

From Lemma 3.8, the J-image of five elements $[\zeta_9]_{12}$, $[\eta_{10}\mu_{11}]_{12}$, $[\eta_{11}]\overline{\nu}_{12}$, $[\eta_{11}]\epsilon_{12}$ and $[\mu_{11}]$ of $\pi_{20}(R_{12})$ are independent on $\pi_{32}(S^{12})$. Therefore we have

$$\begin{aligned}\pi_{20}(R_{12}) &= \{[\zeta_9]_{12}\} \oplus \{[\eta_{10}\mu_{11}]_{12}\} \oplus \{[\eta_{11}]\overline{\nu}_{12}\} \oplus \{[\eta_{11}]\epsilon_{12}\} \oplus \{[\mu_{11}]\} \\ &\cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2\end{aligned}$$

In the exact sequence $(20)_{12}$, Ker $\Delta = 0$ by Lemma 3.1. It follows that

$$p_* : \mathrm{Coker}\ \Delta \to \pi_{20}(R_{13}) \tag{19}$$

is the isomorphism, where $\Delta : \pi_{21}(S^{12}) \to \pi_{20}(R_{12})$ and $\pi_{21}(S^{12}) \cong Z_2\{\nu_{12}^3\} \oplus Z_2\{\mu_{12}\} \oplus Z_2\{\eta_{12}\epsilon_{13}\}$ by Theorem 7.2 of [18].

Lemma 3.9. *For the homomorphism $\Delta : \pi_{21}(S^{12}) \to \pi_{20}(R_{12})$, we have*
(1) $\Delta(\eta_{12}\epsilon_{13}) = [\zeta_9]_{12}$,
(2) $\Delta(\mu_{12}) \equiv [\eta_{10}\mu_{11}]_{12} \mod [\zeta_9]_{12}$,
(3) $\Delta(\nu_{12}^3) = 0$.

Proof. By use of the diagram (3), we have

$$\begin{array}{rcll} J\Delta(\eta_{12}\epsilon_{13}) & = & P(\eta_{25}\epsilon_{26}) & \\ & = & (\Sigma\theta')\epsilon_{24} & \text{by (7.30) of [18]} \\ & = & J([\zeta_9]_{12}) & \text{by Lemma 3.8 (1).} \end{array}$$

Since the J-homomorphism $J : \pi_{20}(R_{12}) \to \pi_{32}(S^{12})$ is a monomorphism by Lemma 3.8, we obtain (1). Also we have

$$\begin{array}{rcll} J\Delta(\mu_{12}) & = & P(\mu_{25}) & \\ & = & \Sigma\beta'' & \text{by Lemma16.2 of [10]} \\ & \equiv & J([\eta_{10}\mu_{11}]) \ \bmod (\Sigma\theta')\epsilon_{24} & \text{by Lemma 3.8 (2).} \end{array}$$

Since the J-homomorphism $J : \pi_{20}(R_{12}) \to \pi_{32}(S^{12})$ is a monomorphism by Lemma 3.8, we obtain (2).

From [6], we recall the element $[\nu_{12}^2] \in \pi_{18}(R_{13})$. Then $[\nu_{12}^2]\nu_{18} \in \pi_{21}(R_{13})$ satisfies $p_*([\nu_{12}^2]\nu_{18}) = \nu_{12}^3$. So we have

$$\Delta(\nu_{12}^3) = \Delta(p_*([\nu_{12}^2]\nu_{18})) = 0.$$

□

Then, it follows from (19) that

$$\pi_{20}(R_{13}) = \{[\eta_{11}]_{13}\overline{\nu}_{12}\} \oplus \{[\eta_{11}]_{13}\epsilon_{12}\} \oplus \{[\mu_{11}]_{13}\} \cong Z_2 \oplus Z_2 \oplus Z_2$$

Consider the homomorphism $\Delta : \pi_{21}(S^{13}) \to \pi_{20}(R_{13})$, where $\pi_{21}(S^{13}) \cong Z_2\{\overline{\nu}_{13}\} \oplus Z_2\{\epsilon_{13}\}$ by Theorem 7.1 of [18]. By use of $\Delta(\iota_{13}) = [\eta_{11}]_{13}$ and the formula (5), we have

$\Delta(\overline{\nu}_{13}) = \Delta(\iota_{13})\overline{\nu}_{12} = [\eta_{11}]_{13}\overline{\nu}_{12}$,
$\Delta(\epsilon_{13}) = [\eta_{11}]_{13}\epsilon_{12}$.

Then, from the exact sequence $(20)_{13}$ and Lemma 3.1, we obtain that the sequence

$$0 \to Z_2\{[\mu_{11}]_{13}\} \xrightarrow{i_*} \pi_{20}(R_{14}) \xrightarrow{p_*} Z_8\{2\sigma_{13}\} \to 0 \tag{20}$$

is exact. From Lemma 1 of [17], we have

$$p_*\Delta(\sigma_{14}) = 2\sigma_{13} = p_*([2\iota_{13}]\sigma_{13}),$$

where $\Delta : \pi_{21}(S^{14}) \to \pi_{20}(R_{14})$. By the exactness of the sequence (20), we obtain that

$$\Delta(\sigma_{14}) \equiv [2\iota_{13}]\sigma_{13} \ \bmod [\mu_{11}]_{14}. \tag{21}$$

On the other hand, we have

$$HJ([2\iota_{13}]\sigma_{13}) = 2\sigma_{27} = HP(\sigma_{29}).$$

From the exactness of the EHP sequence, we have

$$J([2\iota_{13}]\sigma_{13}) \equiv P(\sigma_{29}) \ \bmod \Sigma\pi_{33}(S^{13}).$$

Since the order of $P(\sigma_{29})$ is 16 by Lemma 16.3 of [10], we have that the order of $[2\iota_{13}]\sigma_{13}$ is 16. Therefore, from the exact sequence (20), we have

$$\pi_{20}(R_{14}) = \{[2\iota_{13}]\sigma_{13}\} \cong Z_{16},$$

with the relations $[\mu_{11}]_{14} = 8[2\iota_{13}]\sigma_{13}$ and $\Delta(\sigma_{14}) = [2\iota_{13}]\sigma_{13}$.

Then, from (21), we have obtained that $\Delta : \pi_{21}(S^{14}) \to \pi_{20}(R_{14})$ is an isomorphism. From the exact sequence $(20)_{14}$ and Lemma 3.1, we have

$$\pi_{20}(R_{15}) = 0.$$

Since $\pi_{20}(S^{15}) = \pi_{20}(S^{16}) = \pi_{21}(S^{16}) = 0$, we have

$$\pi_{20}(R_k) = 0 \qquad \text{for} \quad k = 16, 17$$

from the exactness of the sequences $(20)_{15}$ and $(20)_{16}$ respectively.

From the exact sequence $(20)_{17}$, we easily obtain that

$$\pi_{20}(R_{18}) = \{[2\nu_{17}]\} \cong Z_4,$$

since $\mathrm{Ker}\,\Delta \cong Z_4\{2\nu_{17}\}$ by Lemma 3.1.

Consider the homomorphism $\Delta : \pi_{21}(S^{18}) \to \pi_{20}(R_{18})$, where $\pi_{21}(S^{18}) \cong Z_8\{\nu_{18}\}$. By Lemma 1 of [17], we have

$$\Delta(\nu_{18}) = [2\nu_{17}].$$

Then, from the exact sequence $(20)_{18}$ and Lemma 3.1, we easily obtain that

$$\pi_{20}(R_{19}) = \{[\eta_{18}^2]\} \cong Z_2.$$

Consider the exact sequence $(20)_{19}$. By [7], we obtain that the sequence

$$0 \to \pi_{20}(R_{19}) \to \pi_{20}(R_{20}) \to \pi_{20}(S^{19}) \to 0$$

is a split exact sequence. Therefore we have

$$\pi_{20}(R_{20}) = \{[\eta_{18}^2]_{20}\} \oplus \{[\eta_{19}]\} \cong Z_2 \oplus Z_2$$

and $J([\eta_{18}^2]) = \overline{\beta}, J([\eta_{18}^2]_{20}) = \Sigma\overline{\beta} = P(\eta_{41}), J([\eta_{19}]) \equiv \overline{\overline{\beta}} \mod \Sigma\pi_{39}(S^{19})$ by Lemma 16.4 of [10] and the diagram (3).

By Lemma 16.4 of [10] and the diagram (3), we have $\Delta(\eta_{20}) = [\eta_{18}^2]_{20}$. Then, from the exact sequence $(20)_{20}$, we obtain that

$$\pi_{20}(R_{21}) = \{[\eta_{19}]_{21}\} \cong Z_2.$$

Since $P(\iota_{43}) = \Sigma\overline{\overline{\beta}} = J([\eta_{19}]_{21})$ by Lemma 16.5 of [10] , we have

$$\Delta(\iota_{21}) = [\eta_{19}]_{21}.$$

It follows from the exact sequence $(20)_{21}$ that

$$\pi_{20}(R_{22}) = 0.$$

Additional remark in the proof.

From the fact $\zeta_{10} \in \{\eta_{10}, \eta_{11}\epsilon_{12}, 2\iota_{20}\}_1$ by Lemma 9.1 of [18] and $\Delta(\eta_{10}) = 2[\iota_7]_{10}\nu_7$, by Theorem 1 of [7], we have

$$\begin{aligned} \Delta(\zeta_{10}) &\in \{\Delta(\eta_{10}), \eta_{10}\epsilon_{11}, 2\iota_{19}\} \\ &= \{2[\iota_7]_{10}\nu_7, \eta_{10}\epsilon_{11}, 2\iota_{19}\} \\ &\subset \{2[\iota_7]_{10}, \nu_7\eta_{10}\epsilon_{11}, 2\iota_{19}\} \\ &\equiv 0 \bmod 2\pi_{20}(R_{10}) = \{2[\zeta_9]\} \end{aligned}$$

From (17), we obtain that

$$\Delta(\zeta_{10}) = 2[\zeta_9].$$

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HIDEYUKI KACHI
DEPARTMENT OF MATHEMATICAL SEIENCES
FACULTY OF SEIENCE
SHINSHU UNIVERSITY
MATUMOTO 390-8621, JAPAN

JUNO MUKAI
DEPARTMENT OF MATHEMATICAL SEIENCES
FACULTY OF SEIENCE
SHINSHU UNIVERSITY
MATUMOTO 390-8621, JAPAN

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