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CERTAIN METRICS ON $R^4{}_+$

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Abstract. We studied the pseudo-Riemannian metric

$$ds^2 = \frac{1}{x_4x_4}\Big\{\sum\nolimits_{b,c=1}^{3}\Big(\delta_{bc} - \frac{ax_bx_c}{1+ar^2}\Big)dx_bdx_c - \frac{1}{1+ax_4x_4}dx_4dx_4\Big\}$$

on $R^3 \times R_+$, where $r^2 = \sum_{b=1}^{3} x_bx_b$ and $a =$ constant, which satisfies the Einstein condition (in [1], [2], [3]). The purpose of this work is to find Einstein metrics including the above metric as a special one, which is analogous to the relation between the Schwarzschild metric and the Kerr one in the theory of relativity.

1. Preliminaries and Curvature Tensor

Using the polar coordinates (r, ϑ, φ) of R^3 :

$$x_1 = r\sin\vartheta\cos\varphi, \quad x_2 = r\sin\vartheta\sin\varphi, \quad x_3 = r\cos\vartheta,$$

and setting $x_4 = t$ the above metric can be written as

$$ds^2 = \frac{1}{t^2}\Big\{\frac{1}{1+ar^2}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) - \frac{1}{1+at^2}dt^2\Big\}. \tag{1.1}$$

Now setting the formal coordinates (u_i) as

$$u_1 = r, \quad u_2 = \vartheta, \quad u_3 = \varphi, \quad u_4 = t$$

we consider a pseudo-Riemannian metric

$$ds^2 = \sum\nolimits_{i,j=1}^{4} g_{ij}du_idu_j, \quad g_{ij} = g_{ji} = \frac{1}{u_4u_4}F_{ij} \tag{1.2}$$

with

$$F_{\alpha\beta} = F_{\alpha\beta}(u_1, u_2), \quad F_{\lambda\mu} = F_{\lambda\mu}(u_1, u_2, u_4), \quad F_{12} = F_{\alpha\lambda} = 0, \tag{1.3}$$

where we set $\alpha, \beta, \gamma, \cdots = 1, 2$ and $\lambda, \mu, \nu, \cdots = 3, 4$ in the paper. The Christoffel symbols made by g_{ij} :

$$\{{}_j{}^i{}_h\} = \frac{1}{2}\sum_k g^{ik}\Big(\frac{\partial g_{jk}}{\partial u_h} + \frac{\partial g_{kh}}{\partial u_j} - \frac{\partial g_{jh}}{\partial u_k}\Big), \quad (g^{ij}) = (g_{ij})^{-1},$$

are given exactly as follows :

$$\{{}_\beta{}^\alpha{}_\gamma\} = \frac{1}{2}F^{\alpha\alpha}\Big(\delta^\alpha_\beta\frac{\partial F_{\alpha\alpha}}{\partial u_\gamma} + \delta^\alpha_\gamma\frac{\partial F_{\alpha\alpha}}{\partial u_\beta} - \delta_{\beta\gamma}\frac{\partial F_{\beta\beta}}{\partial u_\alpha}\Big), \quad \{{}_\beta{}^\lambda{}_\gamma\} = \frac{1}{u_4}F^{4\lambda}F_{\beta\gamma},$$

$$(1.4)\quad \{{}_\beta{}^\alpha{}_\mu\} = -\frac{1}{u_4}\delta^\alpha_\beta\delta_{4\mu}, \quad \{{}_\beta{}^\lambda{}_\mu\} = \frac{1}{2}\sum_\nu F^{\lambda\nu}\frac{\partial F_{\nu\mu}}{\partial u_\beta}, \quad (F^{ij}) = (F_{ij})^{-1}$$

$$\{{}_\mu{}^\alpha{}_\nu\} = -\frac{1}{2}F^{\alpha\alpha}\frac{\partial F_{\mu\nu}}{\partial u_\alpha}, \quad \{{}_\mu{}^\lambda{}_\nu\} = \{{}_\mu{}^\lambda{}_\nu\}_\Lambda - \frac{1}{u_4}(\delta^\lambda_\mu\delta_{4\nu} + \delta^\lambda_\nu\delta_{4\mu} - F^{4\lambda}F_{\mu\nu}),$$

where

$$\{{}_3{}^\lambda{}_3\}_\Lambda = -\frac{1}{2}F^{\lambda 4}\frac{\partial F_{33}}{\partial u_4}, \quad \{{}_3{}^\lambda{}_4\}_\Lambda = \frac{1}{2}F^{\lambda 3}\frac{\partial F_{33}}{\partial u_4},$$

$$\{{}_4{}^\lambda{}_4\}_\Lambda = \frac{1}{2}F^{\lambda 4}\frac{\partial F_{44}}{\partial u_4} + F^{\lambda 3}\frac{\partial F_{34}}{\partial u_4}$$

and $\{{}_\beta{}^\alpha{}_\gamma\}$ becomes exacty as

$$\{{}_1{}^1{}_1\} = \frac{1}{2}F^{11}\frac{\partial F_{11}}{\partial u_1}, \quad \{{}_1{}^2{}_1\} = -\frac{1}{2}F^{22}\frac{\partial F_{11}}{\partial u_2}, \quad \{{}_1{}^1{}_2\} = \frac{1}{2}F^{11}\frac{\partial F_{11}}{\partial u_2},$$

$$\{{}_2{}^2{}_2\} = \frac{1}{2}F^{22}\frac{\partial F_{22}}{\partial u_2}, \quad \{{}_2{}^1{}_2\} = -\frac{1}{2}F^{11}\frac{\partial F_{22}}{\partial u_1}, \quad \{{}_1{}^2{}_2\} = \frac{1}{2}F^{22}\frac{\partial F_{22}}{\partial u_1}.$$

Using these expressions, the components of the curvature tensor of the metric g_{ij} :

$$R_j{}^i{}_{hk} = \frac{\partial\{{}_j{}^i{}_k\}}{\partial u_h} - \frac{\partial\{{}_j{}^i{}_h\}}{\partial u_k} + \sum_\ell\{{}_\ell{}^i{}_h\}\{{}_j{}^\ell{}_k\} - \sum_\ell\{{}_\ell{}^i{}_k\}\{{}_j{}^\ell{}_h\}$$

are given as follows :

$$R_\beta{}^\alpha{}_{12} = \frac{1}{2}F^{\alpha\alpha}\Big\{-\delta^\alpha_1\frac{\partial^2 F_{\alpha\alpha}}{\partial u_\beta\partial u_2} + \delta^\alpha_2\frac{\partial^2 F_{\alpha\alpha}}{\partial u_\beta\partial u_1} + \delta_{\beta 1}\frac{\partial^2 F_{\beta\beta}}{\partial u_\alpha\partial u_2} - \delta_{\beta 2}\frac{\partial^2 F_{\beta\beta}}{\partial u_\alpha\partial u_1}\Big\}$$

$$+\frac{1}{4}\delta^\alpha_1 F^{\alpha\alpha}\Big\{\frac{\partial F_{\alpha\alpha}}{\partial u_\beta}\frac{\partial\log(F_{\alpha\alpha}F_{\beta\beta})}{\partial u_2} + \frac{F_{\alpha\alpha}}{\partial u_2}\frac{\partial\log F_{22}}{\partial u_\beta} - \delta_{\beta 2}\sum_\gamma F^{\gamma\gamma}\frac{\partial F_{\alpha\alpha}}{\partial u_\gamma}\frac{\partial F_{\beta\beta}}{\partial u_\gamma}$$

$$(1.5)\quad -\frac{4}{u_4u_4}F_{\alpha\alpha}F^{44}F_{\beta 2}\Big\} - \frac{1}{4}\delta^\alpha_2 F^{\alpha\alpha}\Big\{\frac{\partial F_{\alpha\alpha}}{\partial u_\beta}\frac{\partial\log(F_{\alpha\alpha}F_{\beta\beta})}{\partial u_1} + \frac{\partial F_{\alpha\alpha}}{\partial u_1}\frac{\partial\log F_{11}}{\partial u_\beta}$$

$$-\delta_{\beta 1}\sum_\gamma F^{\gamma\gamma}\frac{\partial F_{\alpha\alpha}}{\partial u_\gamma}\frac{\partial F_{\beta\beta}}{\partial u_\gamma} - \frac{4}{u_4u_4}F_{\alpha\alpha}F^{44}F_{\beta 1}\Big\}$$

$$+\frac{1}{4}\delta_{\beta 2}F^{\alpha\alpha}\Big\{\frac{\partial\log F_{\alpha\alpha}}{\partial u_1}\frac{\partial F_{\beta\beta}}{\partial u_\alpha} + \frac{\partial\log F_{11}}{\partial u_\alpha}\frac{\partial F_{\beta\beta}}{\partial u_1} + \frac{\partial\log F_{22}}{\partial u_\alpha}\frac{\partial F_{22}}{\partial u_1}\Big\}$$

$$-\frac{1}{4}\delta_{\beta 1}F^{\alpha\alpha}\Big\{\frac{\partial\log F_{\alpha\alpha}}{\partial u_2}\frac{\partial F_{\beta\beta}}{\partial u_\alpha} + \frac{\partial\log F_{22}}{\partial u_\alpha}\frac{\partial F_{\beta\beta}}{\partial u_2} + \frac{\partial\log F_{11}}{\partial u_\alpha}\frac{\partial F_{11}}{\partial u_2}\Big\},$$

$$R_\beta{}^\lambda{}_{12} = -\frac{1}{u_4}\delta_{\beta 1}\Big\{\frac{\partial}{\partial u_2}(F^{\lambda 4}F_{11}) + \frac{1}{2}F_{11}\sum_{\nu,\mu}F^{\lambda\nu}F^{\mu 4}\frac{\partial F_{\nu\mu}}{\partial u_2}$$

$$-\frac{1}{2}F_{11}F^{\lambda 4}F^{\beta\beta}\frac{\partial F_{\beta\beta}}{\partial u_2}-\frac{1}{2}F^{\lambda 4}\frac{\partial F_{\beta\beta}}{\partial u_2}\Big\}+\frac{1}{u_4}\delta_{\beta 2}\Big\{\frac{\partial}{\partial u_1}(F^{\lambda 4}F_{22}) \tag{1.6}$$

$$+\frac{1}{2}F_{22}\sum_{\nu,\mu}F^{\lambda\nu}F^{\mu 4}\frac{\partial F_{\nu\mu}}{\partial u_1}-\frac{1}{2}F_{22}F^{\lambda 4}F^{\beta\beta}\frac{\partial F_{\beta\beta}}{\partial u_1}-\frac{1}{2}F^{\lambda 4}\frac{\partial F_{\beta\beta}}{\partial u_1}\Big\},$$

and
(1.7)

$$R_\lambda{}^\alpha{}_{12}=-\frac{1}{2u_4}\Big\{F^{34}\Big(\delta_1^\alpha\frac{\partial F_{3\lambda}}{\partial u_2}-\delta_2^\alpha\frac{\partial F_{3\lambda}}{\partial u_1}\Big)+F^{44}\Big(\delta_1^\alpha\frac{\partial F_{4\lambda}}{\partial u_2}-\delta_2^\alpha\frac{\partial F_{4\lambda}}{\partial u_1}\Big)\Big\},$$

and

$$R_\lambda{}^\mu{}_{12}=\frac{1}{4}\sum_\nu\Big(\frac{\partial F^{\mu\nu}}{\partial u_1}\frac{\partial F_{\nu\lambda}}{\partial u_2}-\frac{\partial F^{\mu\nu}}{\partial u_2}\frac{\partial F_{\nu\lambda}}{\partial u_1}\Big). \tag{1.8}$$

Then

$$R_\beta{}^\alpha{}_{\nu\gamma}=\frac{1}{2u_4}\Big(\delta_\gamma^\alpha\sum_\sigma F^{4\sigma}\frac{\partial F_{\sigma\nu}}{\partial u_\beta}-F^{\alpha\alpha}F_{\beta\gamma}\sum_\sigma F^{4\sigma}\frac{\partial F_{\sigma\nu}}{\partial u_\alpha}\Big), \tag{1.9}$$

$$R_\beta{}^\lambda{}_{\nu\gamma}=\frac{1}{2u_4}\Big(\frac{\partial F^{\lambda 4}}{\partial u_4}\delta_\nu^4+\frac{1}{\Delta}\varepsilon^{\lambda 4}\frac{\partial F_{3\nu}}{\partial u_4}-\frac{2}{u_4}F^{44}\delta_\nu^\lambda\Big)F_{\beta\gamma}$$

$$-\frac{1}{2}\sum_\sigma F^{\lambda\sigma}\frac{\partial^2 F_{\sigma\nu}}{\partial u_\beta\partial u_\gamma}-\frac{1}{4}\sum_\sigma\frac{\partial F^{\lambda\sigma}}{\partial u_\gamma}\frac{\partial F_{\sigma\nu}}{\partial u_\beta} \tag{1.10}$$

$$+\frac{1}{4}\sum_\sigma F^{\lambda\sigma}\Big(\frac{\partial F_{\sigma\nu}}{\partial u_\beta}F^{\beta\beta}\frac{\partial F_{\beta\beta}}{\partial u_\gamma}+\frac{\partial F_{\sigma\nu}}{\partial u_\gamma}F^{\gamma\gamma}\frac{\partial F_{\gamma\gamma}}{\partial u_\beta}-\sum_\alpha\frac{\partial F_{\nu\sigma}}{\partial u_\sigma}F^{\alpha\alpha}\frac{\partial F_{\beta\beta}}{\partial u_\alpha}\delta_{\beta\gamma}\Big),$$

where

$$\Delta=F_{33}F_{44}-F_{34}F_{34},\quad \varepsilon^{33}=\varepsilon^{44}=0,\quad \varepsilon^{34}=-\varepsilon^{43}=1,$$

and

$$R_\lambda{}^\alpha{}_{\nu\gamma}=\delta_\gamma^\alpha\Big\{\frac{1}{4}F^{\alpha\alpha}\Big(\frac{\partial F_{\alpha\alpha}}{\partial u_1}F^{11}\frac{\partial F_{\lambda\nu}}{\partial u_1}+\frac{\partial F_{\alpha\alpha}}{\partial u_2}F^{22}\frac{\partial F_{\lambda\nu}}{\partial u_2}\Big)$$

$$+\frac{1}{2u_4}F^{34}\Big(\frac{\partial F_{\lambda 3}}{\partial u_\nu}+\frac{\partial F_{3\nu}}{\partial u_\lambda}\Big)+\frac{1}{2u_4}F^{44}\Big(\frac{\partial F_{\lambda 4}}{\partial u_\nu}+\frac{\partial F_{4\nu}}{\partial u_\lambda}-\frac{\partial F_{\lambda\nu}}{\partial u_4}\Big) \tag{1.11}$$

$$+\frac{1}{u_4u_4}F^{44}F_{\lambda\nu}\Big\}+\frac{1}{4}\frac{\partial F^{\alpha\alpha}}{\partial u_\gamma}\frac{\partial F_{\lambda\nu}}{\partial u_\alpha}+\frac{1}{2}F^{\alpha\alpha}\frac{\partial^2 F_{\lambda\nu}}{\partial u_\gamma\partial u_\alpha}$$

$$-\frac{1}{4}F^{\alpha\alpha}\sum_{\rho,\sigma}F^{\rho\sigma}\frac{\partial F_{\rho\nu}}{\partial u_\alpha}\frac{\partial F_{\sigma\lambda}}{\partial u_\gamma}-\frac{1}{4}F^{\alpha\alpha}F^{\gamma\gamma}\frac{\partial F_{\gamma\gamma}}{\partial u_\alpha}\frac{\partial F_{\lambda\nu}}{\partial u_\gamma},$$

and

$$R_\lambda{}^\mu{}_{\nu\gamma}=-\frac{\partial}{\partial u_\gamma}\{\lambda{}^\mu{}_\nu\}_\Lambda-\frac{1}{2}\sum_{\rho,\sigma}\{\rho{}^\sigma{}_\nu\}_\Lambda F^{\mu\rho}\frac{\partial F_{\sigma\lambda}}{\partial u_\gamma}-\frac{1}{2}\sum_{\rho,\sigma}\{\lambda{}^\rho{}_\nu\}_\Lambda F^{\mu\sigma}\frac{\partial F_{\sigma\rho}}{\partial u_\gamma}$$

$$+\frac{1}{2}\sum_{\sigma} F^{\mu\sigma}\frac{\partial^2 F_{\sigma\lambda}}{\partial u_\nu \partial u_\gamma} - \frac{1}{2u_4}\Big(\frac{\partial F^{4\mu}}{\partial u_\gamma}F_{\lambda\nu} + F^{4\mu}\frac{\partial F_{\lambda\nu}}{\partial u_\gamma}\Big) \tag{1.12}$$

$$-\frac{1}{2u_4}\delta^\mu_\nu \sum_{\sigma} F^{4\sigma}\frac{\partial F_{\sigma\lambda}}{\partial u_\gamma} + \frac{1}{2u_4}F^{4\mu}\frac{\partial F_{\lambda\nu}}{\partial u_\gamma}.$$

Last

$$R_\beta{}^\alpha{}_{34} = \frac{1}{4}F^{\alpha\alpha}\sum_{\rho,\sigma} F^{\rho\sigma}\Big(\frac{\partial F_{\rho 4}}{\partial u_\alpha}\frac{\partial F_{3\sigma}}{\partial u_\beta} - \frac{\partial F_{\rho 3}}{\partial u_\alpha}\frac{\partial F_{4\sigma}}{\partial u_\beta}\Big), \tag{1.13}$$

$$R_\beta{}^\lambda{}_{34} = -\frac{1}{2}\sum_{\sigma}\frac{\partial F^{\lambda\sigma}}{\partial u_4}\frac{\partial F_{\sigma 3}}{\partial u_\beta} - \frac{1}{2}\sum_{\sigma} F^{\lambda\sigma}\frac{\partial^2 F_{\sigma 3}}{\partial u_4 \partial u_\beta}$$

$$+\frac{1}{2}\sum_{\rho,\sigma} F^{\rho\sigma}\Big(\{{}_\rho{}^\lambda{}_3\}_\Lambda \frac{\partial F_{\sigma 4}}{\partial u_\beta} - \{{}_\rho{}^\lambda{}_4\}_\Lambda \frac{\partial F_{\sigma 3}}{\partial u_\beta}\Big) \tag{1.14}$$

$$-\frac{1}{2u_4}\sum_{\sigma} F^{4\sigma}\Big(\delta^\lambda_3 \frac{\partial F_{\sigma 4}}{\partial u_\beta} - \delta^\lambda_4 \frac{\partial F_{\sigma 3}}{\partial u_\beta}\Big),$$

and

$$R_\lambda{}^\alpha{}_{34} = \frac{1}{2}F^{\alpha\alpha}\frac{\partial^2 F_{\lambda 3}}{\partial u_\alpha \partial u_4} - \frac{1}{2}F^{\alpha\alpha}\sum_{\rho}\Big(\frac{\partial F_{\rho 3}}{\partial u_\alpha}\{{}_\lambda{}^\rho{}_4\}_\Lambda - \frac{\partial F_{\rho 4}}{\partial u_\alpha}\{{}_\lambda{}^\rho{}_3\}_\Lambda\Big) \tag{1.15}$$

$$-\frac{1}{2u_4}F^{\alpha\alpha}\sum_{\rho} F^{4\rho}\Big(\frac{\partial F_{\rho 3}}{\partial u_\alpha}F_{\lambda 4} - \frac{\partial F_{\rho 4}}{\partial u_\alpha}F_{\lambda 3}\Big),$$

and

$$R_\lambda{}^\mu{}_{34} = -\frac{\partial\{{}_\lambda{}^\mu{}_3\}_\Lambda}{\partial u_4} + \sum_{\sigma}(\{{}_\sigma{}^\mu{}_3\}_\Lambda\{{}_\lambda{}^\sigma{}_4\}_\Lambda - \{{}_\sigma{}^\mu{}_4\}_\Lambda\{{}_\lambda{}^\sigma{}_3\}_\Lambda)$$

$$-\frac{1}{u_4}\delta^\mu_3\Big(\{{}_\lambda{}^4{}_4\}_\Lambda + \frac{1}{u_4}F^{44}F_{\lambda 4}\Big) + \frac{1}{u_4}\delta^\mu_4\Big(\{{}_\lambda{}^4{}_3\}_\Lambda + \frac{1}{u_4}F^{44}F_{\lambda 3}\Big) \tag{1.16}$$

$$+\frac{1}{u_4}\sum_{\sigma} F^{\mu\sigma}(F_{\lambda 3}\{{}_\sigma{}^4{}_4\}_\Lambda - F_{\lambda 4}\{{}_\sigma{}^4{}_3\}_\Lambda) + \frac{1}{4}\sum_{\alpha,\sigma} F^{\alpha\alpha}F^{\mu\sigma}$$

$$\times\Big(\frac{\partial F_{\lambda 3}}{\partial u_\alpha}\frac{\partial F_{\sigma 4}}{\partial u_\alpha} - \frac{\partial F_{\lambda 4}}{\partial u_\alpha}\frac{\partial F_{\sigma 3}}{\partial u_\alpha}\Big).$$

2. Ricci Tensor

From (1.5)$\sim$ (1.16) we compute the components of the Ricci tensor $R_{ij} = \sum_{k=1}^4 R_i{}^k{}_{kj}$ of the metric (1.2) and they are written as follows,in dividing three groups. First,

$$R_{\beta\gamma} = \sum_j R_\beta{}^j{}_{j\gamma} = R_\beta{}^1{}_{12}\delta_{2\gamma} - R_\beta{}^2{}_{12}\delta_{1\gamma} + R_\beta{}^3{}_{3\gamma} + R_\beta{}^4{}_{4\gamma}$$

becomes

$$\begin{aligned}
R_{11} = &-\frac{1}{2}F^{22}\Big(\frac{\partial^2 F_{22}}{\partial u_1 \partial u_1} + \frac{\partial^2 F_{11}}{\partial u_2 \partial u_2}\Big) + \frac{1}{4}F^{22}\frac{\partial F_{22}}{\partial u_1}\frac{\partial \log(F_{11}F_{22})}{\partial u_1} \\
&+\frac{1}{4}F^{22}\frac{\partial F_{11}}{\partial u_2}\frac{\partial \log(F_{11}F_{22})}{\partial u_2} - \frac{1}{2}\sum_{\mu,\nu} F^{\mu\nu}\frac{\partial^2 F_{\mu\nu}}{\partial u_1 \partial u_1} - \frac{1}{4}\sum_{\mu,\nu}\frac{\partial F^{\mu\nu}}{\partial u_1}\frac{\partial F_{\mu\nu}}{\partial u_1} \\
&+\frac{1}{4}\sum_{\mu\nu} F^{\mu\nu}\Big(\frac{\partial F_{\mu\nu}}{\partial u_1}F^{11}\frac{\partial F_{11}}{\partial u_1} - \frac{\partial F_{\mu\nu}}{\partial u_2}F^{22}\frac{\partial F_{11}}{\partial u_2}\Big) \\
&+\frac{1}{2u_4}\Big(\frac{\partial F^{44}}{\partial u_4} + \frac{1}{\Delta}\frac{\partial F_{33}}{\partial u_4} - \frac{6}{u_4}F^{44}\Big)F_{11},
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
R_{22} = &-\frac{1}{2}F^{11}\Big(\frac{\partial^2 F_{11}}{\partial u_2 \partial u_2} + \frac{\partial^2 F_{22}}{\partial u_1 \partial u_1}\Big) + \frac{1}{4}F^{11}\frac{\partial F_{11}}{\partial u_2}\frac{\partial \log(F_{11}F_{22})}{\partial u_2} \\
&+\frac{1}{4}F^{11}\frac{\partial F_{22}}{\partial u_1}\frac{\partial \log(F_{11}F_{22})}{\partial u_1} - \frac{1}{2}\sum_{\mu,\nu} F^{\mu\nu}\frac{\partial^2 F_{\mu\nu}}{\partial u_2 \partial u_2} - \frac{1}{4}\sum_{\mu,\nu}\frac{\partial F^{\mu\nu}}{\partial u_2}\frac{\partial F_{\mu\nu}}{\partial u_2} \\
&+\frac{1}{4}\sum_{\mu,\nu} F^{\mu\nu}\Big(\frac{\partial F_{\mu\nu}}{\partial u_2}F^{22}\frac{\partial F_{22}}{\partial u_2} - \frac{\partial F_{\mu\nu}}{\partial u_1}F^{11}\frac{\partial F_{22}}{\partial u_1}\Big) \\
&+\frac{1}{2u_4}\Big(\frac{\partial F^{44}}{\partial u_4} + \frac{1}{\Delta}\frac{\partial F_{33}}{\partial u_4} - \frac{6}{u_4}F^{44}\Big)F_{22},
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
R_{12} = &-\frac{1}{2}\sum_{\mu,\nu} F^{\mu\nu}\frac{\partial^2 F_{\mu\nu}}{\partial u_1 \partial u_2} - \frac{1}{4}\sum_{\mu,\nu}\frac{\partial F^{\mu\nu}}{\partial u_2}\frac{\partial F_{\mu\nu}}{\partial u_1} \\
&+\frac{1}{4}\sum_{\mu,\nu} F^{\mu\nu}\Big(\frac{\partial F_{\mu\nu}}{\partial u_1}\frac{\partial \log F_{11}}{\partial u_2} + \frac{\partial F_{\mu\nu}}{\partial u_2}\frac{\partial \log F_{22}}{\partial u_1}\Big).
\end{aligned} \tag{2.3}$$

Second,

$$R_{\lambda\mu} = \sum_\alpha R_\lambda{}^\alpha{}_{\alpha\mu} + \sum_\sigma R_\lambda{}^\sigma{}_{\sigma\mu} = -\sum_\alpha R_\lambda{}^\alpha{}_{\mu\alpha} + R_\lambda{}^3{}_{34}\delta_{4\mu} - R_\lambda{}^4{}_{34}\delta_{3\mu}$$

becomes

$$R_{\lambda\mu} = -\frac{1}{4}\sum_{\alpha,\beta}\frac{\partial \log F_{\alpha\alpha}}{\partial u_\beta}F^{\beta\beta}\frac{\partial F_{\lambda\mu}}{\partial u_\beta} - \frac{1}{u_4}\sum_{\sigma}F^{4\sigma}\Big(\frac{\partial F_{\lambda\sigma}}{\partial u_\mu} + \frac{\partial F_{\sigma\mu}}{\partial u_\lambda} - \frac{\partial F_{\lambda\mu}}{\partial u_\sigma}\Big)$$

$$-\frac{1}{4}\sum_{\alpha}\frac{\partial F^{\alpha\alpha}}{\partial u_\alpha}\frac{\partial F_{\lambda\mu}}{\partial u_\alpha} - \frac{1}{2}\sum_{\alpha}F^{\alpha\alpha}\frac{\partial^2 F_{\lambda\mu}}{\partial u_\alpha \partial u_\alpha} + \frac{1}{4}\sum_{\alpha,\rho,\sigma}F^{\alpha\alpha}F^{\rho\sigma}\frac{\partial F_{\rho\mu}}{\partial u_\alpha}\frac{\partial F_{\sigma\lambda}}{\partial u_\alpha}$$

$$+\frac{1}{4}\sum_{\alpha}F^{\alpha\alpha}\frac{\partial \log F_{\alpha\alpha}}{\partial u_\alpha}\frac{\partial F_{\lambda\mu}}{\partial u_\alpha} + \delta_{4\mu}\Big[-\frac{\partial\{\lambda{}^3{}_3\}_\Lambda}{\partial u_4} + \sum_{\sigma}\{\sigma{}^3{}_3\}_\Lambda\{\lambda{}^\sigma{}_4\}_\Lambda$$

$$(2.4) \qquad -\sum_{\sigma}\{\sigma{}^3{}_4\}_\Lambda\{\lambda{}^\sigma{}_3\}_\Lambda - \frac{1}{u_4}\{\lambda{}^4{}_4\}_\Lambda + \frac{1}{4}\Big(\sum_{\alpha,\sigma}F^{\alpha\alpha}\frac{\partial F_{\lambda 3}}{\partial u_\alpha}F^{3\sigma}\frac{\partial F_{\sigma 4}}{\partial u_\alpha}$$

$$-\sum_{\alpha,\sigma}F^{\alpha\alpha}\frac{\partial F_{\lambda 4}}{\partial u_\alpha}F^{3\sigma}\frac{\partial F_{\sigma 3}}{\partial u_\alpha}\Big)\Big] - \delta_{3\mu}\Big[-\frac{\partial\{\lambda{}^4{}_3\}_\Lambda}{\partial u_4} + \sum_{\sigma}\{\sigma{}^4{}_3\}_\Lambda\{\lambda{}^\sigma{}_4\}_\Lambda$$

$$-\sum_{\sigma}\{\sigma{}^4{}_4\}_\Lambda\{\lambda{}^\sigma{}_3\}_\Lambda + \frac{1}{u_4}\{\lambda{}^4{}_3\}_\Lambda + \frac{1}{4}\Big(\sum_{\alpha,\sigma}F^{\alpha\alpha}\frac{\partial F_{\lambda 3}}{\partial u_\alpha}F^{4\sigma}\frac{\partial F_{\sigma 4}}{\partial u_\alpha}$$

$$-F^{\alpha\alpha}\frac{\partial F_{\lambda 4}}{\partial u_\alpha}F^{4\sigma}\frac{\partial F_{\sigma 3}}{\partial u_\alpha}\Big)\Big] + \Big(-\frac{3}{u_4 u_4}F^{44} + \frac{1}{2u_4\Delta}\frac{\partial F_{33}}{\partial u_4}\Big)F_{\lambda\mu}$$

$$-\frac{1}{2u_4\Delta}F_{\lambda 3}F_{\mu 3}\sum_{\rho,\sigma}F^{\rho\sigma}\frac{\partial F_{\rho\sigma}}{\partial u_4}.$$

Third,

$$R_{\alpha\lambda} = R_\alpha{}^1{}_{1\lambda} + R_\alpha{}^2{}_{2\lambda} + R_\alpha{}^3{}_{34}\delta_{4\lambda} - R_\alpha{}^4{}_{34}\delta_{3\lambda}$$

becomes

$$R_{\alpha\lambda} = -\frac{1}{u_4}\sum_{\sigma}F^{4\sigma}\frac{\partial F_{\sigma\lambda}}{\partial u_\alpha} - \frac{1}{2}\frac{\partial}{\partial u_4}\Big(\frac{1}{\Delta}F_{4\delta}\frac{\partial F_{33}}{\partial u_\alpha}\Big) + \frac{1}{2}\frac{\partial}{\partial u_4}\Big(\frac{1}{\Delta}F_{3\lambda}\frac{\partial F_{34}}{\partial u_\alpha}\Big)$$

$$(2.5) \qquad +\frac{1}{4\Delta}\frac{\partial F_{33}}{\partial u_\lambda}\frac{\partial F_{4\lambda}}{\partial u_\alpha} + \frac{1}{4\Delta^2}\Big\{F_{4\lambda}\frac{\partial F_{33}}{\partial u_4}\Big(-F_{44}\frac{\partial F_{33}}{\partial u_\alpha} + F_{34}\frac{\partial F_{34}}{\partial u_\alpha}\Big)$$

$$+\Big(F_{3\lambda}\frac{\partial F_{44}}{\partial u_4} - 2F_{4\lambda}\frac{\partial F_{34}}{\partial u_4}\Big)\Big(F_{33}\frac{\partial F_{43}}{\partial u_\alpha} - F_{34}\frac{\partial F_{33}}{\partial u_\alpha}\Big)\Big\}.$$

These expressions of R_{ij} are computed under the condition (1.3) for F_{ij} and here we set further restrictions as

$$(2.6) \qquad \frac{\partial F_{33}}{\partial u_4} = \frac{\partial F_{34}}{\partial u_4} = 0.$$

In fact, we consider F_{44} depends on u_1, u_2, u_4 and other F_{ij} depend only u_1, u_2. Then, some of $\{{}_j{}^i{}_h\}$ given by (1.4) becomes as

$$\{{}_3{}^\lambda{}_3\}_\Lambda = \{{}_3{}^\lambda{}_4\}_\Lambda = 0, \quad \{{}_4{}^\lambda{}_4\}_\Lambda = \frac{1}{2}F^{\lambda 4}\frac{\partial F_{44}}{\partial u_4}. \tag{2.7}$$

Since we have

$$\sum_\sigma F^{4\sigma}\frac{\partial F_{\sigma\lambda}}{\partial u_\alpha} = -\sum_\sigma \frac{\partial F^{4\sigma}}{\partial u_\alpha}F_{\sigma\lambda} = \frac{\partial}{\partial u_\alpha}\Big(\frac{F_{34}}{\Delta}\Big)F_{3\lambda} - \frac{\partial}{\partial u_\alpha}\Big(\frac{F_{33}}{\Delta}\Big)F_{4\lambda}$$
$$= \frac{1}{\Delta}\Big(F_{3\lambda}\frac{\partial F_{34}}{\partial u_\alpha} - F_{4\lambda}\frac{\partial F_{33}}{\partial u_\alpha}\Big) + \frac{1}{\Delta}\delta_{4\lambda}\frac{\partial \Delta}{\partial u_\alpha},$$

(2.5) is turned into the expression

$$R_{\alpha\lambda} = -\frac{1}{u_4\Delta}\Big(F_{3\lambda}\frac{\partial F_{34}}{\partial u_\alpha} - F_{4\lambda}\frac{\partial F_{33}}{\partial u_\alpha}\Big) + \frac{1}{2}\frac{\partial}{\partial u_4}\Big(\frac{1}{\Delta}\Big(F_{3\lambda}\frac{\partial F_{34}}{\partial u_\alpha} - F_{4\lambda}\frac{\partial F_{33}}{\partial u_\alpha}\Big)\Big) - \frac{1}{u_4\Delta}\delta_{4\lambda}\frac{\partial \Delta}{\partial u_\alpha} + \frac{1}{4\Delta^2}F_{3\lambda}\frac{\partial F_{44}}{\partial u_4}\Big(F_{33}\frac{\partial F_{34}}{\partial u_\alpha} - F_{34}\frac{\partial F_{33}}{\partial u_\alpha}\Big) \tag{2.8}$$

by (2.6).

Now, denoting

$$Y_\alpha = \frac{1}{\Delta}\Big(F_{33}\frac{\partial F_{34}}{\partial u_\alpha} - F_{34}\frac{\partial F_{33}}{\partial u_\alpha}\Big), \quad Z_\alpha = \frac{1}{\Delta}\Big(F_{34}\frac{\partial F_{34}}{\partial u_\alpha} - F_{44}\frac{\partial F_{33}}{\partial u_\alpha}\Big), \tag{2.9}$$

(2.8) is written as

$$\begin{aligned} R_{\alpha 3} &= -\frac{1}{u_4}Y_\alpha + \frac{1}{2}\frac{\partial}{\partial u_4}Y_\alpha + \frac{1}{4\Delta}\frac{\partial \Delta}{\partial u_4}Y_\alpha, \\ R_{\alpha 4} &= -\frac{1}{u_4}Z_\alpha + \frac{1}{2}\frac{\partial}{\partial u_4}Z_\alpha - \frac{1}{u_4}\frac{\partial}{\partial u_\alpha}\log\Delta + \frac{1}{4\Delta}F_{34}\frac{\partial F_{44}}{\partial u_4}Y_\alpha. \end{aligned}$$

Now, we suppose that

$$R_{\alpha\lambda} = 0. \tag{2.10}$$

From $R_{\alpha\lambda} = 0$, we consider the two cases as follows

Case I : $Y_1 = Y_2 = 0$ and Case II : $(Y_1, Y_2) \neq (0,0)$.

Case I. $Y_1 = Y_2 = 0$ implies that $F_{34}/F_{33} = b$, constant, therefore we have

$$\Delta = F_{33}F_{44} - F_{34}F_{34} = F_{33}(F_{44} - b^2F_{33}), \quad Z_\alpha = -\frac{1}{\Delta}(F_{44} - b^2F_{33})\frac{\partial F_{33}}{\partial u_\alpha}$$

and hence

$$Z_\alpha = -\frac{\partial}{\partial u_\alpha}\log F_{33},$$

which implies $\partial Z_\alpha/\partial u_4 = 0$ by (2.6) and

$$\begin{aligned} R_{\alpha 4} &= -\frac{1}{u_4}\Big(Z_\alpha + \frac{\partial}{\partial u_\alpha}\log\Delta\Big) \\ &= \frac{1}{u_4}\Big(\frac{\partial}{\partial u_\alpha}\log F_{33} - \frac{\partial}{\partial u_\alpha}\log(F_{33}(F_{44} - b^2F_{33}))\Big) \\ &= -\frac{1}{u_4}\frac{\partial}{\partial u_\alpha}\log(F_{44} - b^2F_{33}). \end{aligned}$$

Therefore, $R_{\alpha 4} = 0$ implies that

$$F_{44} = b^2F_{33} + \phi(u_4), \quad F_{34} = bF_{33}, \quad \Delta = \phi(u_4)F_{33}, \tag{2.11}$$

where $\phi = \phi(u_4)$ is an integral free function of u_4.

Cade II. Since we have

$$\begin{aligned} R_{\alpha 3} = \frac{Y_\alpha}{4}\Big\{-\frac{\partial}{\partial u_4}\log(u_4)^4 + \frac{\partial}{\partial u_4}\log(Y_\alpha)^2 + \frac{\partial}{\partial u_4}\log|\Delta|\Big\} \\ = \frac{Y_\alpha}{4}\frac{\partial}{\partial u_4}\log\Big(\frac{{Y_\alpha}^2|\Delta|}{(u_4)^4}\Big), \end{aligned}$$

$R_{\alpha 3} = 0$ implies that the function

$$\frac{{Y_\alpha}^2\Delta}{(u_4)^4} = \frac{1}{(u_4)^4\Delta}\Big(F_{33}\frac{\partial F_{34}}{\partial u_\alpha} - F_{34}\frac{\partial F_{33}}{\partial u_\alpha}\Big)^2$$

must depend only on u_1 and u_2. Therefore, by (2.6) we see that

$$(u_4)^4\Delta = f(u_1, u_2) \quad \text{or} \quad F_{44} = \frac{F_{34}F_{34}}{F_{33}} + \frac{1}{(u_4)^4}\frac{f}{F_{33}}, \tag{2.12}$$

where f is a function of u_1 and u_2. From $R_{\alpha 4} = 0$ we obtain

$$\begin{aligned} \frac{\partial Z_\alpha}{\partial u_4} &- \frac{2}{u_4}Z_\alpha = \frac{2}{u_4}\frac{\partial}{\partial u_\alpha}\log|\Delta| - \frac{1}{2\Delta}F_{34}\frac{\partial F_{44}}{\partial u_4}Y_\alpha \\ &= \frac{1}{u_4}\frac{\partial}{\partial u_\alpha}f^2 + \frac{1}{2\Delta^2}F_{34}\Big(\frac{4}{(u_4)^5}\frac{f}{F_{33}}\Big)\Big(F_{33}\frac{\partial F_{34}}{\partial u_\alpha} - F_{34}\frac{\partial F_{33}}{\partial u_\alpha}\Big) \\ &= \frac{1}{u_4}\frac{\partial}{\partial u_\alpha}f^2 + \frac{2}{f}(u_4)^3\Big(F_{34}\frac{\partial F_{34}}{\partial u_\alpha} - F_{34}F_{34}\frac{\partial \log|F_{33}|}{\partial u_\alpha}\Big) \\ &= \frac{1}{u_4}A_\alpha + (u_4)^3B_\alpha, \end{aligned}$$

where we set

$$A_\alpha = \frac{\partial}{\partial u_\alpha}\log f^2, \quad B_\alpha = \frac{1}{f}(F_{34})^2\frac{\partial}{\partial u_\alpha}\log\Big(\frac{F_{34}}{F_{33}}\Big)^2$$

and they depend only on u_1 and u_2. Regarding the above expression as an ordinary differential equation on u_4, we see that

$$Z_\alpha = -\frac{1}{2}A_\alpha + \frac{1}{2}(u_4)^4 B_\alpha + C_\alpha (u_4)^2,$$

where C_α is some function of u_1 and u_2. Then we have

$$\begin{aligned} Z_\alpha &= \frac{1}{\Delta}\Big(F_{34}\frac{\partial F_{34}}{\partial u_\alpha} - F_{44}\frac{\partial F_{33}}{\partial u_\alpha}\Big) \\ &= \frac{(u_4)^4}{f}\Big(F_{34}\frac{\partial F_{34}}{\partial u_\alpha} - \frac{F_{34}F_{34}}{F_{33}}\frac{\partial F_{33}}{\partial u_\alpha} - \frac{1}{(u_4)^4}\frac{f}{F_{33}}\frac{\partial F_{33}}{\partial u_\alpha}\Big) \\ &= -\frac{\partial}{\partial u_\alpha}\log|f| + (u_4)^4\frac{1}{f}F_{34}F_{34}\Big(\frac{1}{F_{34}}\frac{\partial F_{34}}{\partial u_\alpha} - \frac{1}{F_{33}}\frac{\partial F_{33}}{\partial u_\alpha}\Big) + C_\alpha(u_4)^2, \end{aligned}$$

which implies

$$-\frac{\partial \log|F_{33}/f|}{\partial u_\alpha} + C_\alpha (u_4)^2 = 0.$$

By the assumption (2.6), it must be

$$\frac{\partial \log|F_{33}/f|}{\partial u_\alpha} = 0 \quad \text{and} \quad C_\alpha = 0.$$

Therefore we obtain for Case II

$$F_{33} = af, \quad F_{44} = \frac{1}{a}\Big(\frac{F_{34}F_{34}}{f} + \frac{1}{(u_4)^4}\Big), \tag{2.13}$$

$$\Delta = F_{33}F_{44} - F_{34}F_{34} = \frac{f}{(u_4)^4},$$

where $f = f(u_1, u_2)$ and a is an integral constant.

In the following sections we shall investigate the solution

$$g_{ij} = \frac{1}{u_4 u_4}F_{ij},$$

which satisfies the Einstein condition :

$$R_{ij} = \frac{1}{4}Rg_{ij}, \quad R = \sum_{i,j} g^{ij}R_{ij} \tag{2.14}$$

under the conditions (1.3) and (2.6).

3. Solutions for Case I

We can express the Einstein condition (2.14) by

$$R_{ij} = \lambda F_{ij}. \tag{3.1}$$

Since $F_{\alpha\lambda} = 0$, it must be $R_{\alpha\lambda} = 0$. Thus, we divide the argument into two cases I and II. In this section, we search for solutions for Case I. Since we have

$$F_{34} = bF_{33}, \quad F_{44} = b^2 F_{33} + \phi, \quad \Delta = F_{33}F_{44} - F_{34}F_{34} = \phi F_{33}, \tag{3.2}$$

$$F^{33} = \frac{F_{44}}{\Delta} = \frac{1}{F_{33}} + \frac{b^2}{\phi}, \quad F^{34} = -\frac{F_{34}}{\Delta} = -\frac{b}{\phi}, \quad F^{44} = \frac{1}{\phi}, \quad \phi = \phi(u_4),$$

we obtain from (2.4) after long computations

$$\begin{aligned} R_{33} = {} & \frac{1}{4}\Big(\frac{\partial \log(F_{11}/F_{22})}{\partial u_1}\frac{1}{F_{11}}\frac{\partial F_{33}}{\partial u_1} - \frac{\partial \log(F_{11}/F_{22})}{\partial u_2}\frac{1}{F_{22}}\frac{\partial F_{33}}{\partial u_2}\Big) \\ & -\frac{1}{2}\sum_\alpha \frac{1}{F_{\alpha\alpha}}\frac{\partial^2 F_{33}}{\partial u_\alpha \partial u_\alpha} + \frac{1}{4F_{33}}\sum_\alpha \frac{1}{F_{\alpha\alpha}}\frac{\partial F_{33}}{\partial u_\alpha}\frac{\partial F_{33}}{\partial u_\alpha} \\ & -\Big(\frac{1}{2u_4\phi^2}\frac{d\phi}{du_4} + \frac{3}{u_4 u_4 \phi}\Big)F_{33} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} R_{34} = {} & \frac{1}{4}\Big(\frac{\partial \log(F_{11}/F_{22})}{\partial u_1}\frac{1}{F_{11}}\frac{\partial F_{34}}{\partial u_1} - \frac{\partial \log(F_{11}/F_{22})}{\partial u_2}\frac{1}{F_{22}}\frac{\partial F_{34}}{\partial u_2} \\ & - \frac{1}{2}\sum_\alpha \frac{1}{F_{\alpha\alpha}}\frac{\partial^2 F_{34}}{\partial u_\alpha u_\partial u_\alpha} + \frac{1}{4}\sum_\alpha \frac{1}{F_{\alpha\alpha}}\Big\{\Big(\frac{1}{F_{33}} + \frac{b^2}{\phi}\Big)\frac{\partial F_{33}}{\partial u_\alpha}\frac{\partial F_{34}}{\partial u_\alpha} \\ & - \frac{b}{\phi}\Big(\frac{\partial F_{33}}{\partial u_\alpha}\frac{\partial F_{44}}{\partial u_\alpha} + \frac{\partial F_{34}}{\partial u_\alpha}\frac{\partial F_{34}}{\partial u_\alpha}\Big) + \frac{1}{\phi}\frac{\partial F_{34}}{\partial u_\alpha}\frac{\partial F_{44}}{\partial u_\alpha}\Big\} \\ & - \frac{1}{2u_4\Delta^2}F_{33}\frac{\partial F_{44}}{\partial u_4}F_{33}F_{34} - \frac{3}{u_4 u_4 \phi}F_{34} \\ = {} & \frac{b}{4}\Big(\frac{\partial \log(F_{11}/F_{22})}{\partial u_1}\frac{1}{F_{11}}\frac{\partial F_{33}}{\partial u_1} - \frac{\partial \log(F_{11}/F_{22})}{\partial u_2}\frac{1}{F_{22}}\frac{\partial F_{33}}{\partial u_2}\Big) \\ & - \frac{b}{2}\sum_\alpha \frac{1}{F_{\alpha\alpha}}\frac{\partial^2 F_{33}}{\partial u_\alpha \partial u_\alpha} + \frac{b}{4F_{33}}\sum_\alpha \frac{1}{F_{\alpha\alpha}}\frac{\partial F_{33}}{\partial u_\alpha}\frac{\partial F_{33}}{\partial u_\alpha} \\ & - \Big(\frac{1}{2u_4\phi^2}\frac{d\phi}{du_4} + \frac{3}{u_4u_4\phi}\Big)bF_{33} = bR_{33}, \end{aligned}$$

that is

$$R_{34} = bR_{33}, \tag{3.4}$$

and

$$\begin{aligned} R_{44} &= \frac{1}{4}\Big(\frac{\partial\log(F_{11}/F_{22})}{\partial u_1}\frac{1}{F_{11}}\frac{\partial F_{44}}{\partial u_1} - \frac{\partial\log(F_{11}/F_{22})}{\partial u_2}\frac{1}{F_{22}}\frac{\partial F_{44}}{\partial u_2}\Big) \\ &- \frac{1}{2}\sum_\alpha \frac{1}{F_{\alpha\alpha}}\frac{\partial^2 F_{44}}{\partial u_\alpha\partial u_\alpha} + \frac{1}{4}\sum_\alpha\frac{1}{F_{\alpha\alpha}}\Big\{\Big(\frac{1}{F_{33}}+\frac{b}{\phi}\Big)\frac{\partial F_{43}}{\partial u_\alpha}\frac{\partial F_{43}}{\partial u_\alpha} \\ &- \frac{2b}{\phi}\frac{\partial F_{43}}{\partial u_\alpha}\frac{\partial F_{44}}{\partial u_\alpha} + \frac{1}{\phi}\frac{\partial F_{44}}{\partial u_\alpha}\frac{\partial F_{44}}{\partial u_\alpha}\Big\} - \frac{1}{2u_4\Delta\phi}\frac{\partial F_{44}}{\partial u_4}(3\Delta + F_{34}F_{34}) \\ &- \frac{3}{u_4u_4\phi}F_{44} = b^2R_{33} - \frac{3}{2u_4\phi}\frac{d\phi}{du_4} - \frac{3}{u_4u_4}, \end{aligned}$$

that is

$$R_{44} = b^2R_{33} - \frac{3}{2u_4\phi}\frac{d\phi}{du_4} - \frac{3}{u_4u_4}. \tag{3.5}$$

Regarding $R_{\alpha\beta}$, we obtain from (2.1),(2.2) and (2.3)

$$\begin{aligned} R_{11} = &-\frac{1}{2F_{33}}\frac{\partial^2F_{33}}{\partial u_1\partial u_1} - \frac{1}{2F_{22}}\Big(\frac{\partial^2F_{11}}{\partial u_2\partial u_2}+\frac{\partial^2F_{22}}{\partial u_1\partial u_1}\Big) + \frac{1}{4F_{22}}\Big\{\frac{1}{F_{11}}\Big(\frac{\partial F_{11}}{\partial u_2}\frac{\partial F_{11}}{\partial u_2} \\ &+\frac{\partial F_{11}}{\partial u_1}\frac{\partial F_{22}}{\partial u_1}\Big) + \frac{1}{F_{22}}\Big(\frac{\partial F_{11}}{\partial u_2}\frac{\partial F_{22}}{\partial u_2}+\frac{\partial F_{22}}{\partial u_1}\frac{\partial F_{22}}{\partial u_1}\Big)\Big\} \qquad (3.6) \\ &+\frac{1}{4F_{33}}\Big\{\frac{1}{F_{33}}\frac{\partial F_{33}}{\partial u_1}\frac{\partial F_{33}}{\partial u_1} + \frac{1}{F_{11}}\frac{\partial F_{11}}{\partial u_1}\frac{\partial F_{33}}{\partial u_1} - \frac{1}{F_{22}}\frac{\partial F_{11}}{\partial u_2}\frac{\partial F_{33}}{\partial u_2}\Big\} \\ &-\Big(\frac{1}{2u_4\phi^2}\frac{d\phi}{du_4}+\frac{3}{u_4u_4\phi}\Big)F_{11}, \end{aligned}$$

$$\begin{aligned} R_{22} = &-\frac{1}{2F_{33}}\frac{\partial^2F_{33}}{\partial u_2\partial u_2} - \frac{1}{2F_{11}}\Big(\frac{\partial^2F_{11}}{\partial u_2\partial u_2}+\frac{\partial^2F_{22}}{\partial u_1\partial u_1}\Big) + \frac{1}{4F_{11}}\Big\{\frac{1}{F_{11}}\Big(\frac{\partial F_{11}}{\partial u_2}\frac{\partial F_{11}}{\partial u_2} \\ &+\frac{\partial F_{11}}{\partial u_1}\frac{\partial F_{22}}{\partial u_1}\Big) + \frac{1}{F_{22}}\Big(\frac{\partial F_{11}}{\partial u_2}\frac{\partial F_{22}}{\partial u_2}+\frac{\partial F_{22}}{\partial u_1}\frac{\partial F_{22}}{\partial u_1}\Big)\Big\} \qquad (3.7) \\ &+\frac{1}{4F_{33}}\Big\{\frac{1}{F_{33}}\frac{\partial F_{33}}{\partial u_2}\frac{\partial F_{33}}{\partial u_2} + \frac{1}{F_{22}}\frac{\partial F_{22}}{\partial u_2}\frac{\partial F_{33}}{\partial u_2} - \frac{1}{F_{11}}\frac{\partial F_{22}}{\partial u_1}\frac{\partial F_{33}}{\partial u_1}\Big\} \\ &-\Big(\frac{1}{2u_4\phi^2}\frac{d\phi}{du_4}+\frac{3}{u_4u_4\phi}\Big)F_{22}, \end{aligned}$$

and

$$\begin{aligned} R_{12} = &-\frac{1}{2F_{33}}\frac{\partial^2F_{33}}{\partial u_1\partial u_2} + \frac{1}{4F_{33}}\Big\{\frac{1}{F_{33}}\frac{\partial F_{33}}{\partial u_1}\frac{\partial F_{33}}{\partial u_2} + \frac{1}{F_{11}}\frac{\partial F_{11}}{\partial u_2}\frac{\partial F_{33}}{\partial u_1} \\ &+\frac{1}{F_{22}}\frac{\partial F_{22}}{\partial u_1}\frac{\partial F_{33}}{\partial u_2}\Big\} = -\frac{1}{2}\frac{\partial^2\log F_{33}}{\partial u_1\partial u_2} + \frac{1}{4}\Big\{-\frac{\partial\log F_{33}}{\partial u_1}\frac{\partial\log F_{33}}{\partial u_2} \qquad (3.8) \end{aligned}$$

$$+\frac{\partial\log F_{11}}{\partial u_2}\frac{\partial\log F_{33}}{\partial u_1}+\frac{\partial\log F_{22}}{\partial u_1}\frac{\partial\log F_{33}}{\partial u_2}\Big\}.$$

Now, we suppose (3.1) holds for F_{ij} satisfying (3.2)$\sim$ (3.8). From (3.5) we have

$$R_{44}=\lambda b^2F_{33}-\frac{3}{2u_4\phi}\frac{d\phi}{du_4}-\frac{3}{u_4u_4}=\lambda F_{44}-\lambda\phi-\frac{3}{2u_4\phi}\frac{d\phi}{du_4}-\frac{3}{u_4u_4},$$

which implies

$$\lambda=-\frac{3}{2u_4\phi^2}\frac{d\phi}{du_4}-\frac{3}{u_4u_4\phi}. \tag{3.9}$$

From (3.3),(3.9) and $R_{33}=\lambda F_{33}$, we obtain

$$\begin{aligned}&\sum_\alpha\frac{1}{F_{\alpha\alpha}}\frac{\partial^2F_{33}}{\partial u_\alpha\partial u_\alpha}-\frac{1}{2}\Big(\frac{\partial\log(F_{11}/F_{22})}{\partial u_1}\frac{1}{F_{11}}\frac{\partial F_{33}}{\partial u_1}\\&-\frac{\partial\log(F_{11}/F_{22})}{\partial u_2}\frac{1}{F_{22}}\frac{\partial F_{33}}{\partial u_2}\Big)-\frac{1}{2F_{33}}\sum_\alpha\frac{1}{F_{\alpha\alpha}}\frac{\partial F_{33}}{\partial u_\alpha}\frac{\partial F_{33}}{\partial u_\alpha}-\frac{2}{u_4\phi^2}\frac{d\phi}{du_4}F_{33}=0.\end{aligned}\tag{3.10}$$

$R_{34}=\lambda F_{34}$ and $R_{44}=\lambda F_{44}$ are satisfied automatically from the above arguments. From $R_{12}=0$, we obtain

$$\frac{\partial^2F_{33}}{\partial u_1\partial u_2}-\frac{1}{2}\Big(\frac{1}{F_{33}}\frac{\partial F_{33}}{\partial u_1}\frac{\partial F_{33}}{\partial u_2}+\frac{1}{F_{11}}\frac{\partial F_{11}}{\partial u_2}\frac{\partial F_{33}}{\partial u_1}+\frac{1}{F_{22}}\frac{\partial F_{22}}{\partial u_1}\frac{\partial F_{33}}{\partial u_2}\Big)=0. \tag{3.11}$$

Finally from (3.6),(3.7),(3.9) and $R_{\alpha\alpha}=\lambda F_{\alpha\alpha}$, we obtain

$$\begin{aligned}&\frac{1}{F_{33}}\frac{\partial^2F_{33}}{\partial u_1\partial u_1}+\frac{1}{F_{22}}\Big(\frac{\partial^2F_{11}}{\partial u_2\partial u_2}+\frac{\partial^2F_{22}}{\partial u_1\partial u_1}\Big)-\frac{1}{2F_{22}}\Big\{\frac{1}{F_{11}}\Big(\frac{\partial F_{11}}{\partial u_2}\frac{\partial F_{11}}{\partial u_2}\\&+\frac{\partial F_{11}}{\partial u_1}\frac{\partial F_{22}}{\partial u_1}\Big)+\frac{1}{F_{22}}\Big(\frac{\partial F_{11}}{\partial u_2}\frac{\partial F_{22}}{\partial u_2}+\frac{\partial F_{22}}{\partial u_1}\frac{\partial F_{22}}{\partial u_1}\Big)\Big\}\\&-\frac{1}{2F_{33}}\Big\{\frac{1}{F_{33}}\frac{\partial F_{33}}{\partial u_1}\frac{\partial F_{33}}{\partial u_1}+\frac{1}{F_{11}}\frac{\partial F_{11}}{\partial u_1}\frac{\partial F_{33}}{\partial u_1}-\frac{1}{F_{22}}\frac{\partial F_{11}}{\partial u_2}\frac{\partial F_{33}}{\partial u_2}\Big\}\\&-\frac{2}{u_4\phi^2}\frac{d\phi}{du_4}F_{11}=0,\end{aligned}\tag{3.12}$$

$$\begin{aligned}&\frac{1}{F_{33}}\frac{\partial^2F_{33}}{\partial u_2\partial u_2}+\frac{1}{F_{11}}\Big(\frac{\partial^2F_{11}}{\partial u_2\partial u_2}+\frac{\partial^2F_{22}}{\partial u_1\partial u_1}\Big)-\frac{1}{2F_{11}}\Big\{\frac{1}{F_{11}}\Big(\frac{\partial F_{11}}{\partial u_2}\frac{\partial F_{11}}{\partial u_2}\\&+\frac{\partial F_{11}}{\partial u_1}\frac{\partial F_{22}}{\partial u_1}\Big)+\frac{1}{F_{22}}\Big(\frac{\partial F_{11}}{\partial u_2}\frac{\partial F_{22}}{\partial u_2}+\frac{\partial F_{22}}{\partial u_1}\frac{\partial F_{22}}{\partial u_1}\Big)\Big\}\\&-\frac{1}{2F_{33}}\Big\{\frac{1}{F_{33}}\frac{\partial F_{33}}{\partial u_2}\frac{\partial F_{33}}{\partial u_2}+\frac{1}{F_{22}}\frac{\partial F_{22}}{\partial u_2}\frac{\partial F_{33}}{\partial u_2}-\frac{1}{F_{11}}\frac{\partial F_{22}}{\partial u_1}\frac{\partial F_{33}}{\partial u_1}\Big\}\\&-\frac{2}{u_4\phi^2}\frac{d\phi}{du_4}F_{22}=0.\end{aligned}\tag{3.13}$$

Using the formula :

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial u_\alpha \partial u_\beta} = \frac{\partial^2 \log\Phi}{\partial u_\alpha \partial u_\beta} + \frac{\partial \log\Phi}{\partial u_\alpha}\frac{\partial \log\Phi}{\partial u_\beta},$$

the equalities (3.10)$\sim$ (3.13) can be written as follows,

$$\begin{aligned}(3.10') \quad &\frac{1}{F_{11}}\Big\{\frac{\partial^2 \log F_{33}}{\partial u_1\partial u_1} + \frac{1}{2}\Big(\frac{\partial \log F_{33}}{\partial u_1}\frac{\partial \log F_{33}}{\partial u_1} - \frac{\partial \log(F_{11}/F_{22})}{\partial u_1}\frac{\partial \log F_{33}}{\partial u_1}\Big)\Big\}\\ &+\frac{1}{F_{22}}\Big\{\frac{\partial^2 \log F_{33}}{\partial u_2\partial u_2} + \frac{1}{2}\Big(\frac{\partial \log F_{33}}{\partial u_2}\frac{\partial \log F_{33}}{\partial u_2} - \frac{\partial \log(F_{22}/F_{11})}{\partial u_2}\frac{\partial \log F_{33}}{\partial u_2}\Big)\Big\}\\ &-\frac{2}{u_4\phi^2}\frac{d\phi}{du_4} = 0,\end{aligned}$$

$$\begin{aligned}(3.11') \quad &\frac{\partial^2 \log F_{33}}{\partial u_1\partial u_2} + \frac{1}{2}\Big(\frac{\partial \log F_{33}}{\partial u_1}\frac{\partial \log F_{33}}{\partial u_2} - \frac{\partial \log F_{22}}{\partial u_1}\frac{\partial \log F_{33}}{\partial u_2}\\ &-\frac{\partial \log F_{11}}{\partial u_2}\frac{\partial \log F_{33}}{\partial u_1}\Big) = 0,\end{aligned}$$

$$\begin{aligned}(3.12') \quad &\frac{1}{F_{11}}\Big\{\frac{\partial^2 \log F_{33}}{\partial u_1\partial u_1} + \frac{\partial^2 \log F_{22}}{\partial u_1\partial u_1} + \frac{1}{2}\Big(\frac{\partial \log F_{33}}{\partial u_1}\frac{\partial \log F_{33}}{\partial u_1}\\ &+\frac{\partial \log F_{22}}{\partial u_1}\frac{\partial \log F_{22}}{\partial u_1} - \frac{\partial \log F_{11}}{\partial u_1}\frac{\partial \log F_{22}}{\partial u_1} - \frac{\partial \log F_{11}}{\partial u_1}\frac{\partial \log F_{33}}{\partial u_1}\Big)\Big\}\\ &+\frac{1}{F_{22}}\Big\{\frac{\partial^2 \log F_{11}}{\partial u_2\partial u_2} + \frac{1}{2}\Big(\frac{\partial \log F_{11}}{\partial u_2}\frac{\partial \log F_{11}}{\partial u_2} - \frac{\partial \log F_{11}}{\partial u_2}\frac{\partial \log F_{22}}{\partial u_2}\\ &+\frac{\partial \log F_{11}}{\partial u_2}\frac{\partial \log F_{33}}{\partial u_2}\Big)\Big\} - \frac{2}{u_4\phi^2}\frac{d\phi}{du_4} = 0,\end{aligned}$$

$$\begin{aligned}(3.13') \quad &\frac{1}{F_{22}}\Big\{\frac{\partial^2 \log F_{33}}{\partial u_2\partial u_2} + \frac{\partial^2 \log F_{11}}{\partial u_2\partial u_2} + \frac{1}{2}\Big(\frac{\partial \log F_{33}}{\partial u_2}\frac{\partial \log F_{33}}{\partial u_2}\\ &+\frac{\partial \log F_{11}}{\partial u_2}\frac{\partial \log F_{11}}{\partial u_2} - \frac{\partial \log F_{11}}{\partial u_2}\frac{\partial \log F_{22}}{\partial u_2} - \frac{\partial \log F_{22}}{\partial u_2}\frac{\partial \log F_{33}}{\partial u_2}\Big)\Big\}\\ &+\frac{1}{F_{11}}\Big\{\frac{\partial^2 \log F_{22}}{\partial u_1\partial u_1} + \frac{1}{2}\Big(\frac{\partial \log F_{22}}{\partial u_1}\frac{\partial \log F_{22}}{\partial u_1} - \frac{\partial \log F_{11}}{\partial u_1}\frac{\partial \log F_{22}}{\partial u_1}\\ &+\frac{\partial \log F_{22}}{\partial u_1}\frac{\partial \log F_{33}}{\partial u_1}\Big)\Big\} - \frac{2}{u_4\phi^2}\frac{d\phi}{du_4} = 0,\end{aligned}$$

supposing $F_{11} \neq 0, F_{22} \neq 0, F_{33} \neq 0$.

If we can find $F_{11}(u_1, u_2), F_{22}(u_1, u_2), F_{33}(u_1, u_2)$ and $\phi = \phi(u_4)$ satisfying $(3.10') \sim (3.13')$, we obtain the metric $g_{ij} = \frac{1}{u_4 u_4}F_{ij}$ satisfying the

Einstein condition (3.1). In the following we show that we can obtain such solutions under the additional restriction :

$$\frac{\partial F_{11}}{\partial u_2} = \frac{\partial F_{22}}{\partial u_2} = 0 \quad \text{and} \quad F_{33} = \psi(u_1)\sin^2 u_2. \tag{3.14}$$

Then (3.10′) is reduced to

$$\frac{1}{F_{11}}\Big\{\frac{d^2\log\psi}{du_1^2} + \frac{1}{2}\Big(\frac{d\log\psi}{du_1}\frac{d\log\psi}{du_1} - \frac{d\log(F_{11}/F_{22})}{du_1}\frac{d\log\psi}{du_1}\Big)\Big\} \\ -\frac{2}{F_{22}} - \frac{2}{u_4\phi^2}\frac{d\phi}{du_4} = 0 \tag{3.15}$$

and (3.11′) is reduced to

$$\Big(\frac{d\log\psi}{du_1} - \frac{d\log F_{22}}{du_1}\Big)\frac{\cos u_2}{\sin u_2} = 0. \tag{3.16}$$

(3.12′) and (3.13′) are reduced respectively to

$$\frac{1}{F_{11}}\Big\{\frac{d^2\log\psi}{du_1^2} + \frac{d^2\log F_{22}}{du_1^2} + \frac{1}{2}\Big(\frac{d\log\psi}{du_1}\frac{d\log\psi}{du_1} \\ +\frac{d\log F_{22}}{du_1}\frac{d\log F_{22}}{du_1} - \frac{d\log F_{11}}{du_1}\frac{d\log F_{22}}{du_1} - \frac{d\log F_{11}}{du_1}\frac{d\log\psi}{du_1}\Big)\Big\} \\ -\frac{2}{u_4\phi^2}\frac{d\phi}{du_4} = 0, \tag{3.17}$$

$$\frac{1}{F_{11}}\Big\{\frac{d^2\log\psi}{du_1^2} + \frac{1}{2}\Big(\frac{d\log F_{22}}{du_1}\frac{d\log F_{22}}{du_1} - \frac{d\log F_{11}}{du_1}\frac{d\log F_{22}}{du_1} \\ +\frac{d\log F_{22}}{du_1}\frac{d\log\psi}{du_1}\Big)\Big\} - \frac{2}{F_{22}} - \frac{2}{u_4\phi^2}\frac{d\phi}{du_4} = 0. \tag{3.18}$$

Then, we see firstly from (3.16) that

$$\psi(u_1) = cF_{22}(u_1), \quad c = \text{constant}. \tag{3.19}$$

Using this relation we obtain from (3.15),(3.17) and (3.18) respectively

(3.15′)

$$\frac{1}{F_{11}}\Big\{\frac{d^2\log F_{22}}{du_1^2} + \frac{1}{2}\Big(\frac{d\log F_{22}}{du_1}\frac{d\log F_{22}}{du_1} - \frac{d\log(F_{11}/F_{22})}{du_1}\frac{d\log F_{22}}{du_1}\Big)\Big\} \\ -\frac{2}{F_{22}} - \frac{2}{u_4\phi^2}\frac{d\phi}{du_4} = 0,$$

$$\frac{1}{F_{11}}\Big\{2\frac{d^2\log F_{22}}{du_1^2} + \frac{d\log F_{22}}{du_1}\frac{d\log F_{22}}{du_1} - \frac{d\log F_{11}}{du_1}\frac{d\log F_{22}}{du_1}\Big\} \tag{3.17′}$$

$$-\frac{2}{u_4\phi^2}\frac{d\phi}{du_4}=0,$$

$$\text{(3.18}')\qquad \frac{1}{F_{11}}\Big\{\frac{d^2\log F_{22}}{du_1^2}+\frac{d\log F_{22}}{du_1}\frac{d\log F_{22}}{du_1}-\frac{1}{2}\frac{d\log F_{11}}{du_1}\frac{d\log F_{22}}{du_1}\Big\}$$
$$-\frac{2}{F_{22}}-\frac{2}{u_4\phi^2}\frac{d\phi}{du_4}=0.$$

$(3.17')-(3.15')$ becomes

$$\frac{1}{F_{11}}\Big\{\frac{d^2\log F_{22}}{du_1^2}-\frac{1}{2}\frac{d\log F_{11}}{du_1}\frac{d\log F_{22}}{du_1}\Big\}+\frac{2}{F_{22}}=0$$

and $(3.17')-(3.15')\times 2$ becomes

$$\text{(3.20)}\qquad -\frac{1}{F_{11}}\Big(\frac{d\log F_{22}}{du_1}\Big)^2+\frac{4}{F_{22}}+\frac{2}{u_4\phi^2}\frac{d\phi}{du_4}=0.$$

Thus, we see that to solve the system of differential equations $(3.15')$, $(3.17')$ and $(3.18')$ on F_{11}, F_{22} and ϕ they can be replaced by

$$\text{(3.17}')\qquad \frac{1}{F_{22}}\frac{d^2F_{22}}{du_1^2}-\Big(\frac{1}{F_{22}}\frac{dF_{22}}{du_1}\Big)^2-\frac{1}{2F_{11}F_{22}}\frac{dF_{11}}{du_1}\frac{dF_{22}}{du_1}+2\frac{F_{11}}{F_{22}}=0,$$

$$\text{(3.20}')\qquad -\Big(\frac{1}{F_{22}}\frac{dF_{22}}{du_1}\Big)^2+4\frac{F_{11}}{F_{22}}+2\frac{F_{11}}{u_4\phi^2}\frac{d\phi}{du_4}=0,$$

$$\text{(3.18}'')\qquad \frac{1}{F_{22}}\frac{d^2F_{22}}{du_1^2}-\frac{1}{2F_{11}F_{22}}\frac{dF_{11}}{du_1}\frac{dF_{22}}{du_1}-2\frac{F_{11}}{F_{22}}-2F_{11}\frac{1}{u_4\phi^2}\frac{d\phi}{du_4}=0.$$

Now, setting

$$q=q(u_4):=\frac{2}{u_4\phi^2}\frac{d\phi}{du_4},\quad y=F_{22},\quad z=\frac{dF_{22}}{du_1},$$

the above expressions become respectively

$$\begin{aligned}\frac{1}{y}z' \quad &-\quad \Big(\frac{z}{y}\Big)^2-\frac{1}{2}\frac{F_{11}{}'}{F_{11}}\frac{z}{y}+2\frac{F_{11}}{y}=0,\\ &-\quad \Big(\frac{z}{y}\Big)^2+4\frac{F_{11}}{y}+qF_{11}=0,\\ \frac{1}{y}z' \quad &-\quad \frac{F_{11}{}'}{2F_{11}}\frac{z}{y}-2\frac{F_{11}}{y}-F_{11}q=0.\end{aligned}$$

From the third one of the above equations we obtain

$$\text{(3.21)}\qquad y'=z\quad\text{and}\quad z'=qF_{11}y+\frac{F_{11}{}'}{2F_{11}}z+2F_{11}$$

and the second and first ones become

$$z^2 = F_{11}(qy^2 + 4y), \tag{3.22}$$

$$z' = \frac{z^2}{y} + \frac{F_{11}{}'}{2F_{11}} z - 2F_{11}, \tag{3.23}$$

respectively. Now, let y and z satisfy (3.21), then we obtain

$$\begin{aligned}
\frac{d}{du_1}\{z^2 &- F_{11}(qy^2+4y)\} = 2zz' - F_{11}{}'(qy^2+4y) - F_{11}(2qyz+4z) \\
&= 2z(qF_{11}y + \frac{F_{11}{}'}{2F_{11}}z + 2F_{11}) - F_{11}{}'(qy^2+4y) - 2F_{11}(qyz+2z) \\
&= \frac{1}{F_{11}}\frac{dF_{11}}{du_1}\{z^2 - F_{11}(qy^2+4y)\},
\end{aligned}$$

which implies the equation

$$z^2 - F_{11}(qy^2+4y) = F_{11} \times \text{constant}$$

and hence we see that if y and z satisfy (3.22) at some value of u_1, then it holds for all its near values. We see easily that (3.23) and the second expression of (3.21) imply (3.22). Therefore, it is sufficient to treat only (3.22) for our purpose. From (3.2), (3.22), q does not depend on u_4, hence we obtain

$$\phi = \frac{1}{c_0 + c_1 u_4 u_4}, \quad c_0, c_1 = \text{constants and} \quad q = -4c_1.$$

then (3.22) can be written as

$$z^2 = 4F_{11}y(1 - c_1 y). \tag{3.24}$$

Hence it must be $F_{11}y(1-c_1y) \geq 0$, and therefore

$$\frac{dy}{du_1} = \pm 2\sqrt{F_{11}y(1-c_1y)} \quad \text{or} \quad \frac{dy}{\sqrt{\varepsilon y(1-c_1y)}} = \pm 2\sqrt{\varepsilon F_{11}}du_1,$$

where $\varepsilon = 1$ for $F_{11} > 0$ and $\varepsilon = -1$ for $F_{11} < 0$. We denote indefinite integrals of the left and right hand sides by

$$\Phi(y) = \int \frac{dy}{\sqrt{\varepsilon y(1-c_1y)}} \quad \text{and} \quad h(u_1) = \int \sqrt{\varepsilon F_{11}(u_1)}du_1 \tag{3.25}$$

and we obtain

$$\Phi(y) = \pm 2(h(u_1) + c_2), \quad c_2 = \text{constant}.$$

Denoting the inverse function of Φ by Ψ, we obtain the following theorem.

Theorem 1. The system (3.9)∼(3.12) satisfying the Einstein condition is given by a solution of

$$(3.26)\quad F_{11} = F_{11}(u_1), \quad F_{22} = \Psi(\pm 2(h(u_1) + c_2)), \quad F_{33} = cF_{22}\sin^2 u_2,$$
$$F_{34} = bF_{33}, \quad F_{44} = b^2 F_{33} + \frac{1}{c_0 + c_1 u_4 u_4},$$

where b, c, c_0, c_1, c_2 are constants and F_{11} is any smooth function of u_1 .

Example 1. Let $b = 0, c = 1, c_0 = -1, c_1 = 0$ and $F_{11} = 1$, then $\varepsilon = 1$ and

$$\Phi(y) = \int \frac{dy}{\sqrt{y}} = 2\sqrt{y}, \quad h(u_1) = \int du_1 = u_1, \quad \text{and} \quad 2\sqrt{y} = \pm 2(u_1 + c_2),$$

hence $y = (u_1 + c_2)^2$. Putting $c_2 = 0$ and so $y = F_{22} = u_1^2, F_{33} = u_1^2 \sin^2 u_2, F_{34} = 0, F_{44} = -1$. Therefore we obtain the metric (1.1) with $a = 0$:

$$ds^2 = \frac{1}{u_4 u_4}(du_1{}^2 + u_1{}^2 du_2{}^2 + u_1{}^2 \sin^2 u_2 du_3{}^2 - du_4{}^2).$$

Example 2. Let $b = 0, c = 1, c_0 = -1, c_1 = -1$ and $F_{11} = \frac{1}{1+u_1u_1}$, then $\varepsilon = 1$ and

$$\Phi(y) = \int \frac{dy}{\sqrt{y(1+y)}} = \log(2y + 1 + 2\sqrt{y(1+y)}) \quad (\text{with}\, \Phi(0) = 0),$$

and $\quad h(u_1) = \int \frac{du_1}{\sqrt{1+u_1u_1}} = \log(u_1 + \sqrt{1 + u_1 u_1})$ (with $h(0) = 0$), and

$$\log(2y + 1 + 2\sqrt{y(1+y)}) = 2\log(u_1 + \sqrt{1 + u_1u_1})) \quad (\text{with}\, c_2 = 0),$$

from which we obtain

$$2y + 1 + 2\sqrt{y(1+y)} = 2u_1u_1 + 1 + 2u_1\sqrt{1 + u_1u_1}$$

and

$$2y + 1 - 2\sqrt{y(1+y)} = 2u_1u_1 + 1 - 2u_1\sqrt{1 + u_1u_1},$$

hence we obtain

$$y = F_{22} = u_1{}^2, \quad F_{33} = u_1{}^2 \sin^2 u_2, \quad F_{34} = 0, \quad F_{44} = -\frac{1}{1 + u_4u_4}.$$

Therefore, we obtain the metric (1.1) with $a = 1$:

$$ds^2 = \frac{1}{u_4u_4}\Big(\frac{1}{1 + u_1u_1} du_1{}^2 + u_1{}^2 du_2{}^2 + u_1{}^2 \sin^2 u_2 du_3{}^2 - \frac{1}{1 + u_4u_4} du_4{}^2\Big).$$

Analogously, if we set $b = 0, c = 1, c_0 = -1, c_1 = 1$ and $F_{11} = \frac{1}{1-u_1u_1}$, we obtain the metric (1.1) with $a = -1$.

4. Analysis and a Conclusion for Case II

For Case II described in Section 2, we have (2.13) and we set $F_{34} = fh$, then we obtain

$$F_{33} = af, \quad F_{34} = fh, \quad F_{44} = \frac{1}{a}\Big(fh^2 + \frac{1}{(u_4)^4}\Big),$$

$$F^{33} = \frac{F_{44}}{\Delta} = \frac{1}{a}\Big((u_4)^4h^2 + \frac{1}{f}\Big), \quad F^{34} = -\frac{F_{34}}{\Delta} = -(u_4)^4h, \tag{4.1}$$

$$F^{44} = \frac{F_{33}}{\Delta} = a(u_4)^4, \quad \Delta = F_{33}F_{44} - F_{34}F_{34} = \frac{f}{(u_4)^4},$$

where $a \neq 0$, constant and f, h are functions of u_1, u_2.

Using (4.1) for (2.3), we obtain

$$R_{12} = -\frac{1}{2f}\frac{\partial^2 f}{\partial u_2 \partial u_1} + \frac{1}{4f^2}\frac{\partial f}{\partial u_1}\frac{\partial f}{\partial u_2} + \frac{1}{4f}\frac{\partial \log F_{11}}{\partial u_2}\frac{\partial f}{\partial u_1} \tag{4.2}$$

$$+\frac{1}{4f}\frac{\partial \log F_{22}}{\partial u_1}\frac{\partial f}{\partial u_2} - \frac{1}{2}(u_4)^4 f\frac{\partial h}{\partial u_1}\frac{\partial h}{\partial u_2}.$$

If we consider the Einstein condition (3.1), it must be $R_{12} = 0$, hence from (4.2) we have

$$f\frac{\partial h}{\partial u_1}\frac{\partial h}{\partial u_2} = 0.$$

Since $\Delta \neq 0$,it must be $f \neq 0$ and h is not constant for Case II, therefore we have

$$\text{(i)} \qquad \frac{\partial h}{\partial u_1} \neq 0 \quad \text{and} \quad \frac{\partial h}{\partial u_2} = 0$$

or

$$\text{(ii)} \qquad \frac{\partial h}{\partial u_1} = 0 \quad \text{and} \quad \frac{\partial h}{\partial u_2} \neq 0.$$

In the following, we consider the case (i). By (4.1) and (2.7), we obtain from (2.4)

$$R_{33} = \frac{a}{4}F^{11}\frac{\partial \log(F_{11}/F_{22})}{\partial u_1}\frac{\partial f}{\partial u_1} - \frac{a}{4}F^{22}\frac{\partial \log(F_{11}/F_{22})}{\partial u_2}\frac{\partial f}{\partial u_2} \tag{4.3}$$

$$-\frac{a}{2}\sum_\alpha F^{\alpha\alpha}\Big(\frac{\partial^2 f}{\partial u_\alpha \partial u_\alpha} - \frac{1}{2f}\frac{\partial f}{\partial u_\alpha}\frac{\partial f}{\partial u_\alpha}\Big) + \frac{a}{2}(u_4)^4F^{11}f^2\frac{\partial h}{\partial u_1}\frac{\partial h}{\partial u_1} - au_4u_4F_{33}.$$

Next, analogously from (2.4) we obtain

$$R_{34} = \frac{1}{4}F^{11}\frac{\partial \log(F_{11}/F_{22})}{\partial u_1}\Big(h\frac{\partial f}{\partial u_1} + f\frac{\partial h}{\partial u_1}\Big) - \frac{1}{4}F^{22}\frac{\partial \log(F_{11}/F_{22})}{\partial u_2}h\frac{\partial f}{\partial u_2}$$

$$-\frac{1}{2}F^{11}\Big\{\Big(\frac{\partial^2 f}{\partial u_1 \partial u_1} - \frac{1}{2f}\frac{\partial f}{\partial u_1}\frac{\partial f}{\partial u_1}\Big)h + f\frac{\partial^2 h}{\partial u_1 \partial u_1} + \frac{3}{2}\frac{\partial f}{\partial u_1}\frac{\partial h}{\partial u_1}\Big)\Big\} \tag{4.4}$$

$$-\frac{1}{2}F^{22}\Big(\frac{\partial^2 f}{\partial u_2\partial u_2}-\frac{1}{2f}\frac{\partial f}{\partial u_2}\frac{\partial f}{\partial u_2}\Big)h+\frac{1}{2}(u_4)^4F^{11}f^2h\frac{\partial h}{\partial u_1}\frac{\partial h}{\partial u_1}$$
$$-au_4u_4F_{34}.$$

Finally, from (2.4) we obtain

$$R_{44}=\frac{1}{4a}F^{11}\frac{\partial\log(F_{11}/F_{22})}{\partial u_1}\Big(h^2\frac{\partial f}{\partial u_1}+2fh\frac{\partial h}{\partial u_1}\Big)$$
$$-\frac{1}{4a}F^{22}\frac{\partial\log(F_{11}/F_{22})}{\partial u_2}h^2\frac{\partial f}{\partial u_2}-\frac{1}{2a}F^{11}\Big\{h^2\Big(\frac{\partial^2 f}{\partial u_1\partial u_1}-\frac{1}{2f}\frac{\partial f}{\partial u_1}\frac{\partial f}{\partial u_1}\Big)$$
$$+3h\frac{\partial f}{\partial u_1}\frac{\partial h}{\partial u_1}+2fh\Big(\frac{\partial^2 h}{\partial u_1\partial u_1}+\frac{1}{2h}\frac{\partial h}{\partial u_1}\frac{\partial h}{\partial u_1}\Big)\Big\} \tag{4.5}$$
$$-\frac{1}{2a}F^{22}h^2\Big(\frac{\partial^2 f}{\partial u_2\partial u_2}-\frac{1}{2f}\frac{\partial f}{\partial u_2}\frac{\partial f}{\partial u_2}\Big)$$
$$+\frac{1}{2a}(u_4)^4F^{11}f^2h^2\frac{\partial h}{\partial u_1}\frac{\partial h}{\partial u_1}+\frac{4}{u_4u_4}-au_4u_4F_{44}.$$

Now, considering the Einstein condition (3.1) we put

$$\lambda=R_{33}/F_{33}=R_{33}/(af) \tag{4.6}$$
$$=\frac{1}{4}F^{11}\frac{\partial\log(F_{11}/F_{22})}{\partial u_1}\frac{\partial\log f}{\partial u_1}-\frac{1}{4}F^{22}\frac{\partial\log(F_{11}/F_{22})}{\partial u_2}\frac{\partial\log f}{\partial u_2}$$
$$-\frac{1}{2}\sum_\alpha F^{\alpha\alpha}\Big(\frac{1}{f}\frac{\partial^2 f}{\partial u_\alpha\partial u_\alpha}-\frac{1}{2f^2}\frac{\partial f}{\partial u_\alpha}\frac{\partial f}{\partial u_\alpha}\Big)+\frac{1}{2}(u_4)^4F^{11}f\frac{\partial h}{\partial u_1}\frac{\partial h}{\partial u_1}-au_4u_4$$

and we obtain from (4.3) and (4.4)

$$R_{34}-\lambda F_{34}=R_{34}-\frac{h}{a}R_{33} \tag{4.7}$$
$$=\frac{1}{4}F^{11}\frac{\partial\log(F_{11}/F_{22})}{\partial u_1}f\frac{\partial h}{\partial u_1}-\frac{1}{2}F^{11}\Big(f\frac{\partial^2 h}{\partial u_1\partial u_1}+\frac{3}{2}\frac{\partial f}{\partial u_1}\frac{\partial h}{\partial u_1}\Big).$$

Since $F^{11}\neq 0, R_{34}-\lambda F_{34}=0$ is equivalent to

$$\frac{\partial\log(F_{11}/F_{22})}{\partial u_1}f\frac{\partial h}{\partial u_1}-2f\frac{\partial^2 h}{\partial u_1\partial u_1}-3\frac{\partial f}{\partial u_1}\frac{\partial h}{\partial u_1}=0, \tag{4.8}$$

from which we obtain the relation

$$F_{11}/F_{22}=f^3\Big(\frac{\partial h}{\partial u_1}\Big)^2\rho(u_2)$$

by integration, where ρ is a function of u_2.

Next, we obtain from (4.3) and (4.5) the equality :

$$R_{44} - \lambda F_{44} = R_{44} - \frac{\lambda}{a}\Big(fh^2 + \frac{1}{(u_4)^4}\Big) = R_{44}$$
$$- \frac{1}{a^2 f}R_{33}\Big(fh^2 + \frac{1}{(u_4)^4}\Big) = R_{44} - \frac{h^2}{a^2}R_{33} - \frac{1}{a^2 f(u_4)^4}R_{33}$$

$$(4.9) \quad = -\frac{1}{2a}F^{11}\Big\{-\frac{\partial \log(F_{11}/F_{22})}{\partial u_1} fh\frac{\partial h}{\partial u_1} + 3h\frac{\partial f}{\partial u_1}\frac{\partial h}{\partial u_1} + 2fh\frac{\partial^2 h}{\partial u_1 \partial u_1}$$

$$+ 2f\frac{\partial h}{\partial u_1}\frac{\partial h}{\partial u_1}\Big\} + \frac{4}{u_4 u_4} + \frac{1}{a(u_4)^4}\Big\{-\frac{1}{4}F^{11}\frac{\partial \log(F_{11}/F_{22})}{\partial u_1}\frac{\partial \log f}{\partial u_1}$$
$$+ \frac{1}{4}F^{22}\frac{\partial \log(F_{11}/F_{22})}{\partial u_2}\frac{\partial \log f}{\partial u_2} + \frac{1}{2}\sum_\alpha F^{\alpha\alpha}\Big(\frac{1}{f}\frac{\partial^2 f}{\partial u_\alpha \partial u_\alpha} - \frac{1}{2f^2}\frac{\partial f}{\partial u_\alpha}\frac{\partial f}{\partial u_\alpha}\Big)\Big\},$$

which cannot vanish as a function of u_1, u_2, u_4. Hence we see that there exist no solutions for the case (i).

For the case (ii) : $\frac{\partial h}{\partial u_1} = 0$ and $\frac{\partial h}{\partial u_2} \neq 0$, analogously to the case (i) we obtain the equalities:

$$R_{33} = \frac{a}{4}F^{11}\frac{\partial \log(F_{11}/F_{22})}{\partial u_1}\frac{\partial f}{\partial u_1}$$

$$(4.3') \quad -\frac{a}{4}F^{22}\frac{\partial \log(F_{11}/F_{22})}{\partial u_2}\frac{\partial f}{\partial u_2} - \frac{a}{2}\sum_\alpha F^{\alpha\alpha}\Big(\frac{\partial^2 f}{\partial u_\alpha \partial u_\alpha} - \frac{1}{2f}\frac{\partial f}{\partial u_\alpha}\frac{\partial f}{\partial u_\alpha}\Big)$$

$$+\frac{a}{2}(u_4)^4 F^{22} f^2 \frac{\partial h}{\partial u_2}\frac{\partial h}{\partial u_2} - a u_4 u_4 F_{33},$$

$$R_{34} = \frac{1}{4}F^{11}\frac{\partial \log(F_{11}/F_{22})}{\partial u_1} h \frac{\partial f}{\partial u_1}$$

$$(4.4') \quad -\frac{1}{4}F^{22}\frac{\partial \log(F_{11}/F_{22})}{\partial u_2}\Big(h\frac{\partial f}{\partial u_2} + f\frac{\partial h}{\partial u_2}\Big) - \frac{1}{2}F^{11}\Big(\frac{\partial^2 f}{\partial u_1 \partial u_1}$$

$$-\frac{1}{2f}\frac{\partial f}{\partial u_1}\frac{\partial f}{\partial u_1}\Big)h - \frac{1}{2}F^{22}\Big\{\Big(\frac{\partial^2 f}{\partial u_2 \partial u_2} - \frac{1}{2f}\frac{\partial f}{\partial u_2}\frac{\partial f}{\partial u_2}\Big)h$$

$$+f\frac{\partial^2 h}{\partial u_2 \partial u_2} + \frac{3}{2}\frac{\partial f}{\partial u_2}\frac{\partial h}{\partial u_2}\Big\} + \frac{1}{2}(u_4)^4 F^{22} f^2 h \frac{\partial h}{\partial u_2}\frac{\partial h}{\partial u_2} - a u_4 u_4 F_{34},$$

and

$$
\begin{aligned}
R_{44} &= \frac{1}{4a}F^{11}\frac{\partial\log(F_{11}/F_{22})}{\partial u_1}h^2\frac{\partial f}{\partial u_1}\\
&- \frac{1}{4a}F^{22}\frac{\partial\log(F_{11}/F_{22})}{\partial u_2}\Big(h^2\frac{\partial f}{\partial u_2}+2fh\frac{\partial h}{\partial u_2}\Big)-\frac{1}{2a}F^{11}h^2\Big(\frac{\partial^2 f}{\partial u_1\partial u_1}\\
&\quad -\frac{1}{2f}\frac{\partial f}{\partial u_1}\frac{\partial f}{\partial u_1}\Big)-\frac{1}{2a}F^{22}\Big\{h^2\Big(\frac{\partial^2 f}{\partial u_2\partial u_2}-\frac{1}{2f}\frac{\partial f}{\partial u_2}\frac{\partial f}{\partial u_2}\Big)\\
&\quad +3h\frac{\partial f}{\partial u_2}\frac{\partial h}{\partial u_2}+2fh\Big(\frac{\partial^2 h}{\partial u_2\partial u_2}+\frac{1}{2h}\frac{\partial h}{\partial u_2}\frac{\partial h}{\partial u_2}\Big)\Big\}\\
&\quad +\frac{1}{2a}(u_4)^4F^{22}f^2h^2\frac{\partial h}{\partial u_2}\frac{\partial h}{\partial u_2}+\frac{4}{u_4u_4}-au_4u_4F_{44}.
\end{aligned}
\tag{4.5'}
$$

Now, we put

$$
\begin{aligned}
\lambda = R_{33}/F_{33} = R_{33}/(af) &= \frac{1}{4}F^{11}\frac{\partial\log(F_{11}/F_{22})}{\partial u_1}\frac{\partial\log f}{\partial u_1}\\
-\frac{1}{4}F^{22}\frac{\partial\log(F_{11}/F_{22})}{\partial u_2}\frac{\partial\log f}{\partial u_2} &- \frac{1}{2}\sum_{\alpha}F^{\alpha\alpha}\Big(\frac{1}{f}\frac{\partial^2 f}{\partial u_\alpha\partial u_\alpha}-\frac{1}{2f^2}\frac{\partial f}{\partial u_\alpha}\frac{\partial f}{\partial u_\alpha}\Big)\\
&+\frac{1}{2}(u_4)^4F^{22}f\frac{\partial h}{\partial u_2}\frac{\partial h}{\partial u_2}-au_4u_4,
\end{aligned}
\tag{4.6'}
$$

and we obtain from $(4.3'), (4.4')$ and $(4.5')$ the equations

$$
\begin{aligned}
R_{34}-\lambda F_{34} &= -\frac{1}{4}F^{22}\frac{\partial\log(F_{11}/F_{22})}{\partial u_2}f\frac{\partial h}{\partial u_2}\\
&-\frac{1}{2}F^{22}\Big(f\frac{\partial^2 h}{\partial u_2\partial u_2}+\frac{3}{2}\frac{\partial f}{\partial u_2}\frac{\partial h}{\partial u_2}\Big)
\end{aligned}
\tag{4.7'}
$$

and

$$
\begin{aligned}
R_{44}-\lambda F_{44} &= -\frac{1}{2a}F^{22}\Big\{\frac{\partial\log(F_{11}/F_{22})}{\partial u_2}fh\frac{\partial h}{\partial u_2}+3h\frac{\partial f}{\partial u_2}\frac{\partial h}{\partial u_2}\\
&+2fh\frac{\partial^2 h}{\partial u_2\partial u_2}+2f\frac{\partial h}{\partial u_2}\frac{\partial h}{\partial u_2}\Big\}+\frac{4}{u_4u_4}
\end{aligned}
\tag{4.9'}
$$

$$+ \quad \frac{1}{a(u_4)^4}\Big\{-\frac{1}{4}F^{11}\frac{\partial\log(F_{11}/F_{22})}{\partial u_1}\frac{\partial\log f}{\partial u_1}+\frac{1}{4}F^{22}\frac{\partial\log(F_{11}/F_{22})}{\partial u_2}\frac{\partial\log f}{\partial u_2}$$
$$+ \quad \frac{1}{2}\sum_{\alpha}F^{\alpha\alpha}\Big(\frac{1}{f}\frac{\partial^2 f}{\partial u_\alpha\partial u_\alpha}-\frac{1}{2f^2}\frac{\partial f}{\partial u_\alpha}\frac{\partial f}{\partial u_\alpha}\Big)\Big\},$$

which cannot vanish as a function of u_1, u_2, u_4.Hence we see that there exist no solutions for the case (ii). Thus, we obtain the following conclusion.

Theorem 2. We cannot find solutions g_{ij} satisfying the Einstein condition for Case II:

$F_{34}/F_{33} \neq$ constant, where

$$F_{\alpha\beta} = F_{\alpha\beta}(u_1, u_2), \quad F_{12} = 0, \quad F_{\alpha\lambda} = 0, \quad \alpha, \beta = 1, 2; \lambda = 3, 4,$$

and

$$F_{3\lambda} = F_{3\lambda}(u_1, u_2), \quad F_{44} = F_{44}(u_1, u_2, u_4).$$

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