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A NOTE ON THE DEGENERATE MORSE INEQUALITIES

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Abstract. In this paper we give an analytic proof of the degenerate Morse inequalities in the spirit of E. Witten. The max-min methods are used to estimate the number of 'small' eigenvalues of Witten's deformed Laplacian.

1. Introduction

E. Witten explained in [8] how Morse functions, non-degenerate and degenerate, naturally determine subcomplexes of the de Rham complex which also computes the de Rham cohomology. Such complexes therefore produce an abundant supply of so called Morse inequalities.

In this paper we give an analytic proof of the degenerate Morse inequalities in the spirit of E. Witten. The max-min methods are used to estimate the number of 'small' eigenvalues of Witten's deformed Laplacian. In [2], Bismut proved the nondegenerate and the degenerate Morse inequalities by using heat equation and probability. The methods used in this paper are closely related to Simon et al [6], see also Chang [5] and Roe [7, Chapter 9,14] for reference. Witten's arguments can be formulated as Theorem 4 and 6 of sections 2 and 3 respectively. Theorem 4 concerns the relationship of the cohomology of Witten complex on vector bundle and the twisted cohomology of the base manifold. This theorem was proved in [2] by a local to global argument using Mayer and Vietoris patching arguments over a good cover. Lemma 5 of §2 gives a simpler proof of its starting case. Theorem 6 shows that the number of 'small' eigenvalues of the Witten's deformed Laplacian can be estimated by the dimension of the twisted cohomology of critical submanifolds. The similar theorem for nondegenerate cases was proved in [6].

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2. Preliminaries

The purpose of this section is to study the operators determined by the Witten's deformed Laplacian in neighborhoods of critical submanifolds of a degenerate Morse function.

Let M be a compact Riemannian manifold of dimension n, $\pi : E \to M$ be a Riemannian vector bundle with Riemannian connection $\overline{\nabla}$. Suppose the metric of M is locally defined by $\sum \omega^i \otimes \omega^i$, the structure equations are $d\omega^i = \sum \omega^j \wedge \overline{\omega}^i_j$, $\overline{\omega}^j_i + \overline{\omega}^i_j = 0$. Let $(y^1, \cdots, y^m)$ and $\overline{\omega}^\beta_\alpha$ be fibre coordinates and connection forms of $\overline{\nabla}$ with respect to some orthonormal sections of E respectively, $m = \text{rank}E$, $\overline{\Omega}^\beta_\alpha = d\overline{\omega}^\beta_\alpha - \sum \overline{\omega}^\gamma_\alpha \wedge \overline{\omega}^\beta_\gamma = \sum \frac{1}{2} R^\beta_{\alpha\ ij} \omega^i \wedge \omega^j$ be the curvature forms. We have used the following indices:

$$1 \leq A, B, \cdots \leq n+m;\ 1 \leq i, j, \cdots \leq n;\ n+1 \leq \alpha, \beta, \cdots \leq n+m.$$

By the theory of connection, the tangent space of E can be decomposed as a direct sum of vertical and horizontal subspaces which can be determined by the one forms ω^i, $\omega^\alpha = dy^\alpha + \sum y^\beta \overline{\omega}^\alpha_\beta$. For each $t > 0$, we define metric $g(t)$ on E locally by

$$ds^2(t) = \frac{1}{t} \sum_i \omega^i \otimes \omega^i + \sum_\alpha \omega^\alpha \otimes \omega^\alpha.$$

Lemma 1. *The structure equations of Riemannian manifold $(E, g(t))$ are*

$$d(\omega^i/\sqrt{t}) = \sum \omega^j/\sqrt{t} \wedge \omega^i_j(t) + \sum \omega^\alpha \wedge \omega^i_\alpha(t),$$

$$d\omega^\alpha = \sum \omega^i/\sqrt{t} \wedge \omega^\alpha_i(t) + \sum \omega^\beta \wedge \omega^\alpha_\beta,$$

where $\omega^i_j(t) = \overline{\omega}^i_j + \sum \frac{1}{2} t R^\alpha_{\beta\ ji} y^\beta \omega^\alpha$, $\omega^i_\alpha(t) = -\omega^\alpha_i(t) = \sum \frac{1}{2} t R^\beta_{\alpha\ ij} y^\beta \omega^j/\sqrt{t}$, $\omega^\beta_\alpha = \overline{\omega}^\beta_\alpha$.

The proof is a direct computation, so we omit it.

Assuming that the bundle E is split into two subbundles, $E = E^+ \oplus E^-$, y^+, y^- are fibre coordinates defined as above. Let $h(x,y) = \frac{1}{2}|y^+|^2 - \frac{1}{2}|y^-|^2$ be a function on E (x be local coordinates on M). Let $\delta(t)$ be the formal adjoint of d with respect to the inner product $(,)_t$ on $\bigwedge(E)$ and $d_t = e^{-th} d e^{th}$, $\delta_t = e^{th}\delta(t)e^{-th}$ as defined by Witten [8]. The Laplacian of operator $d_t + \delta_t$ is

$$H_t = (d_t + \delta_t)^2 = (d + \delta(t))^2 + t^2 |dh|^2 + tG.$$

where G is a bundle homomorphism of $\bigwedge E$.

Lemma 2. (1) $< \varphi^*_{\sqrt{t}} u, \varphi^*_{\sqrt{t}} v >_t = t^r \varphi^*_{\sqrt{t}}(< u, v >_1)$, $u, v \in \bigwedge^r(E)$;

(2) $H_t = \varphi^*_{\sqrt{t}} \cdot tH_1 \cdot \varphi^*_{1/\sqrt{t}}$,
where the map φ_s *is defined by* $\varphi_s(e) = se,\ e \in E$.

Proof. (2) is an easy consequence of (1). $\square$

In the following lemma, D is the Levi-Civita connection of metric $g(t)$.

Lemma 3. *If* $u \in L^2(\bigwedge(E))$, $(H_t u, u)_t \leq tK_0(u,u)_t$ *for some constant* $K_0 > 0$, *then we have*

(1) $\int_K < u, u >_t \leq \frac{1}{\varepsilon^2 t}(C_1 + K_0)(u,u)_t$,
(2) $\int_{|y|\leq 1} < Du, Du >_t \leq (t^2 C_2 + tK_0)(u,u)_t$ *for* $t \geq 1$,

where $K = \{e \in E|\ |dh|(e) \geq \varepsilon\}$, C_1 *and* C_2 *are constants.*

Proof. Obviously,

$$t^2\varepsilon^2 \int_K < u, u >_t \leq t^2(|dh|^2 u, u)_t \leq (H_t u, u)_t - t(Gu, u)_t.$$

Denote $h = \sum \frac{1}{2}\varepsilon_\alpha (y^\alpha)^2$, $\varepsilon^\alpha = \pm 1$. Using Lemma 1, we can show that $G = \sum \varepsilon_\alpha \omega^\alpha i(e_\alpha) - \sum \varepsilon_\alpha i(e_\alpha)\omega^\alpha$. Then there is a constant C_1 such that $|(Gu,u)_t| \leq C_1(u,u)_t$. (1) has been proved.

(2) can be proved as follows: Let ρ_1, ρ_2 be cut-off functions on E such that $0 \leq \rho_1, \rho_2 \leq 1$, $\rho_1^2 + \rho_2^2 = 1$, the support of ρ_1 is compact and the set $\{|y| \leq 1\}$ is contained in the subset $\rho_1^{-1}(1)$. Then from (see [6], Theorem 3.2)

$$H_t = \sum_{l=1}^{2} \rho_l H_t \rho_l - \sum_{l=1}^{2} (D\rho_l)^2,$$

we have

$$\begin{aligned}
&((d + \delta(t))^2 \rho_1 u, \rho_1 u)_t \\
&= (D\rho_1 u, D\rho_1 u)_t + \sum (\omega^A i(e_B) R^D_{CAB} \omega^C i(e_D)\rho_1 u, \rho_1 u)_t \\
&\leq tK_0(u,u)_t - t(Gu,u)_t + \sum_l ((D\rho_l)^2 u, u)_t,
\end{aligned}$$

where $\{e_A\}$ are orthonormal basis and R^D_{CAB} are curvature components of D. By Lemme 1, there is a constant C_2' such that, on the support of ρ_1, $|R^D_{CAB}| \leq t^2 C_2'$. This completes the proof. $\square$

Theorem 4. *The kernel of* $H_t : L^2(\bigwedge^{p+n^-}(E)) \longrightarrow L^2(\bigwedge^{p+n^-}(E))$ *is isomorphic to the twisted cohomology* $H^p(M, o(E^-))$ *for each* p, *where* $o(E^-)$ *is the orientation bundle of* E^- *and* $n^- = rankE^-$.

Let N be an open set in M whose closure $\overline{N}$ is diffeomorphic to a closed ball in $\mathbf{R}^n$, $\overline{E} = E|_{\overline{N}}$. Define

$$H_t^k(\overline{E}) = \ker(d_t|L^2(\bigwedge^k(\overline{E})))/\mathrm{im}(d_t|L^2(\bigwedge^{k-1}(\overline{E}))).$$

Since N and $\overline{N}$ are homotopically equivalent spaces, the cohomology $H_t^*(\overline{E})$ and $H_t^*(E|_N)$ are isomorphic. As in Bismut [2], we need only to prove the following lemma.

Lemma 5. *The cohomology group $H_t^k(\overline{E})$ is isomorphic to the twisted cohomology group $H^{k-n^-}(\overline{N}, o(\overline{E}^-))$.*

Proof. Let δ_t be the formal adjoint of d_t with respect to the trivial metric of $\overline{E} \cong \overline{N} \times \mathbf{R}^m$. Set $\bigwedge_T^k(\overline{E}) = \{\xi \in \bigwedge^k(\overline{E})|\ i^*(\star\xi) = 0\}$, here $i : \partial\overline{E} \to \overline{E}$ is the inclusion map and $\star$ is the Hodge star operator. By Stokes theorem, we can show that the following decomposition holds (cf. [1], p.444)

$$L^2(\bigwedge^k(\overline{E})) = L^2(d_t(\bigwedge^{k-1}(\overline{E}))) \oplus \{\xi \in L^2(\bigwedge_T^k(\overline{E}))|\delta_t\xi = 0\}.$$

Then the cohomology group $H_t^k(\overline{E})$ is isomorphic to the space of harmonic k-forms $\{\xi \in L^2(\bigwedge_T^k(\overline{E}))|\ d_t\xi = \delta_t\xi = 0\}$. Let $d_t + \delta_t = P_1 + P_2 + tdh + ti(dh)$, where P_1 and P_2 are operators on $\overline{N}$ and the fibres of $\overline{E}$ respectively. The operators P_1 and $P_2 + tdh + ti(dh)$ are formal selfadjoint and we also have

$$H_t = P_1^2 + P_2^2 + t^2|dh|^2 + tG.$$

If $\xi \in L^2(\bigwedge_T^k(\overline{E}))$ is a harmonic form, that is $H_t\xi = 0$, then

$$-P_1\xi = (P_2 + tdh + ti(dh))\xi \in \bigwedge_T(\overline{E}).$$

Hence

$$\begin{aligned}(H_t\xi, \xi)_t &= (P_1^2\xi, \xi)_t + ((P_2^2 + t^2|dh|^2 + tG)\xi, \xi)_t \\ &= \|P_1\xi\|_t^2 + \|(P_2 + tdh + ti(dh))\xi\|_t^2.\end{aligned}$$

Note that $P_2^2 + t^2|dh|^2 + tG$ are harmonic oscillators acting on the fibres of $\overline{E}$. Then from

$$P_1^2\xi = 0, \quad (P_2^2 + t^2|dh|^2 + tG)\xi = 0,$$

we have $\xi = \pi^*\overline{\xi} \wedge \Psi$, $\Psi = \exp(-\frac{1}{2}t|y|^2) \cdot dy^1 \wedge \cdots \wedge dy^{n^-}$, $P_1\overline{\xi} = 0$, $\overline{\xi} \in \Gamma(\bigwedge_T(\overline{N}))$, Since $\overline{N}$ is diffeomorphic to a closed disk of $\mathbf{R}^n$, $\overline{\xi}$ is constant on $\overline{N}$. The lemma is proved. □

3. The proof of degenerate Morse inequalities

Let M be a compact smooth manifold of dimension n and h be a degenerate Morse function of M (see Bott [3], [4]), Let $M_1, \cdots, M_r$ be the connected critical submanifolds of h and E_l the normal bundle of M_l in M. Let $g(t)$ be a family of metrics on M, $t > 0$. We can make the following assumptions:

(1) Each bundle E_l split into two subbundles: $E_l = E_l^+ \oplus E_l^-$, formed by positive (negative) eigenvectors of Hessian d^2h respectively, $n_l^- = \text{rank} E_l^-$. In a neighborhood of M_l in M

$$h(x,y) = c + \frac{1}{2}(|y^+|^2 - |y^-|^2),$$

where c is a constant and (y^+, y^-) are bundle coordinates on E_l.

(2) For each l, there is a family of metric on E_l defined as in section 2 which consistent with $g(t)$ in a neighborhood of M_l.

As Witten [8], define a family of operators, $d_t = e^{-th} d e^{th}$, $\delta_t = e^{th} \delta e^{-th}$, where $\delta = \delta(t)$ is the adjoint of d with metric $g(t)$. Then we have

$$H_t = (d_t + \delta_t)^2 = \triangle + t^2 |dh|^2 + tG,$$

where $\triangle = (d+\delta)^2$ and G is a differential operator of order zero.

As in section 2, let $H^*(M_l, o(E_l^-))$ be the twisted cohomology of M_l. Denote $m_k = \sum\limits_{l=1}^{r} \sum\limits_{p+n_l^-=k} \dim H^p(M_l, o(E_l^-))$. If h is a non-degenerate Morse function, m_k is the number of critical points of h with index k. Let $E_q^k(t)$ be q-th eigenvalue of H_t acting on $\Gamma(\bigwedge^k(M))$ counting multiplicity. The Morse inequalities follows from the following theorem directly.

Theorem 6. *(1)* $\lim\limits_{t\to\infty} E_q^k(t)/t = 0, \quad q = 1, 2, \cdots, m_k;$

(2) $\lim\limits_{t\to\infty} E_q^k(t)/t > a, \quad q > m_k,$

where $a > 0$ *is a constant.*

Proof. The methods used here are essentially due to [2] and [6]. Let $\rho : [0, \infty) \longrightarrow [0,1]$ be a smooth function such that $\rho(s) = 1$, if $s \leq \frac{1}{2}$; $\rho(s) = 0$ if $s \geq 1$. Define functions J_l on E_l by $J_l(x,y) = \rho(|y|)$, then $J_0, J_1, \cdots, J_r$ also can be looked as functions on M, $J_0 = (1 - \sum\limits_{l=1}^{r} J_l^2)^{\frac{1}{2}}$. By Theorem 4, $\dim \oplus_{l=1}^r \ker[H_t^l|_{L^2(\wedge^k(E_l))}] = m_k$, where H_t^l is determined by H_t in a neighborhood of M_l, $l = 1, \cdots, r$. Let $\varphi^1, \cdots, \varphi^{m_k}$ be an orthonormal basis of $\oplus_{l=1}^r \ker[H_t^l|_{L^2(\wedge^k(E_l))}]$ and $\overline{\varphi}^q = J_l \varphi^q$ if φ^q in the kernel of H_t^l. Let V be a subspace of $\Gamma(\bigwedge^k(M))$ generated by $\overline{\varphi}^1, \cdots, \overline{\varphi}^{m_k}$.

Let D be the Levi-Civita connection of metric $g(t)$. By Lemma 1,

$$H_t\overline{\varphi}^q = -2\sum \frac{\partial J_l}{\partial y^\alpha} D_{e_\alpha}\varphi^q + \triangle J_l \cdot \varphi^q.$$

Since the support of $\frac{\partial J_l}{\partial y^\alpha}$ is in the set $\{1/2 \leq |y| \leq 1\}$, from Lemma 3, we have

$$(H_t\overline{\varphi}^p, \overline{\varphi}^q)_t = O(t)O(1/\sqrt{t}) = O(\sqrt{t}),$$
$$(\overline{\varphi}^p, \overline{\varphi}^q)_t = \delta_{pq} + O(1/t).$$

(1) has been proved.

On the other hand, if $u \perp V$, we have $(u, J_l\varphi^q)_t = (J_l u, \varphi^q)_t = 0$. By Lemma 2 of section 2, there is a constant a such that $\inf\limits_{v\perp \ker H_t^l} \frac{(H_t^l v,v)_t}{(v,v)_t} \geq ta$ for each l. Using

$$(H_t u, u)_t = \sum_{l=0}^{r} (H_t(J_l u), J_l u)_t - \sum_{l=0}^{r} ((DJ_l)^2 u, u)_t,$$

we can show that $(H_t u, u)_t \geq \frac{1}{2} ta(u,u)_t$ for large t, see also [6]. As the operators H_t^l have discrete spectrum (see Bismut [2], p.228), we know that $a > 0$. $\square$

As explained in Witten [8], the zero and the 'small' eigenspaces of H_t^l gives a finite dimensional subcomplex of the de Rham complex which also computes the de Rham cohomology. Let c_k be the k-th Betti number of M, m_k be defined as above. We have proved the degenerate Morse inequalities for Morse function h.

Theorem 7. *For any k with $1 \leq k \leq n$ the following inequality holds:*

$$c_k - c_{k-1} + \cdots + (-1)^k c_0 \leq m_k - m_{k-1} + \cdots (-1)^k m_0.$$

For $k = n$, the equality holds.

Let $\chi(M)$ be the Euler number of M, $\chi(M_l, o(E_l^-))$ be the Euler number of cohomology $H^*(M_l, o(E_l^-))$. From Theorem 7 we have

Corollary 8. $\chi(M) = \sum\limits_{l=1}^{r} (-1)^{n_l^-} \chi(M_l, o(E_l^-))$, *where n_l^- is the Morse index of M_l.*

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