Math. J. Okayama Univ. **42** (2000), 115-122

A CONVENIENT AXIOM TO CONVENIENT CATEGORIES FOR HOMOTOPY THEORY

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Introduction

As a suitable topology for spaces of cotinuous maps, Brown ([1]) does not adopt the usual compact-open topology but the test-open topology. The adoptation of this topology provides for example that if Y is locally compact and Hausdorff then the natural exponential map $C_2(X \times Y, Z) \to C_2(X, C_2(Y, Z))$ is a homeomorphism, where $C_2(X, Y)$ is the set $\mathcal{T}op(X, Y)$ endowed with the test-open topology. The exponential law is a generalization of the classical Hausdorff type one (Corollary 12 (i)). However, it seems for us that these conditions on Y is surplus and incongruous to the test-open topology. We replace these conditions with a more natural local condition (Theorem 9 (ii)). Moreover, we introduce an axiom which is slightly different from Axiom 2 of Vogt [5]. For each category $\mathcal{C}$ which satisfies our axiom we have both the exponential law in the categry of $k_{\mathcal{C}}$-spaces and similar one described above in $\mathcal{T}op$, by making use of Brown's method. For example, the second exponential law with respect to $\mathcal{C}_3$, the category of compact regular spaces, is a generalization of the classical regular type one.

1. $k_{\mathcal{C}}$-spaces, $\mathcal{C}$-test-open topology and Axiom

Let $\mathcal{C}$ be a non-empty full subcategory of the category $\mathcal{T}op$ of small topological spaces ([2]). By a $\mathcal{C}$-test map on X we mean a continuous map φ with the souce $S\varphi \in O\mathcal{C}$ and the target $T\varphi = X$. Let $\mathcal{C}_X$ denote the class of all $\mathcal{C}$-test maps on X. (The class is a set which may not be small in general ([2]).) Given a space X, we define a space $k_{\mathcal{C}}X$ to be the same underlying set endowed with the final topology with respect to all $\varphi \in \mathcal{C}_X$. The identity map $\varepsilon_X : k_{\mathcal{C}}X \to X$ is cotinuous and induces a bijection $\varepsilon_{X*} : \mathcal{C}_{k_{\mathcal{C}}X} \to \mathcal{C}_X$, i.e. for any $\varphi \in \mathcal{C}_X$ there exists an unique lifting $'\varphi \in \mathcal{C}_{k_{\mathcal{C}}X}$ such that $\varepsilon_X \circ {}'\varphi = \varphi$. These constructions define an idempotent functor $k_{\mathcal{C}} : \mathcal{T}op \to \mathcal{T}op$ and a natural transformation $\varepsilon : k_{\mathcal{C}} \dot{\to} 1_{\mathcal{T}op}$. A space X is called a $k_{\mathcal{C}}$-space if $k_{\mathcal{C}}X = X$. Let $\mathcal{K}_{\mathcal{C}}$ be the full subcategory of $\mathcal{T}op$ consisting of all $k_{\mathcal{C}}$-spaces. $\mathcal{C}$ is a subcategory of $\mathcal{K}_{\mathcal{C}}$.

Let $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_\omega$ be the category of compact Hausdorff spaces, of compact regular spaces and of compact spaces respectively. Our $\mathcal{K}_2 = \mathcal{K}_{\mathcal{C}_2}$ is the Brown's category of k-spaces which is denoted by $\mathcal{HG}$ in Vogt's paper [5].

The following proposition which is due to Brown ([1]) in $\mathcal{C}_2$ case is a characterization of $k_{\mathcal{C}}$-spaces.

Proposition 1. *X is a $k_{\mathcal{C}}$-space if and only if there exists a subfamily $(\varphi_\lambda)_{\lambda\in\Lambda}$ of $\mathcal{C}_X$ with small index set Λ such that $\pi : \coprod_{\lambda\in\Lambda} S\varphi_\lambda \to X$, $\pi_{|S\varphi_\lambda} = \varphi_\lambda$, is an identification.*

Proof. Since X is a $k_{\mathcal{C}}$-space, $\{(\varphi, B) | \varphi^{-1}B \notin \mathcal{F}_{S\varphi}, \varphi \in \mathcal{C}_X, B \notin \mathcal{F}_X\} \to \mathcal{P}_X - \mathcal{F}_X$, $(\varphi, B) \mapsto B$ is a surjection, where $\mathcal{P}_X$ is the power set of X and $\mathcal{F}_X, \mathcal{F}_{S\varphi}$ are the sets of closed sets in $X, S\varphi$ respectively. Applying Axiom of Choice to this surjection, we have a family $(\varphi_B)_{B\in\mathcal{P}_X-\mathcal{F}_X}$ of $\mathcal{C}_X$ such that $\varphi_B^{-1}B$ is not closed in $S\varphi$. Since $\mathcal{C}$ is not empty, for every $x \in X$ there exists a constant map c_x with the souce in $O\mathcal{C}$ and the value x. The function $\Lambda = (\mathcal{P}_X - \mathcal{F}_X) \coprod X \to \mathcal{C}_X$, given by $B \mapsto \varphi_B$ and $x \mapsto c_x$, defines a family $(\varphi_\lambda)_{\lambda\in\Lambda}$ which we need. Since π is a continuous surjection and every non-closed subset of X is not closed with respect to the identification topology, π is an identification. The converse is obvious. □

A space or its subspace in general is called $\mathcal{C}$-imaged if it is the image of a $\mathcal{C}$-test map and is called $\mathcal{C}$-objective if it is $\mathcal{C}$-imaged by a $\mathcal{C}$-test map which is an embedding. A space X is called $\mathcal{C}$-reflective if every $\mathcal{C}$-imaged subset of X is always $\mathcal{C}$-objective. With respect to such a property P it is usual to say a space X is locally P if each point of X has a fundamental system of neighbourhoods with the property P. We call X locally P in the weak sence if each point of X has a neighbourhood with the property P. We have the following

Proposition 2. *Suppose that X is locally $\mathcal{C}$-reflective in the weak sence. Then X is locally $\mathcal{C}$-objective if and only if it is locally $\mathcal{C}$-imaged.*

Proof. Let X be locally $\mathcal{C}$-imaged. Let W_x be a $\mathcal{C}$-reflective neighbourhood admitted by each $x \in X$. Then there exists a $\mathcal{C}$-test map φ such that $x \in \mathrm{Int}(\mathrm{Im}\varphi) \subset \mathrm{Im}\varphi \subset W_x$. Hence $\mathrm{Im}\varphi$ becomes $\mathcal{C}$-objective. The converse is trivial. □

McCord ([3]) proved that every weak Hausdorff space is $\mathcal{C}_2$-reflective. The inverse is also true. A subset A of a topological space X is closed if and only if A is closed in $A \cup \{b\}$ for every $b \notin A$. In particular let A be

$\mathcal{C}_2$-imaged in a $\mathcal{C}_2$-reflective space X. Then $A \cup \{b\}$ is also $\mathcal{C}_2$-imaged and therefore is Hausdorff. A compact set A is closed in $A \cup \{b\}$. Hence X is weak Hausdorff.

Since every compact Hausdorff space is weak Hausdorff we conclude that X is locally $\mathcal{C}_2$-objective if and only if it is locally $\mathcal{C}_2$-imaged and locally weak Hausdorff. The set of non-negative real numbers $[0, \infty)$ with the set of open sets $\{[0, a) | 0 \leq a \leq \infty\}$ is an example of locally $\mathcal{C}_2$-imaged space which is not locally $\mathcal{C}_2$-objective.

The following proposition is due to Oshima in $\mathcal{C}_2$ case ([4]).

Proposition 3. *Every space which is locally $\mathcal{C}$-objective in the weak sence is a $k_{\mathcal{C}}$-space.*

Proof. Let X be a space which is locally $\mathcal{C}$-objective in the weak sence and let N_x denote a $\mathcal{C}$-objective neighbourhood admitted by each $x \in X$. Let U be an open set in $k_{\mathcal{C}}X$. We remark that if A is open in a subspace N then $A \cap \mathrm{Int}N$ is open in X. This implies that $U \cap \mathrm{Int}N_x$ is open in X and therefore $U = \bigcup_{x \in X}(U \cap \mathrm{Int}N_x)$ is open in X, which complete the proof. □

Let $C(X, Y)$ be the set $\mathcal{T}op(X, Y)$ endowed with the $\mathcal{C}$-test-open topology, i.e. the topology generated by the subbase $\{W(\varphi, U) | \varphi \in \mathcal{C}_X, U \in \mathcal{O}_Y\}$, where $\mathcal{O}_Y$ is the set of open sets in Y and $W(\varphi, U) = W(\mathrm{Im}\varphi, U) = \{f \in \mathcal{T}op(X, Y) | f(\mathrm{Im}\varphi) \subset U\}$. The topology of $C_2(X, Y)$ is the Brown's test-open topology and that of $C_\omega(X, Y)$ is the usual compact-open topology.

Lemma 4. (i) *If $\iota : B \to Y$ is an embedding, so is $\iota_* : C(X, B) \to C(X, Y)$.*

(ii) *$f : X \to Z$ induces an embedding $f^* : C(Z, Y) \to C(X, Y)$ if for any $\psi \in \mathcal{C}_Z$ there exists a $\varphi \in \mathcal{C}_X$ such that $f(\mathrm{Im}\varphi) = \mathrm{Im}\psi$. Especially ${\varepsilon_X}^* : C(X, Z) \to C(k_{\mathcal{C}}X, Z)$ is an embedding.*

Proof. Let P, Q are topological spaces and $j : P \to Q$ be an injective set function. We remark that j is an embedding if (and only if) there exists a subbase $\mathcal{S}$ of Q such that $j^{-1}\mathcal{S} = \{j^{-1}S | S \in \mathcal{S}\}$ is a subbase of P. It is easy to see that the subbase $\{W(\varphi, U) | \varphi \in \mathcal{C}_X, U \in \mathcal{O}_Y\}$ satisfies the if-condition of the remark in both cases (i), (ii). It remains to complete the proof of (ii) by showing that $f^* : C(Z, Y) \to C(X, Y)$ is injective. Since every $\mathcal{C}$-imaged subset in Z is contained in $\mathrm{Im}f$ by the condition of (ii) and every one point subset is $\mathcal{C}$-imaged by a constant $\mathcal{C}$-map, f is surjective and hence f^* is injective. □

We now introduce the following axiom which is slightly different from Axiom 2 of Vogt [5].

Axiom. (i) The cartesian product of two spaces in $\mathcal{C}$ is again in $\mathcal{C}$.

(ii) Every object in $\mathcal{C}$ is compact and locally $\mathcal{C}$-imaged.

Since every object of $\mathcal{C}_2$, $\mathcal{C}_3$ is regular, these categories satisfy our axiom. If $\mathcal{C}$ satisfies Axiom, so is the full subcategory $\mathcal{C}' \subset \mathcal{T}op$ of compact locally $\mathcal{C}$-imaged spaces. We therefore have an increasing sequence, indexed by the ordinals, of categories which satisfy Axiom.

Proposition 5. *Suppose that $\mathcal{C}$ satisfies Axiom. Then the cartesian product of a $k_{\mathcal{C}}$-space and a space which is locally $\mathcal{C}$-objective in the weak sence is a $k_{\mathcal{C}}$-space.*

Proof. At first we remark that a space which is locally $\mathcal{C}$-objective in the weak sence is locally $\mathcal{C}$-imaged, and therefore is locally compact. Secondly, by making use of Proposition 1, 3, we deduce that P is a $k_{\mathcal{C}}$-space if and only if there exists an identification $L \to P$ such that L is locally $\mathcal{C}$-objective in the weak sence. Now let X be a $k_{\mathcal{C}}$-space and L_2 be locally $\mathcal{C}$-objective in the weak sence. Following the above second remark, we take an identification $\pi : L_1 \to X$ where L_1 is locally $\mathcal{C}$-objective in the weak sence. Since L_2 is localy compact by the first remark, $\pi \times L_2 : L_1 \times L_2 \to X \times L_2$ is also an identification ([1]). Moreover $L_1 \times L_2$ is locally $\mathcal{C}$-objective in the weak sence. Hence $X \times L_2$ is a $k_{\mathcal{C}}$-space. $\square$

Lemma 6. *Suppose that $\mathcal{C}$ satisfies Axiom. Then*

(i) *the topology of $C(k_{\mathcal{C}}(X \times Y), Z)$ is generated by the subbase $\{W('(\varphi \times \psi), U) | \varphi \in \mathcal{C}_X, \psi \in \mathcal{C}_Y, U \in \mathcal{O}_Z\}$,*

(ii) *the topology of $C(X \times Y, Z)$ is generated by the subbase $\{W(\varphi \times \psi, U) | \varphi \in \mathcal{C}_X, \psi \in \mathcal{C}_Y, U \in \mathcal{O}_Z\}$,*

(iii) *the topology of $C(X, T)$ is generated by $\{W(\varphi, S) | \varphi \in \mathcal{C}_X, S \in \mathcal{S}\}$ if the topology of T is generated by $\mathcal{S}$.*

Proof. (i) It suffices to prove that for every $W('\Phi, U), \Phi \in \mathcal{C}_{X \times Y}, U \in \mathcal{O}_Z$ and every $f \in W('\Phi, U)$ there exist $\varphi_p \in \mathcal{C}_X, \psi_p \in \mathcal{C}_Y, 1 \le p \le N$ such that $f \in \bigcap_{p=1}^N W('(\varphi_p \times \psi_p), U) \subset W('\Phi, U)$. We have the decomposition $\Phi = (\varphi, \psi) = (\varphi \times \psi) \circ \Delta$, where $\varphi \in \mathcal{C}_X, \psi \in \mathcal{C}_Y$ and $\Delta : S\Phi \to S\Phi \times S\Phi$ is the diagonal map. Since $\mathrm{Im}\Delta \subset {}'(\varphi \times \psi)^{-1} f^{-1} U$, for every $k \in S\varphi$ there exists an open set U_k such that $(k, k) \in U_k \times U_k \subset {}'(\varphi \times \psi)^{-1} f^{-1} U$. Since $S\Phi$ is locally $\mathcal{C}$-imaged by Axiom, there exists $\sigma_k \in \mathcal{C}_{S\Phi}$ such that $k \in \mathrm{Int}(\mathrm{Im}\sigma_k) \subset \mathrm{Im}\sigma_k \subset U_k$. We have an open covering $(\mathrm{Int}(\mathrm{Im}\sigma_k))_{k \in S\Phi}$ of the compact space $S\Phi$. Therefore we can chose a finite set $\{1, 2, \cdots, N\} \subset S\Phi$ which satisfies $S\Phi = \bigcup_{p=1}^N \mathrm{Im}\sigma_p$. Applying ${}'(\varphi \times \psi)$ to the sequence of subsets $\mathrm{Im}\Delta \subset \bigcup_{p=1}^N (\mathrm{Im}\sigma_p \times \mathrm{Im}\sigma_p) \subset {}'(\varphi \times \psi)^{-1} f^{-1} U$, we have the sequence $\mathrm{Im}'\Phi \subset \bigcup_{p=1}^N \mathrm{Im}\,'(\varphi_p \times \psi_p) \subset f^{-1} U$, where $\varphi_p = \varphi \circ \sigma_p$ and

$\psi_p = \psi \circ \sigma_p$. It is easy to see that $\bigcup_{p=1}^N \mathrm{Im}\,'(\varphi_p \times \psi_p) \subset f^{-1}U$ is equivalent to $f \in \bigcap_{p=1}^N W('(\varphi_p \times \psi_p), U)$ and that $\mathrm{Im}'\Phi \subset \bigcup_{p=1}^N \mathrm{Im}\,'(\varphi_p \times \psi_p)$ and $g \in \bigcap_{p=1}^N W('(\varphi_p \times \psi_p), U)$ induces $g \in W('\Phi, U)$, in other word $\bigcap_{p=1}^N W('(\varphi_p \times \psi_p), U) \subset W('\Phi, U)$.

(ii) Since $\varepsilon_{X\times Y}{}^* : C(X \times Y, Z) \to C(k_{\mathcal{C}}(X \times Y), Z)$ is an embedding, the topology of $C(X \times Y, Z)$ generated by the subbase $(\varepsilon_{X\times Y}{}^*)^{-1}\{W('(\varphi \times \psi), U) | \varphi \in \mathcal{C}_X, \psi \in \mathcal{C}_Y, U \in \mathcal{O}_Z\}$, which coincides with $\{W(\varphi \times \psi, U) | \varphi \in \mathcal{C}_X, \psi \in \mathcal{C}_Y, U \in \mathcal{O}_Z\}$.

(iii) Let $\mathcal{B}$ denote the base generated by $\mathcal{S}$. Let $W(\varphi, U)$ be an arbitrary element of the canonical subbase of $C(X, T)$, let $f \in W(\varphi, U)$ and let $k \in S\varphi$. Since $f(\varphi(k)) \in U$ and $\mathcal{B}$ is a base, there exists $B_k \in \mathcal{B}$ such that $f(\varphi(k)) \in B_k \subset U$. Since $S\varphi$ is locally $\mathcal{C}$-imaged, there exists $\psi_k \in \mathcal{C}_{S\varphi}$ such that $k \in \mathrm{Int}(\mathrm{Im}\psi_k) \subset \mathrm{Im}\psi_k \subset \phi^{-1} f^{-1} B_k$. We have an open covering $(\mathrm{Int}(\mathrm{Im}\psi_k))_{k\in S\varphi}$ of the compact space $S\varphi$. Therefore we can chose a finite subset $\{1, 2, \cdots, N\} \subset S\varphi$ and have $S\varphi = \bigcup_{p=1}^N \mathrm{Im}\psi_p,\ f(\phi(\mathrm{Im}\psi_p)) \subset B_p \subset U, (1 \le p \le N)$. Hence we have $f \in \cap_{p=1}^N W(\varphi_p, B_p) \subset \cap_{p=1}^N W(\varphi_p, U) = W(\varphi, U)$, where $\varphi_p = \varphi \circ \psi_p$. Since B_p is an intersection of finite elements of $\mathcal{S}$, $\cap_{p=1}^N W(\varphi_p, B_p)$ is also an intersection of finite sets of the form $W(\varphi, S), \varphi \in \mathcal{C}_X, S \in \mathcal{S}$. This completes the proof. □

Lemma 7. (i) *If Y is locally $\mathcal{C}$-imaged then $ev_{Y,Z} : C(Y, Z) \times Y \to Z, ev_{Y,Z}(f, y) = f(y)$, is continuous.*

(ii) *Suppose $\mathcal{C}$ satisfies Axiom, $ev_{Y,Z} \circ \varepsilon_{C(Y,Z)\times Y}$ is continuous.*

Proof. (i) Let $(f, y) \in C(Y, Z) \times Y$ be an arbitrary element, and let W be an open set such that $f(y) \in W$. Then there exists a $\mathcal{C}$-test map φ such that $y \in \mathrm{Int}(\mathrm{Im}\varphi) \subset \mathrm{Im}\varphi \subset f^{-1}W$. Hence we have a neighbourhood $W(\varphi, W) \times \mathrm{Im}\varphi$ of (f, y) such that $ev_{Y,Z}(W(\varphi, W) \times \mathrm{Im}\varphi) \subset W$.

(ii) It suffices to prove that $ev_{Y,Z} \circ \Phi$ is continuous for every $\Phi \in \mathcal{C}_{C(Y,Z)\times Y}$. We have the decompoxition $ev_{Y,Z} \circ \Phi = ev_{S\Phi,Z} \circ ((\psi^* \circ \varphi) \times 1_{S\Phi}) \circ \Delta_{S\Phi}$, where $\Phi = (\varphi, \psi)$. The right side of the equation is continuous by (i). □

2. EXPONENTIAL LAWS

In $\mathcal{S}et$, the category of small sets, we have a natural bijection $e_{\mathcal{S}et} : \mathcal{S}et(X \times Y, Z) \to \mathcal{S}et(X, \mathcal{S}et(Y, Z)), f \mapsto \tilde{f}, \tilde{f}(x)(y) = f(x, y)$. If X, Y and Z are topological spaces and $\mathcal{C} \subset \mathcal{C}_\omega$, we have sequences of subsets

$$\mathcal{T}op(X \times Y, Z) \subset \mathcal{T}op(k_{\mathcal{C}}(X \times Y), Z) \subset \mathcal{S}et(X \times Y, Z),$$

and

$$\mathcal{T}op(X, C_\omega(Y,Z)) \subset \mathcal{T}op(X, C(Y,Z)) \subset \mathcal{T}op(X, C(k_{\mathcal{C}}Y, Z))$$
$$\subset \mathcal{T}op(k_{\mathcal{C}}X, C(k_{\mathcal{C}}Y,Z)) \subset \mathcal{S}et(X, \mathcal{T}op(k_{\mathcal{C}}Y,Z)) \subset \mathcal{S}et(X, \mathcal{S}et(Y,Z)).$$

Lemma 8. (i) $e_{\mathcal{S}et}(\mathcal{T}op(k_{\mathcal{C}}(X\times Y), Z)) \subset \mathcal{T}op(k_{\mathcal{C}}X, C(k_{\mathcal{C}}Y,Z))$.
(ii) $e_{\mathcal{S}et}(\mathcal{T}op(X\times Y, Z)) \subset \mathcal{T}op(X, C(Y,Z))$.

Proof. (i) Let $f : X \times Y \to Z$ be a function such that $f \circ \varepsilon$ is continuous, and let $x \in X$. Then we have $e_{\mathcal{S}et}(f)(x) \circ \psi = f \circ (c_x, \psi)$ and the right side of the equation is continuous for every $\psi \in \mathcal{C}_Y$ where $c_x : S\psi \to X$ is the constant map with the value x. This shows $e_{\mathcal{S}et}(\mathcal{T}op(k_{\mathcal{C}}(X \times Y), Z)) \subset \mathcal{S}et(X, \mathcal{T}op(k_{\mathcal{C}}Y, Z))$. Moreover let φ be an arbitrary $\mathcal{C}$-test map, let $k \in S\varphi$ and let $W(\mathrm{Im}\psi, W)$ be a neighbourhood of $e_{\mathcal{S}et}(f)(\varphi(k))$. Since $f \circ (\varphi \times \psi)(k \times S\phi) \subset W$, there exists a neighbourhood U of k such that $f \circ (\varphi \times \psi)(U \times S\phi) \subset W$ by the tube lemma. This proves that $e_{\mathcal{S}et}(f) \circ \varphi$ is continuous for every φ. The proof is completed. (ii) It is well known that $e_{\mathcal{S}et}(\mathcal{T}op(X\times Y, Z)) \subset \mathcal{T}op(X, C_\omega(Y,Z))$ is also proved by the tube lemma. □

Therefore we have two restrictions of $e_{\mathcal{S}et}$, $e'_{\mathcal{C}} : \mathcal{T}op(k_{\mathcal{C}}(X\times Y), Z) \to \mathcal{T}op(k_{\mathcal{C}}X, C(k_{\mathcal{C}}Y, Z))$ and $e_{\mathcal{C}} : \mathcal{T}op(X \times Y, Z) \to \mathcal{T}op(X, C(Y,Z))$. The following proposition (i) is due to Brown in $\mathcal{C}_2$ case ([1]).

Theorem 9. (Main Theorem) *Suppose that $\mathcal{C}$ satisfies Axiom. Then*
(i) $e'_{\mathcal{C}} : C(k_{\mathcal{C}}(X \times Y), Z) \to C(k_{\mathcal{C}}X, C(k_{\mathcal{C}}Y, Z))$ *is a natural homeomorphism,*
(ii) $e_{\mathcal{C}} : C(X\times Y, Z) \to C(X, C(Y,Z))$ *is a natural embedding and is a homeomorphism if Y is locally $\mathcal{C}$-imaged or $X \times Y$ is a $k_{\mathcal{C}}$-space.*

Proof. Applying lemma 6 and the remark in the proof of Lemma 4, we deduce that the two injections $e'_{\mathcal{C}}, e_{\mathcal{C}}$ are embeddings. (i) Let g be an arbitrary map in $C(k_{\mathcal{C}}X, C(k_{\mathcal{C}}Y, Z))$. We define a map f by $f = ev_{k_{\mathcal{C}}Y,Z} \circ (g \times k_{\mathcal{C}}Y) \circ \varepsilon_{k_{\mathcal{C}}X \times k_{\mathcal{C}}Y} \circ k_{\mathcal{C}}(\varepsilon_X \times \varepsilon_Y)^{-1}$. Since the lifting of the continuous map $(k_{\mathcal{C}}pr_1, k_{\mathcal{C}}pr_2) : k_{\mathcal{C}}(X\times Y) \to k_{\mathcal{C}}X \times k_{\mathcal{C}}Y$ is equal to $k_{\mathcal{C}}(\varepsilon_X \times \varepsilon_Y)^{-1}$, it is contimuous. By making use of Lemma 7 (ii) and the commutativity $(g \times k_{\mathcal{C}}Y) \circ \varepsilon_{k_{\mathcal{C}}X\times k_{\mathcal{C}}Y} = \varepsilon_{C(k_{\mathcal{C}}Y,Z)),k_{\mathcal{C}}Y} \circ k_{\mathcal{C}}(g \times k_{\mathcal{C}}Y)$, $ev_{k_{\mathcal{C}}Y,Z} \circ (g \times k_{\mathcal{C}}Y) \circ \varepsilon_{k_{\mathcal{C}}X\times k_{\mathcal{C}}Y}$ is continuous. These show that f is certainly an element of $C(k_{\mathcal{C}}(X \times Y), Z)$. Since ε's are identities as set functions, it is eaey to see $f(x,y) = g(x)(y)$, in other word $e_{\mathcal{S}et}(f) = g$. Hence $e'_{\mathcal{C}}$ is surjective. (ii) If $X \times Y$ is a $k_{\mathcal{C}}$-space then ${\varepsilon_X}^{-1} = k_{\mathcal{C}}pr_1 \circ {\varepsilon_{X\times Y}}^{-1} \circ (1_X, c_y)$ is continuous and hence X is a $k_{\mathcal{C}}$-space, so also is Y. We have $e_{\mathcal{C}} = e'_{\mathcal{C}}$, which complete the proof of this case. Let Y be locally $\mathcal{C}$-imaged. By making use of Lemma 7 (i), similar argument in (i) proves that $e_{\mathcal{C}}$ is surjective. □

Lemma 10. *Suppose that $\mathcal{C}$ satisfies Axiom. Then*
(i) $k_{\mathcal{C}}C_\omega(k_{\mathcal{C}}X, Z) \to k_{\mathcal{C}}C(k_{\mathcal{C}}X, Z)$ *is a natural homeomorphism,*
(ii) *so is* $k_{\mathcal{C}}C(k_{\mathcal{C}}X, k_{\mathcal{C}}Z) \to k_{\mathcal{C}}C(k_{\mathcal{C}}X, Z)$.

Proof. If K is an object of $\mathcal{C}$ we have a $k_{\mathcal{C}}$-space $K \times k_{\mathcal{C}}X$ by Proposition 5 and hence the exponential map $e_{\mathcal{C}} : C(K \times k_{\mathcal{C}}X, Z) \to C(K, C(k_{\mathcal{C}}X, Z))$ is a homeomorphism for every Z. (i) Since $e_{\mathcal{C}}$ is defined to be the composition $\mathcal{T}op(K \times k_{\mathcal{C}}X, Z) \to \mathcal{T}op(K, C_\omega(k_{\mathcal{C}}X, Z)) \subset \mathcal{T}op(K, C(k_{\mathcal{C}}X, Z))$ as a set map, $\mathcal{T}op(K, C_\omega(k_{\mathcal{C}}X, Z)) \to \mathcal{T}op(K, C(k_{\mathcal{C}}X, Z))$ is bijective. This says that the continuous bijection $C_\omega(k_{\mathcal{C}}X, Z) \to C(k_{\mathcal{C}}X, Z)$ induces a bijection $\mathcal{C}_{C_\omega(k_{\mathcal{C}}X,Z)} \to \mathcal{C}_{C(k_{\mathcal{C}}X,Z)}$, which complete the proof of (i). Let us now turn to (ii). Since $K \times k_{\mathcal{C}}X$ is a $k_{\mathcal{C}}$-space, $\varepsilon_* : C(K \times k_{\mathcal{C}}X, k_{\mathcal{C}}Z) \to C(K \times k_{\mathcal{C}}X, Z)$ is bijective. By the naturality of exponential homeomorphisms, $(\varepsilon_*)_* : \mathcal{T}op(K, C(k_{\mathcal{C}}X, k_{\mathcal{C}}Z)) \to \mathcal{T}op(K, C(k_{\mathcal{C}}X, Z))$ is bijective for every $K \in O\mathcal{C}$. This complete the proof. □

In the category of $k_{\mathcal{C}}$-spaces $\mathcal{K}_{\mathcal{C}}$ the product and the function space of spaces X, Y are defined to be $X \times Y = k_{\mathcal{C}}(X \times Y)$ and $map(X, Y) = k_{\mathcal{C}}C(X, Y)$. Mixing up Theorem 9 (i) and Lemma 10 (ii) we have the following exponential law in $\mathcal{K}_{\mathcal{C}}$.

Corollary 11. *Suppose that $\mathcal{C}$ satisfies Axiom. Then* $map(X \times Y, Z) \to map(X, map(Y, Z))$ *is a natural homeomorphism.*

Suppose that $\mathcal{C} = \mathcal{C}_2, \mathcal{C}_3$ then the assumptions of the following each classical exponential law are strong enough to deduce the first assumption of Theorem 9 (ii) and that the $\mathcal{C}$-test-open topology coincides with the compact-open topology.

Corollary 12. (i) *If X, Y are Hausdorff spaces and Y is locally compact then* $C_\omega(X \times Y, Z) \to C_\omega(X, C_\omega(Y, Z))$ *is a natural homeomorphism.*

(ii) *If X, Y are regular and Y is locally compact then* $C_\omega(X \times Y, Z) \to C_\omega(X, C_\omega(Y, Z))$ *is a natural homeomorphism.*

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(*Received December 18, 2000*)