Math. J. Okayama Univ. **42** (2000), 73-81

LIE ALGEBRAS REPRESENTED AS A SUM OF TWO SUBALGEBRAS

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ABSTRACT. Let L be a Lie algebra represented as a sum of two subalgebras A and B. We prove that if L belongs to a subclass of the class of locally finite Lie algebras over a field of characteristic $\neq 2$ and both A and B are locally nilpotent, then L is locally soluble. We also prove that if L is a serially finite Lie algebra over a field of characteristic zero, then any common serial subalgebra of A and B is serial in L.

1. Introduction

Groups G factorized by two subgroups A and B, i.e. $G = AB$, have been investigated by many authors for some decades. Among the works Kegel [8] and Wielandt [15] established the well-known theorem: if G is finite, and A and B are nilpotent, then G is soluble.

In Lie algebras there is a corresponding result: If a finite-dimensional Lie algebra L over a field $\mathfrak{k}$ of characteristic $\neq 2$ is represented as a sum of two nilpotent subalgebras A and B, then L is soluble. Goto [4] proved the case of char $\mathfrak{k} = 0$ and Panyukov [10] did the case of char $\mathfrak{k} = p > 2$. On the other hand, Aldosray [2] showed that if $L = A + B$ is an ideally finite Lie algebra over a field of characteristic zero, then any common ascendant subalgebra of both A and B is ascendant in L.

In this paper we shall generalize the result of Goto and Panyukov to a certain class of infinite-dimensional Lie algebras and extend the result of Aldosray to a wider class than that of ideally finite Lie algebras.

In Section 2 we shall show that in a locally finite Lie algebra L a common weakly serial subalgebra of each subalgebra X_i of L for $i \in I$ is always a weakly serial subalgebra of $\langle X_i \mid i \in I\rangle$ (Theorem 2). Let L be a Lie algebra represented as a sum of two subalgebras A and B. In Section 3 we shall prove that if L is a serially finite Lie algebra (resp. a hyperfinite, serially finite Lie algebra) over a field of characteristic zero, then any common serial (resp. ascendant) subalgebra of A and B is serial (resp. ascendant) in L (Theorem 8 (resp. Corollary 9)). In Section 4 we shall verify that if L belongs to the subclass $\text{L}(\text{wser})\mathfrak{F}$ of the class of locally

1991 *Mathematics Subject Classification.* 17B65, 17B30.

finite Lie algebras over a field of characteristic $\neq 2$ and both A and B are locally nilpotent, then L is locally soluble (Theorem 15).

2. Notation and terminology

Throughout the paper Lie algebras are not necessarily finite-dimensional over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified. We mostly follow [3] for the use of notation and terminology.

Let L be a Lie algebra over $\mathfrak{k}$ and let H be a subalgebra of L. For a totally ordered set Σ, a series (resp. a weak series) from H to L of type Σ is a collection $\{\Lambda_\sigma, V_\sigma \mid \sigma \in \Sigma\}$ of subalgebras (resp. subspaces) of L such that

(1) $H \subseteq V_\sigma \subseteq \Lambda_\sigma$ for all $\sigma \in \Sigma$,
(2) $\Lambda_\tau \subseteq V_\sigma$ if $\tau < \sigma$,
(3) $L \backslash H = \cup_{\sigma\in\Sigma}(\Lambda_\sigma \backslash V_\sigma)$,
(4) $V_\sigma \lhd \Lambda_\sigma$ (resp. $[\Lambda_\sigma, H] \subseteq V_\sigma$) for all $\sigma \in \Sigma$.

H is a serial (resp. a weakly serial) subalgebra of L, which we denote by $H\mathrm{ser}L$ (resp. $H\mathrm{wser}L$), if there exists a series (resp. a weak series) from H to L. For an ordinal σ, H is a σ-step ascendant (resp. weakly ascendant) subalgebra of L, denoted by $H \lhd^\sigma L$ (resp. $H \leq^\sigma L$), if there exists an ascending chain $(H_\alpha)_{\alpha\leq\sigma}$ of subalgebras (resp. subspaces) of L such that

(1) $H_0 = H$ and $H_\sigma = L$,
(2) $H_\alpha \lhd H_{\alpha+1}$ (resp. $[H_{\alpha+1}, H] \subseteq H_\alpha$) for any ordinal $\alpha < \sigma$,
(3) $H_\lambda = \cup_{\alpha<\lambda} H_\alpha$ for any limit ordinal $\lambda \leq \sigma$.

H is an ascendant (resp. a weakly ascendant) subalgebra of L, denoted by $H\mathrm{asc}L$ (resp. $H\mathrm{wasc}L$), if $H \lhd^\sigma L$ (resp. $H \leq^\sigma L$) for some ordinal σ. When σ is finite, H is a subideal (resp. a weak subideal) of L and denoted by $H\mathrm{si}L$ (resp. $H\mathrm{wsi}L$). For an ordinal α, we denote by $L^{(\alpha)}$ the α-th term of the transfinite derived series of L. A subspace H of L invariant under all derivations of L is said to be a characteristic ideal and denoted by $H\mathrm{ch}L$.

Let $\mathfrak{X}, \mathfrak{Y}$ be classes of Lie algebras and let Δ be any of the relations $\leq, \lhd, \mathrm{ch}, \mathrm{si}, \mathrm{asc}, \mathrm{ser}, \mathrm{wser}$. $\mathfrak{X}\mathfrak{Y}$ is the class of Lie algebras L having an ideal $I \in \mathfrak{X}$ such that $L/I \in \mathfrak{Y}$. A Lie algebra L is said to lie $\mathrm{L}(\Delta)\mathfrak{X}$ if for any finite subset X of L there exists an $\mathfrak{X}$-subalgebra H of L such that $X \subseteq H \ \Delta \ L$. In particular we write $\mathrm{L}\mathfrak{X}$ for $\mathrm{L}(\leq)\mathfrak{X}$. When $L \in \mathrm{L}\mathfrak{X}$ (resp. $\mathrm{L}(\mathrm{ser})\mathfrak{X}$), L is called a locally (resp. a serially) $\mathfrak{X}$-algebra. $\mathfrak{F}, \mathfrak{A}, \mathfrak{N}, \mathfrak{Z}$ and $\mathrm{E}\mathfrak{A}$ are the classes of Lie algebras which are finite-dimensional, abelian, nilpotent, hypercentral and soluble respectively. The $\mathfrak{X}$-residual $\lambda_{\mathfrak{X}}(L)$ of L is the intersection of the ideals I of L such that $L/I \in \mathfrak{X}$. $\acute{\mathrm{E}}_\mu(\Delta)\mathfrak{X}$ is the class of Lie algebras L having an ascending series $(L_\alpha)_{\alpha\leq\mu}$ of Δ-subalgebras such that

(1) $L_0 = 0$ and $L_\mu = L$,
(2) $L_\alpha \lhd L_{\alpha+1}$ and $L_{\alpha+1}/L_\alpha \in \mathfrak{X}$ for any ordinal $\alpha < \mu$,
(3) $L_\lambda = \cup_{\alpha<\lambda} L_\alpha$ for any limit ordinal $\lambda \leq \mu$.

We define $\acute{\mathrm{E}}(\Delta)\mathfrak{X} = \cup_{\mu>0}\acute{\mathrm{E}}_\mu(\Delta)\mathfrak{X}$. In particular we write $\acute{\mathrm{E}}\mathfrak{X}$ for $\acute{\mathrm{E}}(\leq)\mathfrak{X}$. When $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{X}$, L is called a hyper $\mathfrak{X}$-algebra. The Hirsch-Plotkin radical $\rho(L)$ of L is the unique maximal locally nilpotent ideal of L. For a locally finite Lie algebra L the locally soluble radical $\sigma(L)$ of L is the unique maximal locally soluble ideal of L. The set of left Engel elements of L is denoted by $\mathfrak{e}(L)$.

3. Common weakly serial subalgebras

Before considering a common ascendant subalgebra of two permutable subalgebras in Section 4, we shall state more general forms in the following interesting theorem, which is a generalization of [12, Theorem 7]. To do this we need the following useful result.

Lemma 1 ([5, Theorem 2.12]). *Let H be a subalgebra of a locally finite Lie algebra L. Then $H \mathrm{wser} L$ if and only if $\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L$ and $H/\lambda_{\mathrm{L}\mathfrak{N}}(H) \subseteq \mathfrak{e}(L/\lambda_{\mathrm{L}\mathfrak{N}}(H))$.*

Theorem 2. *Let L be a locally finite Lie algebra over any field and let $\{X_i\}_{i\in I}$ be a collection of subalgebras of L. If H is a common weakly serial subalgebra of X_i for any $i \in I$, then H is a weakly serial subalgebra of $\langle X_i \mid i \in I\rangle$.*

Proof. We may put $L = \langle X_i \mid i \in I\rangle$. Using Lemma 1 we have $\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd X_i$ for any $i \in I$, and so $\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L$. We may also assume that $\lambda_{\mathrm{L}\mathfrak{N}}(H) = 0$ by $\lambda_{\mathrm{L}\mathfrak{N}}(H/\lambda_{\mathrm{L}\mathfrak{N}}(H)) = 0$ and [5, Proposition 2.5]. Then we get $H \subseteq \mathfrak{e}(X_i)$ for all $i \in I$ by using Lemma 1.

On the other hand, L is spanned by the elements of a form $[x_1, x_2, \dots, x_n]$, where each x_k belongs to $\cup_{i\in I} X_i$. For any $h \in H$, there is an $m \in \mathbb{N}$ such that $x_k(\mathrm{ad}\ h)^m = 0$ for $1 \leq k \leq n$. Then we can show that

$$[x_1, x_2, \dots, x_n](\mathrm{ad}\ h)^{nm} = 0$$

by induction on n, using Leibniz formula. Therefore we have $H \subseteq \mathfrak{e}(L)$. Thus it follows from Lemma 1 that $H \mathrm{wser} L$. □

As a direct result of Theorem 2, we have the following:

Corollary 3. *Let L be a Lie algebra over any field and let $\{X_i\}_{i\in I}$ be a collection of subalgebras of L.*

(1) *If $L \in \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F})$ and $H \mathrm{asc} X_i$ for any $i \in I$, then $H \mathrm{asc}\langle X_i \mid i \in I\rangle$.*
(2) *If $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}$ and $H \mathrm{wasc} X_i$ for any $i \in I$, then $H \mathrm{wasc}\langle X_i \mid i \in I\rangle$.*
(3) *If $L \in \mathrm{L}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ and $H \mathrm{ser} X_i$ for any $i \in I$, then $H \mathrm{ser}\langle X_i \mid i \in I\rangle$.*

Proof. (1) Since $\acute{\mathrm{E}}(\lhd)\mathfrak{F} \leq \mathrm{L}\mathfrak{F}$ by [7, Corollary 3.3] we obtain $L \in \mathrm{L}\mathfrak{F}$. Hence Theorem 2 implies that $H \mathrm{wser} \langle X_i \mid i \in I \rangle$. Because $\langle X_i \mid i \in I \rangle \in \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F})$ we conclude from [6, Proposition 2] that $H \mathrm{asc} \langle X_i \mid i \in I \rangle$.

(2) and (3) follow from [6, Theorem 1] and [5, Theorem 2.7] respectively as in the proof of (1). □

4. Common ascendant subalgebras

Let L be a Lie algebra and let A, B be subalgebras of L. As in groups we say that L is *factorized* by A and B if $L = A + B$.

Let L be factorized by A and B, and let $H \leq A \cap B$. In this section we shall consider some conditions under which $H \mathrm{asc} A$ and $H \mathrm{asc} B$ implies $H \mathrm{asc} L$. First we easily see the following:

Lemma 4. *Let $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$ and let $L = A+B$ be the sum of two subalgebras A and B. If $H \mathrm{asc} A$ and $H \mathrm{asc} B$, then $H \mathrm{asc} L$.*

Proof. Since $H \mathrm{wasc} A$ and $H \mathrm{wasc} B$, it is evident that $H \mathrm{wasc} L$. Therefore [12, Corollary to Theorem 2] indicates $H \mathrm{asc} L$. □

Remark. As in the proof of Lemma 4, we can show the following, which is a generalization of [9, Corollary to Proposition 2] : Let $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$ and let $H \leq X_i$ $(i = 1, 2, \ldots, n)$ be subalgebras of L such that $\langle X_1, \ldots, X_n \rangle = X_1 + \cdots + X_n$. If $H \mathrm{asc} X_i$ for any i, then $H \mathrm{asc} \langle X_1, \ldots, X_n \rangle$.

The following is originally due to Tôgô and is a generalization of [9, Remark to Lemma 4], for it is clear that $\mathrm{E}\mathfrak{A} \cup \mathfrak{Z} \leq \acute{\mathrm{E}}(\mathrm{ch})\mathfrak{A}$ over any field.

Lemma 5. *Let L be a Lie algebra such that $L = H+K$ with $H \leq L, K \lhd L$ and $K \in \acute{\mathrm{E}}_\mu(\mathrm{ch})\mathfrak{A}$. If $H \leq^\lambda L$, then $H \lhd^{\lambda\mu} L$.*

Proof. Let $(K_\alpha)_{\alpha \leq \mu}$ be an ascending abelian series of characteristic ideals of K. We note that $K_\alpha \lhd L$ by [3, Lemma 1.4.4]. Therefore $K_\alpha \lhd H+K_\alpha \leq L$ for all $\alpha \leq \mu$. Now for any $\alpha < \mu$ we put $\overline{H} = (H + K_\alpha)/K_\alpha, \overline{K}_{\alpha+1} = K_{\alpha+1}/K_\alpha$. Then

$$\overline{K}_{\alpha+1} \lhd \overline{H} + \overline{K}_{\alpha+1} \quad \text{and} \quad \overline{K}_{\alpha+1} \in \mathfrak{A}.$$

On the other hand, we have $\overline{H} \leq^\lambda \overline{H} + \overline{K}_{\alpha+1}$ as $H \leq^\lambda H + K_{\alpha+1}$. By virtue of [12, Lemma 3], we obtain $\overline{H} \lhd^\lambda \overline{H} + \overline{K}_{\alpha+1}$. Hence $H + K_\alpha \lhd^\lambda H + K_{\alpha+1}$ for all $\alpha < \mu$. For any limit ordinal $\beta \leq \mu$ it is trivial that $H + K_\beta = \cup_{\alpha<\beta}(H + K_\alpha)$. Therefore it follows that $H \lhd^{\lambda\mu} L$. □

Now using Lemma 5 we can generalize [9, Proposition 5] to the following:

Proposition 6. *Let L be a Lie algebra such that $L = A + B = H + K$ with $A, B, H \leq L, K \lhd L$ and $K \in \acute{\mathrm{E}}_\mu(\mathrm{ch})\mathfrak{A}$. If $H \leq^\lambda A$ and $H \leq^\lambda B$, then $H \lhd^{\lambda\mu} L$.*

Proof. It is evident that $H \leq^\lambda L$. Therefore we conclude the assertion from Lemma 5. □

The following corresponds to [9, Theorem 6].

Proposition 7. *Let $\mathfrak{X}$ be a class of Lie algebras and suppose that $L = A + B \in \mathfrak{X}$ with $A, B \leq L, H\mathrm{asc}A$ and $H\mathrm{asc}B$, always implies that $H\mathrm{asc}L$. Then $L = A + B \in (\acute{\mathrm{E}}(\mathrm{ch})\mathfrak{A})\mathfrak{X}$ with $H\mathrm{asc}A$ and $H\mathrm{asc}B$ always implies that $H\mathrm{asc}L$.*

Proof. Let $L = A + B \in (\acute{\mathrm{E}}(\mathrm{ch})\mathfrak{A})\mathfrak{X}$ with $H \lhd^\lambda A$ and $H \lhd^\lambda B$. Then there exists an ideal K of L such that $K \in \acute{\mathrm{E}}_\mu(\mathrm{ch})\mathfrak{A}$ and $L/K \in \mathfrak{X}$. Here we denote images under the natural map $L \longrightarrow L/K$ by bars. Then

$$\overline{L} = \overline{A} + \overline{B} \in \mathfrak{X},\ \overline{H} \lhd^\lambda \overline{A} \ \text{ and } \ \overline{H} \lhd^\lambda \overline{B}.$$

By the hypothesis, there exists an ordinal $\alpha = \alpha(H, \lambda)$ such that $\overline{H} \lhd^\alpha \overline{L}$, so $H + K \lhd^\alpha L$. On the other hand, $H \leq^\lambda L$ since $H \leq^\lambda A$ and $H \leq^\lambda B$. Hence $H \leq^\lambda H+K$. On account of Lemma 5, it follows that $H \lhd^{\lambda\mu} H+K$. Thus we can reach that $H \lhd^{\lambda\mu+\alpha} L$. □

Let L be factorized by A and B over a field of characteristic zero and let $H\mathrm{asc}A$ and $H\mathrm{asc}B$. Then Aldosray proved that if $L \in \mathrm{L}(\lhd)\mathfrak{F}$ then $H\mathrm{asc}L$ ([2, Theorem 6]). We know the facts that $\mathrm{L}(\lhd)\mathfrak{F} \leq \acute{\mathrm{E}}(\lhd)\mathfrak{F}$ ([14, Lemma 4.1]) and that if $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}$, then the notion of serial subalgebras of L coincides with that of ascendant subalgebras of L ([6, Theorem 1]). Now we shall prove the main theorem in this section, which generalize the result of Aldosray.

Theorem 8. *Let L be a serially finite Lie algebra over a field of characteristic zero and let H, A, B be subalgebras of L such that $L = A + B$ and $H \leq A \cap B$. If H is a common serial subalgebra of both A and B, then H is serial in L.*

Proof. From [11, Theorem 5 and Corollary 6] it follows that

$$\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd A \quad \text{and} \quad H/\lambda_{\mathrm{L}\mathfrak{N}}(H) \leq \rho(A/\lambda_{\mathrm{L}\mathfrak{N}}(H)),$$

$$\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd B \quad \text{and} \quad H/\lambda_{\mathrm{L}\mathfrak{N}}(H) \leq \rho(B/\lambda_{\mathrm{L}\mathfrak{N}}(H)).$$

Hence we have $\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L$. Therefore it is enough to show that $H/\lambda_{\mathrm{L}\mathfrak{N}}(H) \leq \rho(L/\lambda_{\mathrm{L}\mathfrak{N}}(H))$. Now since $H\mathrm{wser}L$ by Theorem 2, Lemma 1 indicates

$$H/\lambda_{\mathrm{L}\mathfrak{N}}(H) \subseteq \mathfrak{e}(L/\lambda_{\mathrm{L}\mathfrak{N}}(H)).$$

Here we denote images under the natural map $L \longrightarrow L/\lambda_{\mathrm{L}\mathfrak{N}}(H)$ by bars. Then

$$\overline{L} = \overline{A} + \overline{B} \in \mathrm{L}(\mathrm{ser})\mathfrak{F},\ \ \overline{H}\mathrm{ser}\overline{A},\ \ \overline{H}\mathrm{ser}\overline{B},$$

$$\overline{H} \leq \rho(\overline{A}) \cap \rho(\overline{B}),\ \ \overline{H} \subseteq \mathfrak{e}(\overline{L}),$$

because of [3, Proposition 13.2.4]. Hence we may replace $\overline{L}, \overline{H}, \overline{A}, \overline{B}$ by L, H, A, B.

Then by [13, Theorem 2] L is, so-called, a neoclassical Lie algebra. That is to say, $L = \sigma(L)\dot{+}\Lambda$, where Λ is a direct sum of finite-dimensional, non-abelian simple subalgebras (see [3, Chapter 13]). As the first paragraph of the proof we set $\overline{L} = L/\sigma(L) = \overline{A} + \overline{B}$. Then

$$\overline{L} \cong \Lambda \in \mathrm{L}(\lhd)\mathfrak{F},\ \ \overline{H}\mathrm{ser}\overline{A},\ \ \overline{H}\mathrm{ser}\overline{B}.$$

Moreover $\overline{A}, \overline{B} \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}$ owing to [14, Lemma 4.1]. Hence we have $\overline{H}\mathrm{asc}\overline{A}$, $\overline{H}\mathrm{asc}\overline{B}$ using [6, Theorem 1(1)]. Now we can derive from [2, Theorem 6] that $\overline{H}\mathrm{asc}\overline{L}$, so $H + \sigma(L)\mathrm{asc}L$. Furthermore $H + \rho(L) \lhd H + \sigma(L)$ owing to [3, Corollary 13.3.13]. Hence $H + \rho(L)\mathrm{asc}L$. On the other hand we obtain $H \in \mathrm{L}\mathfrak{N}$ by $H \leq \rho(A) \cap \rho(B)$. As $H \subseteq \mathfrak{e}(L)$, H acts on $\rho(L)$ by nil derivations, which indicates $H + \rho(L) \in \mathrm{L}\mathfrak{N}$ by [3, Theorem 16.3.8(b)]. Thus we can reach $H + \rho(L) \leq \rho(L)$ by using [3, Theorem 13.3.7], that is, $H \leq \rho(L)$. This completes the theorem. □

By making use of Theorem 8 and [6, Theorem 1(1)], we can obtain a better result than [2, Theorem 6].

Corollary 9. *Let L be a hyperfinite, serially finite Lie algebra over a field of characteristic zero and be factorized by A and B. If H is a common ascendant subalgebra of both A and B, then H is ascendant in L.*

Remark. Over any field, $\mathrm{L}(\lhd)\mathfrak{F} < \acute{\mathrm{E}}(\lhd)\mathfrak{F} \cap \mathrm{L}(\mathrm{ser})\mathfrak{F}$. For, let X be an abelian Lie algebra with basis $\{x_0, x_1, \dots\}$ and let σ be the derivation of X defined by $x_0\sigma = 0$ and $x_{i+1}\sigma = x_i$ $(i \geq 0)$. Form the split extension $L = X\dot{+}\langle\sigma\rangle$. Then $L \in \mathfrak{Z} \leq \acute{\mathrm{E}}(\lhd)\mathfrak{F}\cap\mathrm{L}(\mathrm{ser})\mathfrak{F}$ but $L \notin \mathrm{L}(\lhd)\mathfrak{F}$ (see [6, Remark 1]).

Proposition 7 and Corollary 9 directly lead the following:

Corollary 10. *Let L be a Lie algebra belonging to $(\acute{\mathrm{E}}(\mathrm{ch})\mathfrak{A})(\acute{\mathrm{E}}(\lhd)\mathfrak{F} \cap \mathrm{L}(\mathrm{ser})\mathfrak{F})$ over a field of characteristic zero and be factorized by A and B. If $H\mathrm{asc}A$ and $H\mathrm{asc}B$, then $H\mathrm{asc}L$.*

Using Lemma 5 and Corollary 10, we can easily prove the following corollary, which is a generalization of [1, Corollaries 1 and 2].

Corollary 11. *Let L be a Lie algebra belonging to* $(\acute{\mathrm{E}}(\mathrm{ch})\mathfrak{A})(\acute{\mathrm{E}}(\lhd)\mathfrak{F} \cap \mathrm{L}(\mathrm{ser})\mathfrak{F})$ *over a field of characteristic zero and let X_i $(i = 1, 2, ..., n)$ be subalgebras of L such that $L = X_1 + X_2 + \cdots + X_n$ and $\langle X_i, X_j \rangle = X_i + X_j$ for all $i, j = 1, 2, ..., n$.*

(1) *If $H \mathrm{asc} X_i$ for all i, then $H \mathrm{asc} L$.*
(2) *For each i, if $X_i \mathrm{asc} \langle X_i, X_j \rangle$ for all j, then $X_i \mathrm{asc} L$.*

5. A generalization for the result of Goto and Panyukov

In this section we shall generalize the following result.

Lemma 12 (Goto, Panyukov)**.** *Let L be a finite-dimensional Lie algebra over a field of characteristic $\neq 2$. If L is represented as a sum of two nilpotent subalgebras A and B, then L is soluble.*

For our purpose we need the following two lemmas.

Lemma 13. *Let H be a finitely generated subalgebra of a Lie algebra L.*

(1) *If $H \mathrm{wasc} L$, then $H^{(\omega)} \mathrm{ch} L$.*
(2) *Assume that $L \in \mathrm{L}\mathfrak{F}$. If $H \mathrm{wser} L$, then $H^{(\omega)} \lhd L$.*

Proof. (1) Using [12, Theorem 4] we have $H \leq^{\omega} L$. Hence [5, Lemma 2.10] leads $H^{(\omega)} \lhd L$. Next form the split extension $M = L \dot{+} \mathrm{Der} L$. Then $H \mathrm{wasc} M$. The argument above indicates that $H^{(\omega)} \lhd M$, so $H^{(\omega)} \mathrm{ch} L$.

(2) For any $I \lhd H$ such that $H/I \in \mathrm{LE}\mathfrak{A}$, we have $H^{(\omega)} \leq I$ since $H/I \in \mathrm{E}\mathfrak{A}$. Therefore $H^{(\omega)} \leq \lambda_{\mathrm{LE}\mathfrak{A}}(H)$. Since, in general, $\lambda_{\mathrm{LE}\mathfrak{A}}(H) \leq H^{(\omega)}$, it follows from [5, Proposition 2.11] that $H^{(\omega)} = \lambda_{\mathrm{LE}\mathfrak{A}}(H) \lhd L$. □

Lemma 14. *Let L be a Lie algebra over a field of characteristic $\neq 2$ and let $L = A + B$ be a sum of Engel subalgebras A and B.*

(1) *If $H \in \mathfrak{F}$ and $H \mathrm{wasc} L$, then $H \in \mathrm{E}\mathfrak{A}$.*
(2) *Assume that $L \in \mathrm{L}\mathfrak{F}$. If $H \in \mathfrak{F}$ and $H \mathrm{wser} L$, then $H \in \mathrm{E}\mathfrak{A}$.*

Proof. (1) Because $H^{(\omega)}$ is a finite-dimensional ideal of L by Lemma 13, it follows from [3, Corollary 1.4.3] that

$$C_L(H^{(\omega)}) \lhd L \quad \text{and} \quad L/C_L(H^{(\omega)}) \in \mathfrak{F}.$$

Now we denote images under the natural map $L \longrightarrow L/C_L(H^{(\omega)})$ by bars. Then we have $\overline{L} \in \mathfrak{F}$ and $\overline{L} = \overline{A} + \overline{B}$ is a sum of nilpotent subalgebras $\overline{A}$ and $\overline{B}$. Therefore Lemma 12 shows $\overline{L} \in \mathrm{E}\mathfrak{A}$. In particular, $\overline{H} \in \mathrm{E}\mathfrak{A}$, so $H^{(\omega)} \subseteq C_L(H^{(\omega)})$. Hence $H^{(\omega+1)} = [H^{(\omega)}, H^{(\omega)}] = 0$. This concludes that $H \in \mathrm{E}\mathfrak{A}$.

(2) Since $H^{(\omega)} \lhd L$ by Lemma 13, we can show that $H \in \mathrm{E}\mathfrak{A}$ as in the proof of (1). □

Now we shall prove the main theorem in the section, which is a generalization of Lemma 12.

Theorem 15. *Let L be a Lie algebra over a field of characteristic $\neq 2$. If $L \in \mathrm{L}(\mathrm{wser})\mathfrak{F}$ and L is represented as a sum of two locally nilpotent subalgebras A and B, then L is locally soluble.*

Proof. Let X be a finite subset of L. Then there exists a subalgebra H of L such that $X \subseteq H \mathrm{wser} L$ and $H \in \mathfrak{F}$. Therefore it follows from Lemma 14(2) that $H \in \mathrm{E}\mathfrak{A}$. Thus $L \in \mathrm{LE}\mathfrak{A}$. □

Finally we shall state about any subalgebra of the intersection of permutable two locally nilpotent subalgebras.

Corollary 16. *Let L be a Lie algebra over a field of characteristic $\neq 2$ and let L be factorized by two locally nilpotent subalgebras A and B.*

(1) *If $L \in \mathrm{L}(\mathrm{wser})\mathfrak{F}$, then $H \mathrm{ser} L$ for any subalgebra H of $A \cap B$.*
(2) *If $L \in \mathrm{L}(\lhd)\mathfrak{F}$, then $H \lhd^{\omega} L$ for any subalgebra H of $A \cap B$.*

Proof. (1) Using [3, Proposition 13.2.4] we obtain $H \mathrm{ser} A$ and $H \mathrm{ser} B$. Since $L \in \mathrm{LE}\mathfrak{A}$ by Theorem 15 we conclude from Corollary 3 that $H \mathrm{ser} L$.

(2) From (1) we have $H \mathrm{ser} L \in \mathrm{LE}\mathfrak{A} \cap \mathrm{L}(\lhd)\mathfrak{F} = \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$. Therefore $H \lhd^{\omega} L$ in virtue of [5, Theorem 3.3]. □

Acknowledgements. The paper was prepared while the second author was a visiting fellow at the Centre for Mathematics and its Applications at The Australian National University, from March 1998 to March 1999. He would like to thank the staff for their warm hospitality.

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(*Received March 19,2001*)