Math. J. Okayama Univ. **42** (2000), 19-27

# JORDAN DERIVATIONS OF A SKEW MATRIX RING

NAOKI HAMAGUCHI

Abstract. We determine the form of Jordan derivations of a skew matrix ring $\mathrm{M}_2(R;\sigma,q)$ over a ring $R$. Using this result, we also show the properties of Jordan derivations of $\mathrm{M}_2(R)$, and derivations of $\mathrm{M}_2(R;\sigma,q)$. Moreover, we refer to invariant ideals with respect to these derivations.

## 1. Introduction

Let $R$ be a ring. An additive mapping $D : R \to R$ is said to be a *derivation* if $D(xy) = D(x)y+xD(y)$ for all $x,\ y \in R$. An additive mapping $J : R \to R$ is said to be a *Jordan derivation* if $J(x^2) = J(x)x + xJ(x)$ for all $x \in R$. We can easily check that, for all $x,\ y \in R$, $J(xy + yx) = J(x)y + xJ(y) + J(y)x + yJ(x)$. We can also see that any derivation of $R$ is a Jordan derivation.

In [3], I. N. Herstein has shown that every Jordan derivation of a prime ring not of characteristic 2 is a derivation. This result is extended by J. M. Cusack in [2] to the case of a ring $R$ where $2x = 0$ implies $x = 0$ and $R$ is semiprime or $R$ contains a commutator which is not a zero divisor.

In this paper, we give a necessary and sufficient condition for a given mapping $J$ of a skew matrix ring $\mathrm{M}_2(R;\sigma,q)$ into itself to be a Jordan derivation. By using this result, we can show that there are many Jordan derivations of $\mathrm{M}_2(R;\sigma,q)$ which are not derivations. We also refer to the properties of Jordan derivations of $\mathrm{M}_2(R)$, and derivations of $\mathrm{M}_2(R;\sigma,q)$. Moreover, we consider invariant ideals with respect to these derivations.

## 2. Jordan derivations of $\mathrm{M}_2(R;\sigma,q)$

In this paper, we treat a skew matrix ring defined as follows (cf. [4]): Let $R$ be a ring, $q$ an element in $R$ and $\sigma$ an endomorphism of $R$ such that $\sigma(q) = q$ and $\sigma(r)q = qr$ for all $r \in R$. Let $\mathrm{M}_2(R;\sigma,q)$ be the set of $2 \times 2$ matrices over $R$ with usual addition and the following multiplication:

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = \begin{pmatrix} x_1y_1 + x_2y_3q & x_1y_2 + x_2y_4 \\ x_3\sigma(y_1) + x_4y_3 & x_3\sigma(y_2)q + x_4y_4 \end{pmatrix}.$$

We call $\mathrm{M}_2(R;\sigma,q)$ a *skew matrix ring* over $R$. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is denoted by $e_{11}(a)+e_{12}(b)+e_{21}(c)+e_{22}(d)$.

Let $J$ be a Jordan derivation of $\mathrm{M}_2(R;\sigma,q)$. First, we set

$$J(e_{11}(a)) = \begin{pmatrix} f_1(a) & f_2(a) \\ f_3(a) & f_4(a) \end{pmatrix}, \qquad J(e_{12}(b)) = \begin{pmatrix} h_1(b) & h_2(b) \\ h_3(b) & h_4(b) \end{pmatrix},$$
$$J(e_{21}(c)) = \begin{pmatrix} l_1(c) & l_2(c) \\ l_3(c) & l_4(c) \end{pmatrix}, \qquad J(e_{22}(d)) = \begin{pmatrix} g_1(d) & g_2(d) \\ g_3(d) & g_4(d) \end{pmatrix},$$

where $f_i,\ h_i,\ l_i,\ g_i : R \to R$ are additive mappings.

Since $J(e_{11}(a^2)) = J(e_{11}(a))e_{11}(a)+e_{11}(a)J(e_{11}(a))$ and $J(e_{22}(d^2)) = J(e_{22}(d))e_{22}(d)+e_{22}(d)J(e_{22}(d))$, we get the following two lemmas:

**Lemma 2.1.** *For any $a \in R$,*

(1) *$f_1$ is a Jordan derivation of $R$.*
(2) *$f_2(a^2) = af_2(a)$.*
(3) *$f_3(a^2) = f_3(a)\sigma(a)$.*
(4) *$f_4(a^2) = 0$.*

**Lemma 2.2.** *For any $d \in R$,*

(1) *$g_1(d^2) = 0$.*
(2) *$g_2(d^2) = g_2(d)d$.*
(3) *$g_3(d^2) = dg_3(d)$.*
(4) *$g_4$ is a Jordan derivation of $R$.*

Moreover, from $J(e_{11}(a)e_{22}(d)+e_{22}(d)e_{11}(a)) = 0$, we have the following relations:

$$ag_1(d)+g_1(d)a = 0, \tag{2.1}$$
$$f_2(a)d+ag_2(d) = 0, \tag{2.2}$$
$$g_3(d)\sigma(a)+df_3(a) = 0, \tag{2.3}$$
$$f_4(a)d+df_4(a) = 0. \tag{2.4}$$

On the other hand, by the facts that $J(e_{12}(ab)) = J(e_{11}(a)e_{12}(b)+e_{12}(b)e_{11}(a))$ and $J(e_{21}(dc)) = J(e_{22}(d)e_{21}(c)+e_{21}(c)e_{22}(d))$, we have the following:

**Lemma 2.3.** *For any $a,\ b \in R$,*

(1) *$h_1(ab) = ah_1(b)+h_1(b)a+bf_3(a)q$.*
(2) *$h_2(ab) = f_1(a)b+ah_2(b)+bf_4(a)$.*
(3) *$h_3(ab) = h_3(b)\sigma(a)$.*
(4) *$h_4(ab) = f_3(a)\sigma(b)q$.*

**Lemma 2.4.** *For any $c,\ d \in R$,*

(1) $l_1(dc) = g_2(d)cq$.
(2) $l_2(dc) = l_2(c)d$.
(3) $l_3(dc) = g_4(d)c + dl_3(c) + c\sigma(g_1(d))$.
(4) $l_4(dc) = dl_4(c) + l_4(c)d + c\sigma(g_2(d))q$.

Moreover, from $J(e_{12}(b)^2) = 0$ and $J(e_{21}(c)^2) = 0$, we have the following relations:

$$bh_3(b)q = h_3(b)\sigma(b)q = 0, \tag{2.5}$$

$$h_1(b)b + bh_4(b) = 0, \tag{2.6}$$

$$l_2(c)cq = c\sigma(l_2(c))q = 0, \tag{2.7}$$

$$l_4(c)c + c\sigma(l_1(c)) = 0. \tag{2.8}$$

Now we assume that $R$ has identity. Then a Jordan derivation $J$ has the following properties:

**Lemma 2.5.** *Let $R$ be a ring with identity, and $J$ a Jordan derivation of $\mathrm{M}_2(R;\sigma,q)$. Then there exist additive mappings $f_1,\ f_4,\ g_1,\ g_4 : R \to R$ and elements $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta$ in $R$ such that, for all $a,\ b,\ c,\ d \in R$,*

$$J(e_{11}(a)) = \begin{pmatrix} f_1(a) & a\alpha \\ \beta\sigma(a) & f_4(a) \end{pmatrix}, \tag{2.9}$$

$$J(e_{12}(b)) = \begin{pmatrix} -b\beta q & (f_1+f_4)(b) + b\gamma \\ \delta\sigma(b) & \beta qb \end{pmatrix}, \tag{2.10}$$

$$J(e_{21}(c)) = \begin{pmatrix} -\alpha cq & \varepsilon c \\ \sigma(g_1(c)) + g_4(c) + c\zeta & cq\alpha \end{pmatrix}, \tag{2.11}$$

$$J(e_{22}(d)) = \begin{pmatrix} g_1(d) & -\alpha d \\ -d\beta & g_4(d) \end{pmatrix}. \tag{2.12}$$

*Proof.* By Lemma 2.1 (2), (3), we have $f_2(ab+ba) = af_2(b) + bf_2(a)$ and $f_3(ab+ba) = f_3(a)\sigma(b) + f_3(b)\sigma(a)$ for all $a,\ b \in R$. Putting $b = 1$, we get $f_2(a) = af_2(1)$ and $f_3(a) = f_3(1)\sigma(a)$. Put $\alpha = f_2(1)$ and $\beta = f_3(1)$. From (2.2) and (2.3), we have $g_2(d) = -f_2(1)d = -\alpha d$ and $g_3(d) = -df_3(1) = -d\beta$.

By Lemma 2.3, we have $h_1(b) = -b\beta q$, $h_2(a) = f_1(a) + f_4(a) + ah_2(1)$, $h_3(a) = h_3(1)\sigma(a)$ and $h_4(b) = \beta\sigma(b)q = \beta qb$. Put $\gamma = h_2(1)$ and $\delta = h_3(1)$. By Lemma 2.4, we have $l_1(c) = -\alpha cq$, $l_2(d) = l_2(1)d$, $l_3(d) = \sigma(g_1(d)) + g_4(d) + dl_3(1)$ and $l_4(c) = c\sigma(\alpha)q = cq\alpha$. Putting $\varepsilon = l_2(1)$ and $\zeta = l_3(1)$, we have completed the proof of the lemma. □

An additive mapping $F : R \to R$ is said to be *central* if $F(R)$ is contained in $C$, the center of $R$.

**Theorem 2.6.** *Let $R$ be a ring with identity, and $J : \mathrm{M}_2(R;\sigma,q) \to \mathrm{M}_2(R;\sigma,q)$ an additive mapping. Then $J$ is a Jordan derivation if and only if there exist additive mappings $f_1,\ f_4,\ g_1,\ g_4$ and elements $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta$ of $R$ satisfying* (2.9), (2.10), (2.11) *and* (2.12) *with the following conditions: for all $a,\ b,\ c,\ d \in R$,*

(i) *$f_1$ and $g_4$ are Jordan derivations of $R$.*
(ii) *$f_4$ and $g_1$ are central Jordan derivations of $R$ such that $f_4(a^2) = g_1(a^2) = 0$.*
(iii) $\delta q = \varepsilon c^2 q = 0$.
(iv) $f_1(ab) + f_4(ab) = f_1(a)b + af_1(b) + f_4(a)b + af_4(b)$.
(v) $f_1(d) + f_4(d) + d\gamma = g_1(d) + g_4(d) + \gamma d$.
(vi) $d\delta\sigma(b) = \delta\sigma(b)\sigma(d)$.
(vii) $\sigma(g_1(dc)) + g_4(dc) = c\sigma(g_1(d)) + d\sigma(g_1(c)) + g_4(d)c + dg_4(c)$.
(viii) $\sigma(g_1(\sigma(a))) + g_4(\sigma(a)) + \sigma(a)\zeta = \sigma(f_1(a)) + f_4(a) + \zeta\sigma(a)$.
(ix) $\varepsilon cd = \varepsilon dc$.
(x) $\varepsilon c\sigma(a) = a\varepsilon c$.
(xi) $f_1(bcq) + g_1(cqb) = f_1(b)cq + f_4(b)cq + bg_1(c)q + bg_4(c)q + b\gamma cq + bc\zeta q$.
(xii) $f_4(bcq) + g_4(cqb) = cqf_1(b) + cqf_4(b) + g_1(c)qb + g_4(c)qb + cqb\gamma + c\zeta qb$.

*Particularly, a Jordan derivation $J$ of $\mathrm{M}_2(R;\sigma,q)$ is given by*

$$J\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} f_1(a) + g_1(d) - b\beta q - \alpha cq & \begin{matrix}(f_1+f_4)(b) + a\alpha \\ +b\gamma + \varepsilon c - \alpha d\end{matrix} \\ \begin{matrix}\sigma(g_1(c)) + g_4(c) + \beta\sigma(a) \\ +\delta\sigma(b) + c\zeta - d\beta\end{matrix} & f_4(a) + g_4(d) + \beta qb + cq\alpha \end{pmatrix}$$

*with the conditions above.*

*Proof.* ($\Rightarrow$) : Assume that $J$ is a Jordan derivation of $\mathrm{M}_2(R;\sigma,q)$. Then $J$ satisfies (2.9), (2.10), (2.11) and (2.12) for some additive mappings $f_1,\ f_4,\ g_1,\ g_4$ and elements $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta$ of $R$ by Lemma 2.5.

For the conditions, first, we have (i) by Lemma 2.1 (1) and 2.2 (4).

From Lemma 2.1 (4) and (2.4), we get $f_4(a^2) = f_4(a)a + af_4(a) = 0$ and $2f_4(a) = 0$ by $f_4((a+1)^2) = 0$. Since $2df_4(a) = 0$, we also have $f_4(a)d = df_4(a)$. Hence, $f_4$ is a central Jordan derivation of $R$. By Lemma 2.2 (1) and (2.1), we can show that $g_1$ is also a central Jordan derivation, hence, we get (ii).

From (2.5), we have $\delta q = 0$, and from (2.7), we get $\varepsilon c^2 q = 0$. Hence, we have (iii).

By Lemma 2.3 (2) and (2.10), we get (iv), and since $J(e_{12}(bd)) = J(e_{12}(b)e_{22}(d) + e_{22}(d)e_{12}(b))$, we have (v) and (vi). (Note that, from (vi), we have $d\delta = \delta\sigma(d)$, and hence, $\delta\sigma(b)\sigma(a) = b\delta\sigma(a) = \delta\sigma(a)\sigma(b)$.)

By Lemma 2.4 (2), (3) and (2.11), we have (vii) and (ix), and since $J(e_{21}(c\sigma(a))) = J(e_{21}(c)e_{11}(a) + e_{11}(a)e_{21}(c))$, we get (viii) and (x).

Finally, from $J(e_{11}(bcq))+J(e_{22}(cqb)) = J(e_{12}(b)e_{21}(c)+e_{21}(c)e_{12}(b))$, we have (xi) and (xii).

($\Leftarrow$) : If mappings $f_1,\ f_4,\ g_1,\ g_4$ of $R$ and elements $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta \in R$ satisfy the conditions above, then we can show that, for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(R;\sigma,q)$,

$$J(A^2) = J(A)A + AJ(A)$$

by direct calculation. □

When a Jordan derivation $J$ of $\mathrm{M}_2(R;\sigma,q)$ is determined by Jordan derivations $f_1,\ f_4,\ g_1,\ g_4 : R \to R$ and elements $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta \in R$ as in Theorem 2.6, we denote this $J$ by $(f_1,\ f_4,\ g_1,\ g_4, \alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta)$.

Now we give the properties of Jordan derivations of $\mathrm{M}_2(R)$, derivations of $\mathrm{M}_2(R;\sigma,q)$ and derivations of $\mathrm{M}_2(R)$, which are easily proved by Theorem 2.6.

**Corollary 2.7.** *Let $R$ be a ring with identity, and $J : \mathrm{M}_2(R) \to \mathrm{M}_2(R)$ an additive mapping. Then $J$ is a Jordan derivation if and only if there exist additive mappings $f_1,\ f_4 : R \to R$ and elements $\alpha,\ \beta,\ \gamma$ in $R$ such that, for all $a,\ b,\ c,\ d \in R$,*

$$J(e_{11}(a)) = \begin{pmatrix} f_1(a) & a\alpha \\ \beta a & f_4(a) \end{pmatrix}, \tag{2.13}$$

$$J(e_{12}(b)) = \begin{pmatrix} -b\beta & (f_1+f_4)(b)+b\gamma \\ 0 & \beta b \end{pmatrix}, \tag{2.14}$$

$$J(e_{21}(c)) = \begin{pmatrix} -\alpha c & 0 \\ (f_1+f_4)(c)-\gamma c & c\alpha \end{pmatrix}, \tag{2.15}$$

$$J(e_{22}(d)) = \begin{pmatrix} f_4(d) & -\alpha d \\ -d\beta & f_1(d)+d\gamma-\gamma d \end{pmatrix} \tag{2.16}$$

*with the following conditions: for all $a,\ b \in R$,*

(i) *$f_1$ is a Jordan derivation of $R$.*

(ii) *$f_4$ is a central Jordan derivation of $R$ such that $f_4(a^2) = 0$.*

(iii) *$f_1(ab) + f_4(ab) = f_1(a)b + af_1(b) + f_4(a)b + af_4(b)$.*

*Particularly, a Jordan derivation $J$ of $\mathrm{M}_2(R)$ is given by*

$$J\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$= \begin{pmatrix} f_1(a)+f_4(d)-b\beta-\alpha c & (f_1+f_4)(b)+a\alpha+b\gamma-\alpha d \\ (f_1+f_4)(c)+\beta a-\gamma c-d\beta & f_1(d)+f_4(a)+c\alpha+\beta b+d\gamma-\gamma d \end{pmatrix}$$

*with the conditions above.*

*Proof.* ($\Rightarrow$) : Assume that $J$ is a Jordan derivation of $\mathrm{M}_2(R)$. Under the notation in Theorem 2.6, put $\sigma = \mathrm{id}_R$ and $q = 1$. Then we have $\delta = \varepsilon = 0$ and $\zeta = -\gamma$ by putting $b = c = 1$ in Theorem 2.6 (iii) and (xi). Moreover, by putting $c = 1$ in Theorem 2.6 (xi), we get $g_1(a) = f_4(a)$ and, hence, $g_4(a) = f_1(a) + a\gamma - \gamma a$ from Theorem 2.6 (v). The conditions immediately follows from Theorem 2.6 (i), (ii) and (iv).

($\Leftarrow$) : Assume that there exist additive mappings $f_1$, $f_4$ of $R$ and elements $\alpha$, $\beta$, $\gamma \in R$ satisfying the conditions, and consider a Jordan derivation $J_1 = (f_1, f_4, g_1, g_4, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ of $\mathrm{M}_2(R) = \mathrm{M}_2(R; \mathrm{id}_R, 1)$, where $g_1 = f_4$, $g_4(a) = f_1(a) + a\gamma - \gamma a$ (for all $a \in R$), $\delta = \varepsilon = 0$ and $\zeta = -\gamma$. Then we can see that $J = J_1$, hence, $J$ is a Jordan derivation of $\mathrm{M}_2(R)$. □

**Corollary 2.8.** *Let $R$ be a ring with identity, and $D : \mathrm{M}_2(R;\sigma,q) \to \mathrm{M}_2(R;\sigma,q)$ an additive mapping. Then $D$ is a derivation if and only if there exists a derivation $f$ of $R$ and elements $\alpha$, $\beta$, $\gamma$, $\zeta$ in $R$ such that, for all $a$, $b$, $c$, $d \in R$,*

$$D(e_{11}(a)) = \begin{pmatrix} f(a) & a\alpha \\ \beta\sigma(a) & 0 \end{pmatrix}, \tag{2.17}$$

$$D(e_{12}(b)) = \begin{pmatrix} -b\beta q & f(b) + b\gamma \\ 0 & \beta q b \end{pmatrix}, \tag{2.18}$$

$$D(e_{21}(c)) = \begin{pmatrix} -\alpha c q & 0 \\ f(c) + c\gamma - \gamma c + c\zeta & cq\alpha \end{pmatrix}, \tag{2.19}$$

$$D(e_{22}(d)) = \begin{pmatrix} 0 & -\alpha d \\ -d\beta & f(d) + d\gamma - \gamma d \end{pmatrix} \tag{2.20}$$

*with the relations $\sigma(f(a)) - f(\sigma(a)) = \sigma(a)\gamma - \gamma\sigma(a) + \sigma(a)\zeta - \zeta\sigma(a)$, and $f(q) = \gamma q + \zeta q$.*

*Particularly, a derivation $D$ of $\mathrm{M}_2(R;\sigma,q)$ is given by*

$$D\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} f(a) - b\beta q - \alpha c q & f(b) + a\alpha + b\gamma - \alpha d \\ f(c) + \beta\sigma(a) + c\gamma - \gamma c + c\zeta - d\beta & f(d) + \beta q b + cq\alpha + d\gamma - \gamma d \end{pmatrix}$$

*with the relations above.*

*Proof.* ($\Rightarrow$) : First, note that any derivation is a Jordan derivation. Under the notation in Theorem 2.6, since $D(e_{11}(ab)) = D(e_{11}(a))e_{11}(b) + e_{11}(a)D(e_{11}(b))$ and $D(e_{22}(cd)) = D(e_{22}(c))e_{22}(d) + e_{22}(c)D(e_{22}(d))$, we have that $f = f_1$ is a derivation and $f_4 = g_1 = 0$, and hence, $g_4(d) = f(d) + d\gamma - \gamma d$ from Theorem 2.6 (v).

Moreover, since $D(e_{12}(1)) = D(e_{11}(1))e_{12}(1) + e_{11}(1)D(e_{12}(1))$ and $D(e_{21}(1)) = D(e_{22}(1))e_{21}(1) + e_{22}(1)D(e_{21}(1))$, we have $\delta = \varepsilon = 0$.

$(\Leftarrow)$ : If a mapping $f$ and elements $\alpha,\ \beta,\ \gamma,\ \zeta \in R$ satisfy the conditions above, then we can show that, for any $A,\ B \in \mathrm{M}_2(R;\sigma,q)$,

$$D(AB) = D(A)B + AD(B)$$

by direct calculation. □

**Corollary 2.9.** (cf. [1]) *Let $R$ be a ring with identity, and $D : \mathrm{M}_2(R) \to \mathrm{M}_2(R)$ an additive mapping. Then $D$ is a derivation if and only if there exist a derivation $f$ of $R$ and elements $\alpha,\ \beta,\ \gamma$ in $R$ such that, for all $a,\ b,\ c,\ d \in R$,*

$$D(e_{11}(a)) = \begin{pmatrix} f(a) & a\alpha \\ \beta a & 0 \end{pmatrix}, \tag{2.21}$$

$$D(e_{12}(b)) = \begin{pmatrix} -b\beta & f(b) + b\gamma \\ 0 & \beta b \end{pmatrix}, \tag{2.22}$$

$$D(e_{21}(c)) = \begin{pmatrix} -\alpha c & 0 \\ f(c) - \gamma c & c\alpha \end{pmatrix}, \tag{2.23}$$

$$D(e_{22}(d)) = \begin{pmatrix} 0 & -\alpha d \\ -d\beta & f(d) + d\gamma - \gamma d \end{pmatrix}. \tag{2.24}$$

*Particularly, a derivation $D$ of $\mathrm{M}_2(R)$ is given by*

$$D\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} f(a) - b\beta - \alpha c & f(b) + a\alpha + b\gamma - \alpha d \\ f(c) + \beta a - \gamma c - d\beta & f(d) + \beta b + c\alpha + d\gamma - \gamma d \end{pmatrix}.$$

*Proof.* Put $\sigma = \mathrm{id}_R$ and $q = 1$ in Corollary 2.8. □

Now we give an example of a Jordan derivation of $\mathrm{M}_2(R;\sigma,q)$ which is not a derivation.

**Example 1.** Let $K[X]$ be a polynomial ring in one variable $X$ over a field $K$ not of characteristic 2, and put $R = K[X]/(X^2)$ and $x = X + (X^2) \in R$. Let $f : R \to R$ be a $K$-derivation defined by $f(x) = 2x$. We consider a skew matrix ring $\mathrm{M}_2(R;\mathrm{id}_R,x)$.

Let $J = (f_1,\ f_4,\ g_1,\ g_4,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta)$ be a Jordan derivation of $\mathrm{M}_2(R;\mathrm{id}_R,x)$ such that

$$f_1 = g_4 = f,\quad f_4 = g_1 = 0,\quad \alpha = \beta = \gamma = \zeta = 1,\quad \delta = \varepsilon = x.$$

(Note that $J$ satisfies the conditions of Theorem 2.6.) However, since $\delta \neq 0$, $J$ is not a derivation. In fact,

$$\begin{aligned}
&J(e_{11}(1)e_{12}(1)) - J(e_{11}(1))e_{12}(1) - e_{11}(1)J(e_{12}(1)) \\
&= \begin{pmatrix} -x & 1 \\ x & x \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -x & 1 \\ x & x \end{pmatrix} \\
&= \begin{pmatrix} -x & 1 \\ x & x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} -x & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \neq 0.
\end{aligned}$$

## 3. Invariant ideals with respect to derivations

Let $R$ be a ring, and $F : R \to R$ an additive mapping. An ideal $I$ of $R$ is said to be *$F$-invariant* or *invariant with respect to $F$* if $F(I) \subseteq I$.

Let $I_1,\ I_2,\ I_3,\ I_4$ be ideals of $R$ and put $\mathcal{I} = \begin{pmatrix} I_1 & I_2 \\ I_3 & I_4 \end{pmatrix} \subseteq \mathrm{M}_2(R; \sigma, q)$. If $\mathcal{I}$ is an ideal of $\mathrm{M}_2(R; \sigma, q)$ then we have the following conditions:

$$(3.1)\quad I_3 q \subseteq I_1,\ \ \sigma(I_1) \subseteq I_3,\ \ qI_2 \subseteq I_4 \subseteq I_2,\ \ I_2 q \subseteq I_1 \subseteq I_2,\ \ I_3 q \subseteq I_4 \subseteq I_3.$$

**Theorem 3.1.** *Let $R$ be a ring with identity, $I_1,\ I_2,\ I_3,\ I_4$ ideals of $R$ satisfying* (3.1)*, and $J = (f_1,\ f_4,\ g_1,\ g_4,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta)$ a Jordan derivation of $\mathrm{M}_2(R; \sigma, q)$. Then the ideal $\mathcal{I} = \begin{pmatrix} I_1 & I_2 \\ I_3 & I_4 \end{pmatrix}$ of $\mathrm{M}_2(R; \sigma, q)$ is $J$-invariant if and only if $I_1$, $I_2$, $I_3$ and $I_4$ satisfy the following conditions:*

(1) *$I_1$ is $f_1$-invariant, and $g_1(I_4) \subseteq I_1$.*
(2) *$I_2$ is $(f_1 + f_4)$-invariant, and $\varepsilon \in I_2$.*
(3) *$I_3$ is $(\sigma \circ g_1 + g_4)$-invariant, and $\delta \in I_3$.*
(4) *$I_4$ is $g_4$-invariant, and $f_4(I_1) \subseteq I_4$.*

*Proof.* By Theorem 2.6 and the relations (3.1), the result immediately follows. (Note that $\delta\sigma(b) = b\delta$.) □

For derivations of $\mathrm{M}_2(R; \sigma, q)$, we have the following:

**Corollary 3.2.** *Let $R$ be a ring with identity, $I_1,\ I_2,\ I_3,\ I_4$ ideals of $R$ satisfying* (3.1)*, and $D$ a derivation of $\mathrm{M}_2(R; \sigma, q)$ defined by $f$, $\alpha$, $\beta$, $\gamma$, $\zeta$ as in Corollary* 2.8*. Then the ideal $\mathcal{I} = \begin{pmatrix} I_1 & I_2 \\ I_3 & I_4 \end{pmatrix}$ of $\mathrm{M}_2(R; \sigma, q)$ is $D$-invariant if and only if $I_1$, $I_2$, $I_3$ and $I_4$ are $f$-invariant.*

Next, we consider ideals of $\mathrm{M}_2(R)$. In this case, an ideal $\mathcal{I}$ is in the form of $\begin{pmatrix} I & I \\ I & I \end{pmatrix}$, where $I$ is an ideal of $R$. Hence, we have the following:

**Corollary 3.3.** *Let $R$ be a ring with identity, $I$ an ideal of $R$, and $J$ a Jordan derivation of* $\mathrm{M}_2(R)$ *defined by* $f_1$, $f_4$, $\alpha$, $\beta$, $\gamma$ *as in Corollary* 2.7. *Then the ideal* $\mathcal{I} = \begin{pmatrix} I & I \\ I & I \end{pmatrix}$ *of* $\mathrm{M}_2(R)$ *is $J$-invariant if and only if $I$ is invariant with respect to $f_1$ and $f_4$.*

**Corollary 3.4.** *Let $R$ be a ring with identity, $I$ an ideal of $R$, and $D$ a derivation of* $\mathrm{M}_2(R)$ *defined by* $f$, $\alpha$, $\beta$, $\gamma$ *as in Corollary* 2.9. *Then the ideal* $\mathcal{I} = \begin{pmatrix} I & I \\ I & I \end{pmatrix}$ *of* $\mathrm{M}_2(R)$ *is $D$-invariant if and only if $I$ is $f$-invariant.*


## Acknowledgement

The author would like to thank Professor A. Nakajima for suggesting these problems and for valuable comments.


## References


[1] J. Bergen, I. N. Herstein and C. Lanski: Derivations with invertible values, Can. J. Math. **35** (1983), 300–310.

[2] J. M. Cusack: Jordan derivations on rings, Proc. Amer. Math. Soc. **53** (1975), 321–324.

[3] I. N. Herstein: Jordan derivations on prime rings, Proc. Amer. Math. Soc. **8** (1957), 1104–1110.

[4] K. Oshiro: Theories of Harada in artinian rings and applications to classical artinian rings, International Symposium on Ring Theory, Birkhäuer, to appear.



Naoki Hamaguchi
Department of Mathematics
Graduate School of Natural Science and Technology
Okayama University
Tsushima, Okayama 700-8530, JAPAN



(*Received November 13, 2000*)