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COMPACT ADMISSIBLE FUNCTIONAL CALCULI AND DECOMPOSABILITY

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Abstract. We introduce a class of functional calculi that seem particularly appropriate for dealing with unbounded operators. We characterize operators with this type of functional calculus that are decomposable, in terms of both global and local spectral mapping theorems and analytic functional calculi, among other things.

0. INTRODUCTION AND PRELIMINARIES.

The most tractable class of decomposable operators is that with a functional calculus $f \mapsto \mathcal{E}(f)$, for f in an appropriate Banach algebra of functions $\mathcal{F}$. When such an operator is bounded and $\mathcal{F}$ is chosen to be $C^n[a,b]$, for some real numbers a, b, n a nonnegative integer, the operator is sometimes called *generalized scalar* (see [13]); but more general algebras $\mathcal{F}$, what are called *admissible*, or, for unbounded operators, *quasi admissible*, are equally successful (see [6], [20]). One may explicitly construct the desired invariant subspaces that decompose the spectrum, the local spectral subspaces, in terms of the functional calculus.

Functional calculi are also essential in producing other objects of great interest in operator theory and its applications, including semigroups of operators, cosine families and fractional powers of operators.

When T is a bounded generalized scalar operator, it is easy to see that T^2 is also generalized scalar. The following simple example shows that the same is not true for unbounded operators.

Example 0.1. Let $X \equiv BC^1(\mathbf{R})$ and define

$$(Tg)(s) \equiv sg(s) \ (s \in \mathbf{R}, g \in X),$$

with maximal domain.

Then T has a $BC^1(\mathbf{R})$ functional calculus $f \mapsto \mathcal{E}(f)$, given by

$$\mathcal{E}(f)g \equiv fg \ (f, g \in BC^1(\mathbf{R})).$$

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A functional calculus for T^2, call it $\mathcal{E}_2$, must be given by

$$[\mathcal{E}_2(f)g]\,(s) \equiv f(s^2)g(s) \ (g \in X, s \in \mathbf{R});$$

in order that $\mathcal{E}_2(f)$ be bounded, it is necessary that $s \mapsto sf'(s^2)$ be in $BC(\mathbf{R})$.

Thus T^2 is not generalized scalar; that is, for any $n \in \mathbf{N}$, T^2 fails to have a $BC^n(\mathbf{R})$ functional calculus, despite the fact that T has a $BC^1(\mathbf{R})$ functional calculus.

In this paper, we introduce a class of algebras, that we will call *compact admissible*, that seem more appropriate for unbounded operators, in particular, for determining when an operator is decomposable. This includes, but is not limited to, quasi-admissible algebras; for example, $C_c^\infty(\mathbf{R})$ is a compact admissible algebra that is not quasi admissible. We characterize those operators with a compact-admissible functional calculus that are decomposable, in terms of spectral mapping theorems and in terms of an analytic functional calculus.

Section I gives the basic properties of operators with a compact admissible functional calculus. Although they may not be decomposable, their restriction to local spectral subspaces corresponding to compact sets will be bounded and decomposable.

In Section II, we construct a much larger functional calculus, producing closed but not necessarily bounded operators, for operators with a compact admissible functional calculus.

Section III contains many characterizations of operators with a compact admissible functional calculus that are decomposable.

In Section IV we consider the easiest unbounded multiplication operators that are not scalar, multiplication operators on $BC^1(\mathbf{R})$. We characterize those operators that have a $C_c^1(\mathbf{R})$ functional calculus, and those that are decomposable, using the results from Section III. For such an operator to be decomposable, it is necessary but not sufficient that it have a $C_c^1(\mathbf{R})$ functional calculus. Examples are given of multiplication operators that do not have a $C_c^1(\mathbf{R})$ functional calculus, and of multiplication operators that have a $C_c^1(\mathbf{R})$ functional calculus, but are not decomposable.

All operators are linear, on a Banach space X. We will denote by $L(X)$ the space of bounded operators from X to itself. The domain of an operator T will be denoted by $\mathcal{D}(T)$, its image by $\mathrm{Im}(T)$, its null space by $\mathcal{N}(T)$, its spectrum by $\sigma(T)$, and its resolvent by $\rho(T)$.

Partly to standardize terminology, let us now list some preliminary definitions and results that will be needed throughout the paper.

Definition 0.2 ([6], [11]). The closed operator T is said to have the *single valued extension property* (SVEP) if, for every analytic function $f : \Omega_f \to$

$\mathcal{D}(T)$ defined on an open subset of the complex plane Ω_f, the identity

$$(\lambda - T)f(\lambda) \equiv 0$$

for $\lambda \in \Omega_f$, implies $f(\lambda) \equiv 0$.

Equivalently, for each $x \in X$, the analytic solution of

$$(*) \qquad (\lambda - T)f(\lambda) = x \ \ (\lambda \in \Omega_f)$$

is unique. When this property holds, the union of the domains Ω_f of all $\mathcal{D}(T)$-valued analytic functions f that satisfy (*) is called the *local resolvent set* of T at x, and is denoted by $\rho_T(x)$. The local spectrum $\sigma_T(x)$ is define to be the complement of $\rho_T(x)$ in $\mathbf{C}$. In addition, the SVEP of T implies the existence of an analytic function $\lambda \mapsto f(\lambda)$ that satisfies (*) on $\rho_T(x)$.

Definition 0.3 ([11], [12]). Suppose T is a closed operator and $Y \subseteq X$ is an invariant subspace of T; that is, $T(Y \cap \mathcal{D}(T)) \subseteq Y$. If, for any analytic function $f : \Omega_f \to \mathcal{D}(T)$, the condition $(\lambda - T)f(\lambda) \in Y$ implies that $f(\lambda) \in Y$ for all $\lambda \in \Omega_f$, then Y is called an *analytically invariant subspace* for T.

Proposition 0.4 ([6], [11]) If T has the SVEP, then the following holds.

(a) $\sigma(T) = \bigcup_{x \in X} \sigma_T(x)$.

(b) If Y is analytically invariant for T then $\sigma_{T|_Y}(x) = \sigma_T(x)$, hence

$$\sigma(T|_Y) = \bigcup_{x \in Y} \sigma_T(x).$$

Definition 0.5 ([6], [11]). Suppose T is closed and Y is invariant for T. Y is called a *spectral maximal space* of T if for every Z which is invariant for T, the condition $\sigma(T|_Z) \subseteq \sigma(T|_Y)$ implies $Z \subseteq Y$.

Suppose T has the SVEP. For $F \subseteq \mathbf{C}$, denote

$$X(T, F) = \{x \in X \mid \sigma_T(x) \subseteq F\}.$$

Proposition 0.6 ([6], [11]) Suppose T is closed and has the SVEP and $F \subseteq \mathbf{C}$ is closed. If $X(T, F)$ is closed, then it is a spectral maximal space of T and

$$\sigma(T|_{X(T,F)}) \subseteq F \cap \sigma(T).$$

Definition 0.7 ([10], [11]). The closed operator T is said to have the *spectral decomposition property* (SDP) if for every open cover $\{G_i\}_{i=0}^n$ $(n \in \mathbf{N})$ of $\sigma(T)$ with G_i $(i = 1, 2, ..., n)$ relatively compact and G_0 a neighborhood of ∞, there exists a system of invariant subspaces $\{X_i\}_{i=0}^n$ of T such that

(1) $X_i \subseteq \mathcal{D}(T)$, for $i = 1, 2, ..., n$;

(2) $\sigma(T|_{X_i}) \subseteq G_i$, for $i = 0, 1, ..., n$; and

(3) $X = \sum_{i=0}^{n} X_i$.

The operator T is said to be *decomposable* if it has SDP, and the subspaces X_i in (1) may be chosen to be spectral maximal ([11, Definition II.5.21]).

If $T \in B(X)$, then T has SDP if and only if T is decomposable ([11, Corollary II.6.5]). In general, T is decomposable if and only if T has SDP and $X(T, \emptyset)$ is trivial ([11, Theorem II.6.4]).

Proposition 0.8 ([1], [18], [11]) If T has the SDP, then T has the SVEP and for every closed $F \subseteq \mathbf{C}$, $X(T, F)$ is closed.

Definition 0.9 ([11], [15]). The closed operator T is said to have the *spectral decomposition property relative to the identity* (SDI) if for every open cover $\{G_i\}_{i=0}^{n}$ ($n \in \mathbf{N}$) of $\sigma(T)$ with G_i relatively compact for $i = 1, 2, ..., n$, and G_0 a neighborhood of ∞, there exist invariant subspaces $\{X_i\}_{i=0}^{n}$ and $\{P_i\}_{i=0}^{n} \subseteq L(X)$, that commute with T, such that

(1) $\sigma(T|_{X_i}) \subseteq G_i$, for $i = 0, 1, ..., n$;

(2) $X_i \subseteq \mathcal{D}(T)$, for $i = 1, 2, ..., n$;

(3) $\sum_{i=0}^{n} P_i = I$; and

(4) $\mathrm{Im}(P_i) \subseteq X_i$, for $i = 0, 1, ..., n$.

In [15], a bounded T satisfying Definition 0.9 is called superdecomposable. In this paper, we shall use SDI.

It is clear that if T has SDI then it has SDP.

Definition 0.10 ([5], [11]). The closed operator T is said to have *Property* $\mathcal{B}$ if, whenever $f_n : \Omega \to \mathcal{D}(T)$ is a sequence of analytic functions defined on an open subset of the complex plane such that

$$(\lambda - T)f_n(\lambda) \to 0$$

as $n \to \infty$, uniformly on compact subsets of Ω, then $f_n(\lambda) \to 0$ uniformly on compact subsets of Ω.

I. COMPACT-ADMISSIBLE FUNCTIONAL CALCULI AND BOUNDED DECOMPOSABLE RESTRICTIONS

We shall see that not all operators with a compact-admissible functional calculus are decomposable (see Section III for characterizations of those that are). However, their restriction to certain subspaces are bounded decomposable operators (Theorem 1.7, Corollaries 1.9 and 1.25; see also Corollaries 1.12 and 1.26).

When the intersections of the null spaces of $\mathcal{E}(\phi)$, for $\mathcal{E}$ a $\mathcal{U}$ compact-admissible functional calculus, $\phi \in \mathcal{U}$, is trivial, we will say that $\mathcal{E}$ is a *quasi distribution* (Definition 1.17). When T has a $\mathcal{U}$ quasi distribution, we may characterize the local spectral subspaces of T corresponding to compact sets, in terms of $\mathcal{E}$ (Proposition 1.23).

When $\mathcal{U} \equiv C_c^\infty(\mathbf{R})$, a $\mathcal{U}$ (quasi) distribution is introduced in [4] ([8]), and is called a (quasi) spectral distribution.

Definition 1.1. An algebra $\mathcal{U}$ of complex-valued functions defined on a subset Ω of the complex plane is *compact admissible* if it has the following properties.

(1) For every $\phi \in \mathcal{U}$, ϕ has compact support.

(2) $\mathcal{U}$ is *compact normal*; that is, for any compact set $F \subseteq \Omega$, open cover $\{G_i\}_{i=1}^n$ of F, there are $\phi_1, ..., \phi_n \in \mathcal{U}$ such that $0 \leq \phi_i \leq 1$, for $1 \leq i \leq n$, $\sum_{i=1}^n \phi_i \equiv 1$ on F and $supp(\phi_i) \subseteq G_i$, for $1 \leq i \leq n$.

(3) If $\phi \in \mathcal{U}$, then $z \mapsto f_1(z)\phi(z) \equiv z\phi(z)$ and $\phi_\lambda(z) \equiv (\lambda - z)^{-1}\phi(z)$ are in $\mathcal{U}$, for any $\lambda \notin supp(\phi)$.

Example 1.2. Any admissible or quasi-admissible algebra ([20, Definitions IV.7.2 and IV.9.2]) is compact admissible. For n a nonnegative integer, $C_c^n(\mathbf{R})$ is compact admissible, but not normal (see [20, Definition IV.7.1]), hence not admissible or quasi admissible. If we define $C_\ell^n(\mathbf{R})$ to be the space of n-times continuously differentiable complex-valued functions on the one-point compactification of the real line, that is, $\{f \in C^n(\mathbf{R}) \mid \lim_{|s|\to\infty} f^{(k)}(s)$ exists, for $0 \leq k \leq n\}$, then $C_\ell^n(\mathbf{R})$ is quasi admissible, for any nonnegative integer n.

In Section IV, we will give simple examples of operators with compact-admissible functional calculi that do not have quasi-admissible functional calculi (see Remark 4.10).

Definition 1.3. Suppose T is a closed operator and $\mathcal{U}$ is compact admissible as in Definition 1.1. We say that $\mathcal{E}$ is a $\mathcal{U}$ *functional calculus for* T if $\mathcal{E} : \mathcal{U} \to L(X)$ is an algebra homomorphism such that, for all $\phi \in \mathcal{U}$,

$$\mathcal{E}(\phi)T \subseteq T\mathcal{E}(\phi) = \mathcal{E}(f_1\phi).$$

For the remainder of this section, $\mathcal{U}$ will be compact admissible as in Definition 1.1 and $\mathcal{E}$ will be a $\mathcal{U}$ functional calculus for T.

On certain subspaces the restriction of T will be a bounded operator with an admissible functional calculus; in [6, Chapter 3], these are known as $\mathcal{U}$ scalar operators.

Definition 1.4. If F is a compact subset of the complex plane, define

$$X_{\mathcal{E}}(T, F) \equiv$$
$$\{x \in X | \mathcal{E}(\phi)x = x, \text{ whenever } \phi \in \mathcal{U}, \phi = 1 \text{ on a neighborhood of } F\}.$$

If F is a closed subset of the complex plane, define

$$X_{\mathcal{E}_1}(T,F) \equiv \bigcap \{\mathcal{N}(\mathcal{E}(\phi)) \mid \phi \in \mathcal{U}, supp(\phi) \cap F \text{ is empty } \}.$$

Note that $X_{\mathcal{E}}(T,F)$ and $X_{\mathcal{E}_1}(T,F)$ are closed subspaces of X, since $\mathcal{E}(\phi) \in L(X)$, for all $\phi \in \mathcal{U}$.

Proposition 1.5. *Suppose F is a compact subset of the complex plane. Then*

$$X_{\mathcal{E}}(T,F) \subseteq X_{\mathcal{E}_1}(T,F).$$

Proof. Suppose $x \in X_{\mathcal{E}}(T,F)$, and let $\psi \in \mathcal{U}$ be such that $supp(\psi) \cap F$ is empty. Choose $\phi \in \mathcal{U}$ so that $supp(\phi) \cap supp(\psi)$ is empty and $\phi \equiv 1$ on a neighborhood of F. Then

$$\mathcal{E}(\psi)x = \mathcal{E}(\psi)\mathcal{E}(\phi)x = \mathcal{E}(\psi\phi)x = \mathcal{E}(0)x = 0,$$

so that $x \in X_{\mathcal{E}_1}(T,F)$. $\square$

Definition 1.6. If F is a compact subset of the complex plane, let $T_F \equiv T|_{X_{\mathcal{E}}(T,F)}$,

$\mathcal{U}_F \equiv$

$\{\psi | \phi\psi \in \mathcal{U},$ for some $\phi \in \mathcal{U}$ such that $\phi = 1$ on a neighborhood of $F\}$.

Define a $\mathcal{U}_F$ functional calculus for T_F by

$$\mathcal{E}_F(\psi) \equiv \mathcal{E}(\psi\phi),$$

for ϕ as in the definition of $\mathcal{U}_F$. Note that $\mathcal{E}_F$ is well-defined, by Proposition 1.5: if $\psi\phi_j \in \mathcal{U}$, and $\phi_j = 1$ on a neighborhood of F, for $j = 1,2$, then $(\psi\phi_1 - \psi\phi_2) = 0$ on a neighborhood of F, so for $x \in X_{\mathcal{E}}(T,F)$,

$$\mathcal{E}(\psi\phi_1)x - \mathcal{E}(\psi\phi_2)x = \mathcal{E}\left(\psi\phi_1 - \psi\phi_2\right)x = 0.$$

Theorem 1.7. *If F is a compact subset of the complex plane, then T_F is in $L(X_{\mathcal{E}}(T,F))$, $\mathcal{E}_F$ is a $\mathcal{U}_F$ functional calculus for T_F and $\sigma(T_F) \subseteq F$.*

Proof. It's clear that $\mathcal{E}_F$ is an algebra homomorphism.

$\mathcal{E}_F(1) = \mathcal{E}(\phi) = I$, by definition of $X_{\mathcal{E}}(T,F)$, for $\phi \in \mathcal{U}$, $\phi = 1$ on a neighborhood of F.

Using the same ϕ, it becomes clear that $f_1 \in \mathcal{U}_F$, and for $x \in X_{\mathcal{E}}(T,F)$, $x = \mathcal{E}(\phi)x \in \mathcal{D}(T)$, with

$$T_F x = T\mathcal{E}(\phi)x = \mathcal{E}(f_1\phi)x = \mathcal{E}_F(f_1)x.$$

To see that T_F maps $X_{\mathcal{E}}(T,F)$ to itself, suppose $x \in X_{\mathcal{E}}(T,F)$. If $\phi = 1$ on a neighborhood of F, then $\mathcal{E}(\phi)T_F x = T_F\mathcal{E}(\phi)x = T_F x$, so $T_F x \in X_{\mathcal{E}}(T,F)$.

Since T is closed, this implies that $T_F \in L(X_{\mathcal{E}}(T,F))$.

If $\lambda \notin F$, then $g_\lambda \in \mathcal{U}_F$ $(g_\lambda(z) \equiv (\lambda - z)^{-1})$,

$$(\lambda - T_F)\mathcal{E}_F(g_\lambda) = \mathcal{E}((\lambda - f_1)\phi g_\lambda) = \mathcal{E}(\phi) = I,$$

and

$$\mathcal{E}_F(g_\lambda)(\lambda - T_F) = \mathcal{E}_F(g_\lambda \phi)\mathcal{E}_F((\lambda - f_1)\phi) = \mathcal{E}_F(\phi^2) = I,$$

where $\phi = 1$ on a neighborhood of F. This shows that $\sigma(T_F) \subseteq F$. □

Remark 1.8. The map $\mathcal{E}_F : \mathcal{U}_F \to L(X)$ is a well-defined algebra homomorphism for $T|_{X_{\mathcal{E}_1}(T,F)}$. All that it lacks to make it a functional calculus for $T|_{\mathcal{E}_1(T,F)}$ is that $\mathcal{E}_F(1)$ might not be the identity, just some projection on $X_{\mathcal{E}_1(T,F)}$. In fact, $X_{\mathcal{E}}(T,F)$ is the image of that projection, $[\mathcal{E}_F(1)]\,(X_{\mathcal{E}_1}(T,F))$.

Corollary 1.9. *If F is a compact subset of the complex plane, then T_F has the SDI.*

Proof. Since $\mathcal{U}_F$ is admissible, this follows from Theorem 1.7 and [14, Theorem 1.4.2]. □

Definition 1.10. If F is compact, then we say that $\mathcal{U}$ is *inverse closed on* F if every $\phi \in \mathcal{U}$ is continuous on a neighborhood δ_ϕ of F, and, if for all $z \in \delta_\phi$, $\phi(z) \neq 0$, then there exists $\psi \in \mathcal{U}$ such that

$$\psi(z) = \frac{1}{\phi(z)} \ \ \forall z \in \delta_\phi.$$

If $\mathcal{U}$ is inverse closed on every compact F, then we say that $\mathcal{U}$ is *inverse closed.*

Proposition 1.11. *If $\mathcal{U}$ is inverse closed on a compact set F, then for every $\phi \in \mathcal{U}$,*

$$\phi(\sigma(T_F)) = \sigma(\mathcal{E}(\phi)|_{X_{\mathcal{E}}(T,F)}).$$

Proof. Note that $\mathcal{E}(\phi)|_{X_{\mathcal{E}}(T,F)} = \mathcal{E}_F(\phi)$. Since $\mathcal{U}_F$ is admissible and inverse closed, the proposition follows from [6, Theorem 3.2.1]. □

Corollary 1.12. *If $\mathcal{U}$ is inverse closed on a compact set F, then for every $\phi \in \mathcal{U}$, $\mathcal{E}(\phi)|_{X_{\mathcal{E}}(T,F)}$ is decomposable.*

Proof. As with the proof of Proposition 1.11, this follows from [6, Theorem 3.2.4]. □

Definition 1.13. Suppose $G \subseteq \mathbf{C}$ is open. We will say that $\mathcal{E}$ is zero on G if $\mathcal{E}(\phi) = 0$ whenever $\phi \in \mathcal{U}$ is supported in G. The *support* of $\mathcal{E}$ is

$$supp(\mathcal{E}) \equiv \mathbf{C} - \bigcup\{G \,|\, \mathcal{E} \text{ is zero on } G\}.$$

The following proposition is an analogue of [16, Lemma 2.3]. We offer a proof for completeness.

Proposition 1.14. (1) *For every $\phi \in \mathcal{U}$, $\lambda \mapsto \mathcal{E}(\phi_\lambda)$ is analytic on* $\mathbf{C} -$ *supp*(ϕ), *where ϕ_λ is defined in Definition 1.1.*

(2) $\lim_{\lambda\to\infty} \mathcal{E}(\phi_\lambda) = 0$ *in* $L(X)$.

Proof. (1) Fix $\lambda \notin supp(\phi)$. For any $\mu \notin supp(\phi)$, $n \in \mathbf{N}$,

$$\phi_\mu(z) = \sum_{k=0}^{n} (\lambda-\mu)^k (\lambda - z)^{-(k+1)}\phi(z) + (\lambda-\mu)^{n+1}(\lambda-z)^{-(n+1)}\phi_\mu(z),$$

so that

$$\mathcal{E}(\phi_\mu) = \sum_{k=0}^{n} (\lambda-\mu)^k \mathcal{E}(z \mapsto (\lambda-z)^{-(k+1)}\phi) + (\lambda-\mu)^{n+1}\mathcal{E}(z \mapsto (\lambda-z)^{-(n+1)}\phi_\mu).$$

Let $\psi \in \mathcal{U}$ such that $\psi = 1$ on the support of ϕ and $\lambda \notin supp(\psi)$. Then

$$(\lambda-z)^{-(n+1)}\phi_\mu(z) = [\psi_\lambda(z)]^{n+1}\,\phi_\mu(z)$$

and

$$\mathcal{E}(z \mapsto (\lambda-z)^{-(n+1)}\phi_\mu(z)) = [\mathcal{E}(\psi_\lambda)]^{n+1}\,\mathcal{E}(\phi_\mu).$$

This implies that

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \left[|\lambda-\mu|^{n+1}\|\mathcal{E}(z \mapsto (\lambda-z)^{-(n+1)}\phi_\mu(z))\|\right]^{\frac{1}{n}} \\
&\le \overline{\lim_{n\to\infty}} \left[|\lambda-\mu|^{n+1}\|\mathcal{E}(\psi_\lambda)\|^{n+1}\|\mathcal{E}(\phi_\mu)\|\right]^{\frac{1}{n}} \\
&= |\lambda-\mu|\|\mathcal{E}(\psi_\lambda)\| < 1
\end{aligned}$$

if $|\lambda-\mu| < \frac{1}{\|\mathcal{E}(\psi_\lambda)\|}$.

Thus, for $|\lambda-\mu| < \frac{1}{\|\mathcal{E}(\psi_\lambda)\|}$, we have

$$\mathcal{E}(\phi_\mu) = \sum_{k=0}^{\infty} (\lambda-\mu)^k \mathcal{E}(z \mapsto (\lambda-z)^{-(k+1)}\phi(z)).$$

This implies that $\lambda \mapsto \mathcal{E}(\phi_\lambda)$ is analytic on $\mathbf{C} - supp(\phi)$.
(2) Choose $\psi \in \mathcal{U}$ such that $\psi = 1$ on a neighborhood of the support of ϕ. Arguing as in the proof of (1), we have, for $|\lambda|$ sufficiently large,

$$\mathcal{E}(\phi_\lambda) = \sum_{n=0}^{\infty} (\lambda)^{-(n+1)}\,[\mathcal{E}(z \mapsto z\psi(z))]^n\,\mathcal{E}(\phi).$$

This implies (2). ∎

Corollary 1.15. *For any $\phi \in \mathcal{U}, x \in X$,*

$$\sigma_T(\mathcal{E}(\phi)x) \subseteq supp(\phi) \cap \sigma_T(x).$$

Proof. Since $\mathcal{E}(\phi)T \subseteq T\mathcal{E}(\phi)$,

$$\sigma_T(\mathcal{E}(\phi)x) \subseteq \sigma_T(x),$$

by [11, Proposition I.2.6(III)]. For $\lambda \notin supp(\phi)$,

$$(\lambda - T)\mathcal{E}(\phi_\lambda)x = \mathcal{E}(\phi)x,$$

thus by Proposition 1.14,

$$\sigma_T(\mathcal{E}(\phi)x) \subseteq supp(\phi).$$

□

Definition 1.16. For $x \in X$, $\mathcal{E} \otimes x$ is the following X-valued distribution:

$$(\mathcal{E} \otimes x)(\phi) \equiv \mathcal{E}(\phi)x \;\; (\phi \in \mathcal{U}).$$

The support of $\mathcal{E} \otimes x$ is

$$supp(\mathcal{E} \otimes x) \equiv \mathbf{C} - \bigcup \{ \text{ open } G \subseteq \mathbf{C} \,|\, \mathcal{E} \otimes x \text{ is zero on } G\}.$$

Note that, if F is closed, $X_{\mathcal{E}_1}(T, F) = \{x \,|\, supp(\mathcal{E} \otimes x) \subseteq F\}$.

Definition 1.17. (a) The $\mathcal{U}$ functional calculus $\mathcal{E}$ is a $\mathcal{U}$ *quasi distribution* if

$$N(\mathcal{E}) \equiv \bigcap \{\mathcal{N}(\mathcal{E}(\phi)) \,|\, \phi \in \mathcal{U}\} = \{\vec{0}\}.$$

(b) The $\mathcal{U}$ functional calculus $\mathcal{E}$ is a $\mathcal{U}$ *distribution* if there exists a sequence of compact sets $\{\triangle_n\}_{n \in \mathbf{N}} \subseteq \mathbf{C}$ and a sequence of functions $\{\phi_n\}_{n \in \mathbf{N}} \subseteq \mathcal{U}$ such that $\cup_{n=1}^{\infty} \triangle_n = \mathbf{C}$, for each $n \in \mathbf{N}$, $\phi_n = 1$ on a neighborhood of $\triangle_n$, and

$$\mathcal{E}(\phi_n)x \to x \;\; \forall x \in X,$$

as $n \to \infty$.

Proposition 1.18. *If $\mathcal{E}$ is a $\mathcal{U}$ distribution, then $\mathcal{E}$ is a $\mathcal{U}$ quasi distribution.*

Proof. Suppose $x \in N(\mathcal{E})$. Then, for ϕ_n as in Definition 1.17(b),

$$x = \lim_{n \to \infty} \mathcal{E}(\phi_n)x = 0.$$

□

Remark 1.19. Note that $N(\mathcal{E}) = X_{\mathcal{E}_1}(T, \emptyset)$.

Lemma 1.20. *Suppose $\mathcal{E}$ is a $\mathcal{U}$ functional calculus for T, $x_0 \in \mathcal{D}(T)$, and $(\lambda - T)x_0 = 0$.*

(1) *supp*$(\mathcal{E} \otimes x_0) = \{\lambda\}$.
(2) *If* $\mathcal{E}$ *is a* $\mathcal{U}$ *quasi distribution, and* $\phi \in \mathcal{U}$ *equals* 1 *on a neighborhood of* λ*, then*
$$\mathcal{E}(\phi)x_0 = x_0.$$

Proof. (1) Suppose $\psi \in \mathcal{U}$ and λ is not in the support of ψ. Then
$$\mathcal{E}(\psi)x_0 = \mathcal{E}(\psi_\lambda(\lambda - f_1))x_0 = \mathcal{E}(\psi_\lambda)(\lambda - T)x_0 = 0.$$
This shows that *supp*$(\mathcal{E} \otimes x_0) = \{\lambda\}$.
(2) For any $\psi \in \mathcal{U}$,
$$\mathcal{E}(\psi)\left(\mathcal{E}(\phi)x_0 - x_0\right) = \mathcal{E}\left(\psi\phi - \psi\right)x_0 = 0,$$
since $(\psi\phi - \psi) = 0$ on a neighborhood of λ.

This implies that $\mathcal{E}(\phi)x_0 - x_0 = 0$. □

Proposition 1.21. *If there exists a* $\mathcal{U}$ *quasi distribution for* T*, then* T *has the SVEP.*

Proof. Supppose G is an open subset of the complex plane and $f : G \to \mathcal{D}(T)$ is analytic and satisfies
$$(\lambda - T)f(\lambda) = 0 \ \ (\lambda \in G).$$
Without loss of generality, we may assume G is connected. For the sake of contradiction, suppose there exists $\lambda_0 \in G$ such that $f(\lambda_0) \neq 0$. Then there exists a relatively compact neighborhood δ of λ_0 such that for all $\lambda \in \delta$, $f(\lambda) \neq 0$ and $\overline{\delta} \subseteq G$. Choose $\phi \in \mathcal{U}$ such that
$$\phi(\lambda) = 1 \ \ \forall \lambda \in \delta_1,$$
where δ_1 is a neighborhood of $\overline{\delta}$, with $G - \overline{\delta_1}$ nonempty, and
$$\phi(\lambda) = 0 \ \ \forall \lambda \in G - \overline{\delta_1}.$$
By Lemma 1.20,
$$\mathcal{E}(\phi)f(\lambda) = f(\lambda) \ \ \forall \lambda \in \delta_1,$$
and
$$\mathcal{E}(\phi)f(\lambda) = 0 \ \ \forall \lambda \in G - \overline{\delta_1}.$$
Since $\lambda \mapsto \mathcal{E}(\phi)f(\lambda)$ is analytic, this implies that $\mathcal{E}(\phi)f(\lambda) = 0$ for all $\lambda \in G$. Thus $f(\lambda) = 0$ for all $\lambda \in G$, as desired. □

Lemma 1.22. *If there exists a* $\mathcal{U}$ *quasi distribution for* T*, then*

(a)
$$supp(\mathcal{E} \otimes x) \subseteq \sigma_T(x);$$
and

(b)
$$supp(\mathcal{E}) \subseteq \sigma(T).$$

Proof. (a) This is equivalent to showing that $x \in X_{\mathcal{E}_1}(T, \sigma_T(x))$. So suppose $\phi \in \mathcal{U}$ is zero in an open set containing $\sigma_T(x)$. We must show that $\mathcal{E}(\phi)x = 0$. Let $x(\lambda)$ be the local resolvent of T at x. Then

$$(\lambda - T)\mathcal{E}(\phi)x(\lambda) = \mathcal{E}(\phi)x \ \ \forall\lambda \notin \sigma_T(x);$$

and

$$(\lambda - T)\mathcal{E}(\phi_\lambda)x = \mathcal{E}(\phi)x \ \ \forall\lambda \notin supp(\phi).$$

Since T has the SVEP, this implies that the local resolvent $h(\lambda)$ for $\mathcal{E}(\phi)x$ is entire, and for $\lambda \notin supp(\phi)$,

$$h(\lambda) = \mathcal{E}(\phi_\lambda)x.$$

By Proposition 1.14(2), $h(\lambda) \to 0$ as $\lambda \to \infty$. Thus $h(\lambda) \equiv 0$, so that, for $\lambda \notin \sigma_T(x)$,

$$\mathcal{E}(\phi)x = (\lambda - T)h(\lambda) = 0.$$

(b) follows from (a) and the fact that

$$supp(\mathcal{E}) = \bigcup_{x \in X} supp(\mathcal{E} \otimes x) \quad \text{and} \quad \sigma(T) = \bigcup_{x \in X} \sigma_T(x).$$

□

Proposition 1.23. *Suppose $\mathcal{E}$ is a $\mathcal{U}$ functional calculus for T, and $F \subseteq \mathbf{C}$ is compact; then*

$$X_{\mathcal{E}}(T, F) \subseteq X(T, F).$$

If $\mathcal{E}$ is a $\mathcal{U}$ quasi distribution, then

$$X(T, F) = X_{\mathcal{E}_1}(T, F) = X_{\mathcal{E}}(T, F).$$

Proof. Suppose $x \in X_{\mathcal{E}}(T, F)$, and $\phi \in \mathcal{U}$ equals one on a neighborhood of F. Then for $\lambda \notin supp(\phi)$,

$$(\lambda - T)\mathcal{E}(\phi_\lambda)x = \mathcal{E}(\phi)x = x.$$

Thus $\sigma_T(x) \subseteq supp(\phi)$. Since ϕ was arbitrary, this implies that $\sigma_T(x) \subseteq F$, so that $X_{\mathcal{E}}(T, F) \subseteq X(T, F)$.

Now suppose $\mathcal{E}$ is a $\mathcal{U}$ quasi distribution. We will show that

$$X(T, F) \subseteq X_{\mathcal{E}_1}(T, F) \subseteq X_{\mathcal{E}}(T, F).$$

Lemma 1.22(a) implies that $X(T, F) \subseteq X_{\mathcal{E}_1}(T, F)$.

Now suppose $x \in X_{\mathcal{E}_1}(T, F)$, and $\phi = 1$ on a neighborhood of F. For any $\psi \in \mathcal{U}$,

$$\mathcal{E}(\psi)\left(\mathcal{E}(\phi)x - x\right) = \mathcal{E}(\psi\phi - \psi)x = 0,$$

since $(\psi\phi - \psi) = 0$ on a neighborhood of F. Since $\mathcal{E}$ is a quasi distribution, this implies that $\mathcal{E}(\phi)x = x$, so that $x \in X_{\mathcal{E}}(T, F)$. □

Corollary 1.24. *If $\mathcal{E}$ is a $\mathcal{U}$ quasi distribution for T, then $X(T, \emptyset)$ is trivial.*

Proof. This follows from Proposition 1.23; see Remark 1.19. □

Corollary 1.25. *If there exists a $\mathcal{U}$ quasi distribution for T, then for any compact $F \subseteq \mathbf{C}$, $T|_{X(T,F)}$ has the SDI.*

Proof. By Proposition 1.23, $X(T, F) = X_{\mathcal{E}}(T, F)$, so that $T_F = T|_{X(T,F)}$, thus this follows from Corollary 1.9. □

Proposition 1.26. *If there exists a $\mathcal{U}$ quasi distribution for T and $\mathcal{U}$ is inverse closed on a compact set F, then for every $\phi \in \mathcal{U}$,*

$$\phi(\sigma(T_F)) = \sigma(\mathcal{E}(\phi)|_{X(T,F)}).$$

Proof. Propositions 1.23 and 1.11. □

Corollary 1.27. *If there exists a $\mathcal{U}$ quasi distribution for T and $\mathcal{U}$ is inverse closed on a compact set F, then for every $\phi \in \mathcal{U}$, $\mathcal{E}(\phi)|_{X(T,F)}$ is decomposable.*

Proof. Proposition 1.23 and Corollary 1.12. □

II. AUTOMATIC EXTENSIONS OF COMPACT-ADMISSIBLE FUNCTIONAL CALCULI

Throughout this section, assume that $\mathcal{U}$ is compact admissible as in Definition 1.1 and $\mathcal{E}$ is a $\mathcal{U}$ quasi distribution for the closed operator T as in Definition 1.17. We shall construct, in three different ways ($\mathcal{E}_j, j = 0, 1, 2$), a functional calculus for T defined for a much larger class of functions. Theorem 2.3 summarizes in what sense the map $f \mapsto \mathcal{E}_1(f)$ is a functional calculus for T. For $\mathcal{U} = C_c^\infty(\mathbf{R})$, Theorem 2.3 is in [8]; in that case, the enlarged class of functions $\mathcal{M}(\mathcal{U})$ equals $C^\infty(\mathbf{R})$. See [7] for some other methods of extending functional calculi.

Theorem 2.10 expresses our extended functional calculus $\mathcal{E}_1(f)x$ as a Dunford-Taylor-type Cauchy integral formula, for f analytic in a neighborhood of infinity and the local spectrum of T at x.

Definition 2.1. As in [17], denote by $\mathcal{M}(\mathcal{U})$ the set of all functions f such that $\phi f \in \mathcal{U}$, for all $\phi \in \mathcal{U}$. In [17], $\mathcal{U}$ was assumed to be admissible; here we only assume that $\mathcal{U}$ is compact admissible.

We define three maps from $\mathcal{M}(\mathcal{U})$ into the space of (possibly unbounded) linear operators on X.

For $f \in \mathcal{M}(\mathcal{U})$, define $\mathcal{E}_j(f)$, $j = 0, 1, 2$, as follows.

$$\mathcal{D}(\mathcal{E}_0(f)) \equiv \bigcup_{\phi\in\mathcal{U}} \mathrm{Im}(\mathcal{E}(\phi)),$$
$$\mathcal{E}_0(f)(\mathcal{E}(\phi)x) \equiv \mathcal{E}(\phi f)x \quad (\phi \in \mathcal{U}, x \in X).$$

$$\mathcal{D}(\mathcal{E}_1(f)) \equiv \{x \,|\, \text{there exists } y \text{ such that } \mathcal{E}(\phi)y = \mathcal{E}(\phi f)x, \forall \phi \in \mathcal{U}\};$$
$$\mathcal{E}_1(f)x \equiv y.$$

Note that the definition of a quasi distribution guarantees that y is unique.

Finally, $\mathcal{D}(\mathcal{E}_2(f))$ is defined to be the set of all $x \in X$ for which there exists $\phi_n \in \mathcal{U}, y \in X$ such that

$$\mathcal{E}(\phi_n)x \to x, \mathcal{E}(\phi_n f)x \to y \text{ as } n \to \infty,$$

with

$$\mathcal{E}_2(f)x \equiv y.$$

Proposition 2.2. *For all $f \in \mathcal{M}(\mathcal{U})$, $\psi \in \mathcal{U}$,*

(1)
$$\mathcal{E}(\psi)\mathcal{E}_j(f) \subseteq \mathcal{E}_j(f)\mathcal{E}(\psi) = \mathcal{E}(\psi f),$$
for $j = 0, 1, 2$;

(2) *$\mathcal{E}_1(f)$ is closed;*

(3) *$\mathcal{E}_1$ is an extension of $\mathcal{E}$; and*

(4)
$$\mathcal{E}_0(f) \subseteq \mathcal{E}_2(f) \subseteq \overline{\mathcal{E}_0(f)} \subseteq \mathcal{E}_1(f).$$
If $\mathcal{E}$ is a $\mathcal{U}$ distribution, then
$$\mathcal{E}_2(f) = \overline{\mathcal{E}_0(f)} = \mathcal{E}_1(f).$$

Proof. (1) For $j = 0$, note that, for any $x \in X$, $\phi \in \mathcal{U}$,

$$\begin{aligned}\mathcal{E}(\psi)\mathcal{E}_0(f)(\mathcal{E}(\phi)x) &\equiv \mathcal{E}(\psi)\mathcal{E}(f\phi)x = \mathcal{E}(\psi f\phi)x \\ &\equiv \mathcal{E}_0(f)\mathcal{E}(\psi)(\mathcal{E}(\phi)x) = \mathcal{E}(\psi f)(\mathcal{E}(\phi)x).\end{aligned}$$

For $j = 1$, suppose $x \in \mathcal{D}(\mathcal{E}_1(f))$. Then by definition of $\mathcal{D}(\mathcal{E}_1(f))$,

$$\mathcal{E}(\psi)\mathcal{E}_1(f)x = \mathcal{E}(\psi f)x,$$

so $\mathcal{E}(\psi)\mathcal{E}_1(f) \subseteq \mathcal{E}(\psi f)$.

Now suppose $x \in X$. For any $\phi \in \mathcal{U}$,

$$\mathcal{E}(\phi)\left(\mathcal{E}(\psi f)x\right) = \mathcal{E}(\phi\psi f)x = \mathcal{E}(\phi f)\left(\mathcal{E}(\psi)x\right),$$

thus $\mathcal{E}(\psi)x \in \mathcal{D}(\mathcal{E}_1(f))$, and

$$\mathcal{E}_1(f)\left(\mathcal{E}(\psi)x\right) = \mathcal{E}(\psi f)x,$$

so that $\mathcal{E}_1(f)\mathcal{E}(\psi) = \mathcal{E}(\psi f)$, as desired.

For $j = 2$, suppose $x \in \mathcal{D}(\mathcal{E}_2(f))$, with ϕ_n as in the definition of $\mathcal{D}(\mathcal{E}_2(f))$. Then

$$\mathcal{E}(\phi_n)\mathcal{E}(\psi)x = \mathcal{E}(\psi)\mathcal{E}(\phi_n)x \to \mathcal{E}(\psi)x$$

and

$$\mathcal{E}(\phi_n f)\mathcal{E}(\psi)x = \mathcal{E}(\psi)\mathcal{E}(\phi_n f)x \to \mathcal{E}(\psi)y,$$

as $n \to \infty$, since $\mathcal{E}(\psi) \in L(X)$. This shows that $\mathcal{E}(\psi)\mathcal{E}_2(f) \subseteq \mathcal{E}_2(f)\mathcal{E}(\psi)$.

Now suppose $x \in X$. Let $z \equiv \mathcal{E}(\psi)x$, $y \equiv \mathcal{E}(\psi f)x$. To show that $z \in \mathcal{D}(\mathcal{E}_2(f))$, choose $\delta \in \mathcal{U}$ such that $\delta \equiv 1$ on the support of ψ. By Corollary 1.15, both z and y are in $X(T, supp(\psi))$, thus by Proposition 1.23, $\mathcal{E}(\delta)z = z$ and $\mathcal{E}(\delta)y = y$. Thus, choosing $\phi_n \equiv \delta$, for any n, we see that $z \in \mathcal{D}(\mathcal{E}_2(f))$, and $\mathcal{E}_2(f)z = y$, so that $\mathcal{E}_2(f)\mathcal{E}(\psi) = \mathcal{E}(\psi f)$, as desired.
(2) Suppose $\{x_n\}_{n\in\mathbf{N}} \subseteq \mathcal{D}(\mathcal{E}_1(f))$, $\mathcal{E}_1(f)x_n \to y$ and $x_n \to x$ as $n \to \infty$. For any $\phi \in \mathcal{U}$, since $\mathcal{E}(\phi f)$ and $\mathcal{E}(\phi)$ are in $L(X)$, we have

$$\mathcal{E}(\phi f)x_n \to \mathcal{E}(\phi f)x \quad \text{and} \quad \mathcal{E}(\phi f)x_n = \mathcal{E}(\phi)\mathcal{E}(f)x_n \to \mathcal{E}(\phi)y.$$

Thus $\mathcal{E}(\phi)y = \mathcal{E}(\phi f)x$, for all $\phi \in \mathcal{U}$, so that $x \in \mathcal{D}(\mathcal{E}_1(f))$, and $\mathcal{E}_1(f)x = y$, as desired.
(3) It is clear that

$$\mathcal{E}(\phi)\mathcal{E}(\psi)x = \mathcal{E}(\phi\psi)x \quad \forall \phi \in \mathcal{U}, x \in X.$$

Thus $X \subseteq \mathcal{D}(\mathcal{E}_1(\psi))$, and $\mathcal{E}_1(\psi)x = \mathcal{E}(\psi)x$, for all $x \in X$; that is, $\mathcal{E}_1(\psi) = \mathcal{E}(\psi)$, for all $\psi \in \mathcal{U}$, so that $\mathcal{E}_1$ is an extension of $\mathcal{E}$.
(4) It is clear, from the definition of closure of an operator, that $\mathcal{E}_2(f) \subseteq \overline{\mathcal{E}_0(f)}$.

For $j = 1, 2$, the inclusion $\mathcal{E}_0(f) \subseteq \mathcal{E}_j(f)$ follows from the fact that $\mathcal{E}_j(f)\mathcal{E}(\phi) = \mathcal{E}(\phi f)$, for all $\phi \in \mathcal{U}$ (see (1)).

Since $\mathcal{E}_1(f)$ is closed, this also implies that $\overline{\mathcal{E}_0(f)} \subseteq \mathcal{E}_1(f)$. This concludes the proof of (4).

Now suppose $\mathcal{E}$ is a $\mathcal{U}$ distribution. All that remains is to show that $\mathcal{E}_1(f) \subseteq \mathcal{E}_2(f)$. Suppose $x \in \mathcal{D}(\mathcal{E}_1(f))$. Choose ϕ_n as in Definition 1.17(b). Then by (1),

$$\mathcal{E}(f\phi_n)x = \mathcal{E}(\phi_n)\mathcal{E}_1(f)x \to \mathcal{E}_1(f)x$$

as $n \to \infty$, thus $x \in \mathcal{D}(\mathcal{E}_2(f))$ and $\mathcal{E}_2(f)x = \mathcal{E}_1(f)x$, as desired. □

We will be particularly interested in the largest extension $\mathcal{E}_1$.

Theorem 2.3. (1) $\mathcal{E}_1(f_0) = I$ *(*$f_0(z) \equiv 1$*).*

(2) *For any* $f, g \in \mathcal{M}(\mathcal{U})$,

$$\mathcal{E}_1(f) + \mathcal{E}_1(g) \subseteq \mathcal{E}_1(f + g).$$

(3) $\mathcal{M}(\mathcal{U})$ *is an algebra, and for any* $f, g \in \mathcal{M}(\mathcal{U})$, *we have* $\mathcal{E}_1(f)\mathcal{E}_1(g) \subseteq \mathcal{E}_1(fg)$, *with*

$$\mathcal{D}(\mathcal{E}_1(f)\mathcal{E}_1(g)) = \mathcal{D}(\mathcal{E}_1(g)) \cap \mathcal{D}(\mathcal{E}_1(fg)).$$

(4) *For any* $g \in \mathcal{M}(\mathcal{U})$, *complex* λ, *if* $z \mapsto (\lambda - g(z))^{-1} \in \mathcal{M}(\mathcal{U})$, *then* $(\lambda - \mathcal{E}_1(g))$ *is injective, and*

$$(\lambda - \mathcal{E}_1(g))^{-1} = \mathcal{E}_1(z \mapsto (\lambda - g(z))^{-1}).$$

(5) $\mathcal{E}_0(f_1) \subseteq T \subseteq \mathcal{E}_1(f_1)$.
If $\mathcal{E}$ *is a* $\mathcal{U}$ *distribution, then* $T = \mathcal{E}_1(f_1)$.

Proof. (1) is clear from the definition of $\mathcal{E}_1$.
(2) Suppose $x \in \mathcal{D}(\mathcal{E}_1(f)) \cap \mathcal{D}(\mathcal{E}_1(g))$. Then for any $\phi \in \mathcal{U}$,

$$\mathcal{E}(\phi)[(\mathcal{E}_1(f))x + (\mathcal{E}_1(g))x] = \mathcal{E}(\phi f)x + \mathcal{E}(\phi g)x = \mathcal{E}(\phi(f+g))x,$$

so that $x \in \mathcal{D}(\mathcal{E}_1(f+g))$ and

$$(\mathcal{E}_1(f+g))x = (\mathcal{E}_1(f))x + (\mathcal{E}_1(g))x.$$

(3) It is clear from the definition of $\mathcal{M}(\mathcal{U})$ that it is an algebra.
Suppose $x \in \mathcal{D}((\mathcal{E}_1(f))(\mathcal{E}_1(g)))$. Then for any $\phi \in \mathcal{U}$,

$$\mathcal{E}(\phi)[(\mathcal{E}_1(f))(\mathcal{E}_1(g))x] = \mathcal{E}(\phi f)(\mathcal{E}_1(g))x = \mathcal{E}((\phi f)g)x = \mathcal{E}(\phi(fg))x,$$

so that $x \in \mathcal{D}(\mathcal{E}_1(fg))$, with

$$(\mathcal{E}_1(fg))x = (\mathcal{E}_1(f))(\mathcal{E}_1(g))x.$$

Thus $(\mathcal{E}_1(f))(\mathcal{E}_1(g)) \subseteq \mathcal{E}_1(fg)$. Now suppose $x \in \mathcal{D}(\mathcal{E}_1(fg)) \cap \mathcal{D}(\mathcal{E}_1(g))$. To see that $x \in \mathcal{D}((\mathcal{E}_1(f))(\mathcal{E}_1(g)))$, we again consider arbitrary $\phi \in \mathcal{U}$:

$$\mathcal{E}(\phi)[\mathcal{E}_1(fg)x] = \mathcal{E}(\phi(fg))x = \mathcal{E}((\phi f)g)x = \mathcal{E}(\phi f)((\mathcal{E}_1(g))x),$$

so that $(\mathcal{E}_1(g))x \in \mathcal{D}(\mathcal{E}_1(f))$, as desired.
(4) Let $h(s) \equiv (\lambda - g(s))^{-1}$. By (1) and (3),

$$(\mathcal{E}_1(h))(\lambda - \mathcal{E}_1(g)) = I|_{\mathcal{D}(\mathcal{E}_1(g))}, \quad \text{and } (\lambda - \mathcal{E}_1(g))(\mathcal{E}_1(h)) = I|_{\mathcal{D}(\mathcal{E}_1(h))};$$

this proves (4).
(5) The fact that $\mathcal{E}_0(f_1) \subseteq T$ is in the definition of a $\mathcal{U}$ functional calculus for T. To show that $T \subseteq \mathcal{E}_1(f_1)$, suppose $x \in \mathcal{D}(T)$. By Definition 1.2, for any $\phi \in \mathcal{U}$,

$$\mathcal{E}(\phi)Tx = \mathcal{E}(\phi f_1)x,$$

so that $x \in \mathcal{D}(\mathcal{E}_1(f_1))$, and $(\mathcal{E}_1(f_1))x = Tx$, as desired.

When $\mathcal{E}$ is a $\mathcal{U}$ distribution, then Proposition 2.2, (5) of this theorem and the fact that T is closed imply that $T = \mathcal{E}_1(f_1)$. $\square$

Corollary 2.4. *$\mathcal{E}_1(f_1)$ is the maximal T for which $\mathcal{E}$ is a $\mathcal{U}$ quasi distribution and the closure of $\mathcal{E}_0(f_1)$ is the minimal T.*

If $\mathcal{E}$ is a $\mathcal{U}$ distribution, then T is unique, and equals

$$\mathcal{E}_2(f_1) = \overline{\mathcal{E}_0(f_1)} = \mathcal{E}_1(f_1).$$

To guarantee that our $\mathcal{U}$ functional calculus is consistent with the Dunford-Taylor functional calculus, we will need the following continuity hypothesis on $\mathcal{E}$.

Definition 2.5. We will say that a $\mathcal{U}$ functional calculus $\mathcal{E}$ is *analytic* if, whenever $\phi \in \mathcal{U}$, and $\{f_n\}_{n\in\mathbf{N}}$ is a sequence of functions holomorphic on a neighborhood Ω_ϕ of the support of ϕ, such that $f_n\phi \in \mathcal{U}$ for each n, and $f_n \to 0$ uniformly on $\overline{\Omega_\phi}$, then

$$\mathcal{E}(f_n\phi)x \to 0 \quad \text{as } n \to \infty, \;\; \forall x \in X.$$

For the remainder of this section, we will use the terminology

$$\lambda \mapsto R(x, \lambda, T) \;\; (\lambda \in \rho_T(x))$$

for the local resolvent of T at x.

Lemma 2.6. *If $w \in \rho_T(x)$ is outside the support of $\phi \in \mathcal{U}$, then*

$$\mathcal{E}(\phi)R(x, w, T) = \mathcal{E}(\phi_w)x.$$

Proof.

$$(w - T)\mathcal{E}(\phi_w)x = \mathcal{E}((w - f_1)\phi_w)x = \mathcal{E}(\phi)x,$$

thus

$$\mathcal{E}(\phi_w)x = R(\mathcal{E}(\phi)x, w, T).$$

Also

$$(w - T)\mathcal{E}(\phi)R(x, w, T) = \mathcal{E}(\phi)(w - T)R(x, w, T) = \mathcal{E}(\phi)x,$$

thus

$$R(\mathcal{E}(\phi)x, w, T) = \mathcal{E}(\phi)R(x, w, T).$$

□

Proposition 2.7. *Suppose $\mathcal{E}$ is an analytic $\mathcal{U}$ functional calculus for T, $\phi \in \mathcal{U}$, $x \in X$, f is holomorphic in a neighborhood G of the support of ϕ, and $f\phi \in \mathcal{U}$. Then, choosing Γ to be a contour in G that surrounds supp(ϕ),*

(1)

$$\mathcal{E}(\phi f)x = \int_\Gamma f(w)\mathcal{E}(\phi_w)x \, \frac{dw}{2\pi i};$$

and

(2) *if* $\Gamma \subseteq \rho_T(x)$, *then*

$$\mathcal{E}(\phi f)x = \mathcal{E}(\phi)\left[\int_\Gamma f(w)R(x,w,T)\,\frac{dw}{2\pi i}\right].$$

Proof. (1) For each n, let $\mathcal{P}_n \equiv \{w_{j,n}\}_{j=0}^{k_n}$ be a partition of Γ. Suppose that the norm of $\mathcal{P}_n$ goes to zero, as $n \to \infty$, so that, letting f_n be the Riemann sum

$$f_n(z) \equiv \frac{1}{2\pi i}\left[\sum_{j=0}^{k_n} f(w_{j,n})(w_{j,n}-z)^{-1}\,\Delta w_{j,n}\right] \quad (z \in G),$$

we have

$$f(z) = \int_\Gamma f(w)(w-z)^{-1}\,\frac{dw}{2\pi i} = \lim_{n\to\infty} f_n(z),$$

uniformly for z in an open neighborhood, Ω, of the support of ϕ. Since $\mathcal{E}$ is analytic, this implies that

$$\begin{aligned}\int_\Gamma f(w)\mathcal{E}(\phi_w)x\frac{dw}{2\pi i} &= \lim_{n\to\infty}\frac{1}{2\pi i}\sum_{j=0}^{k_n} f(w_{j,n})\mathcal{E}(z \mapsto \phi(z)(w_{j,n}-z)^{-1})x\,\Delta w_{j,n}\\ &= \lim_{n\to\infty}\mathcal{E}(\phi f_n)x = \mathcal{E}(\phi f)x.\end{aligned}$$

(2) Since $\mathcal{E}(\phi) \in L(X)$,

$$\begin{aligned}\mathcal{E}(\phi)\left[\int_\Gamma f(w)R(x,w,T)\,\frac{dw}{2\pi i}\right] &= \int_\Gamma f(w)\mathcal{E}(\phi)R(x,w,T)\,\frac{dw}{2\pi i}\\ &= \int_\Gamma f(w)\mathcal{E}(\phi_w)x\,\frac{dw}{2\pi i}\\ &= \mathcal{E}(\phi f)x,\end{aligned}$$

by (1) and Lemma 2.6. □

Proposition 2.8. *If* $f \in \mathcal{M}(\mathcal{U})$ *is analytic in a neighborhood,* δ, *of* $\sigma_T(x) \cup \{\infty\}$, *and*

$$y \equiv f(\infty)x + \int_\Gamma f(w)R(x,w,T)\,\frac{dw}{2\pi i},$$

where Γ *is a contour contained in* δ *that surrounds* $\sigma_T(x) \cup \{\infty\}$, $\phi \in \mathcal{U}$ *and supp*(ϕ) *is contained in* δ *and disjoint from* Γ, *then*

$$\mathcal{E}(f\phi)x = \mathcal{E}(\phi)y = f(\infty)\mathcal{E}(\phi)x + \mathcal{E}(\phi)\int_\Gamma f(w)R(x,w,T)\,\frac{dw}{2\pi i}.$$

Proof. This is the same as the proof of Proposition 2.7, after writing

$$f(z) = f(\infty) + \int_\Gamma f(w)(w-z)^{-1}\,\frac{dw}{2\pi i},$$

for z in a neighborhood of $\sigma_T(x) \cup \{\infty\}$. □

Lemma 2.9. *If y is as in Proposition 2.8, then $\sigma_T(y) \subseteq \sigma_T(x)$.*

Proof. For $z, w \in \rho_T(x)$, define

$$g(z,w) \equiv \begin{array}{l} -\,x'(w), \quad \text{if } z = w, \\ \\ (w-z)^{-1}(x(z) - x(w)) \quad \text{if } z \neq w, \end{array}$$

where $x(z) \equiv R(x,z,T)$, so that $(z-T)g(z,w) = R(x,w,T)$.
Define, for $z \notin \sigma_T(x)$,

$$R(y,z,T) \equiv f(\infty)R(x,z,T) + \int_\Gamma f(w)g(z,w)\,dw.$$

Then

$$(z-T)R(y,z,T) = y,$$

for $z \notin \sigma_T(x)$, as desired. □

Theorem 2.10. *If $f \in \mathcal{M}(\mathcal{U})$ is analytic in a neighborhood, δ, of $\sigma_T(x) \cup \{\infty\}$, then $x \in \mathcal{D}(\mathcal{E}_1(f))$, and*

$$\mathcal{E}_1(f)x = f(\infty)x + \int_\Gamma f(w)R(x,w,T)\,\frac{dw}{2\pi i},$$

where Γ is a contour contained in δ that surrounds $\sigma_T(x) \cup \{\infty\}$.

Proof. Let y be as in Proposition 2.8. Suppose $\phi \in \mathcal{U}$. We may write

$$\phi = \phi_1 + \phi_2,$$

where $\phi_j \in \mathcal{U}, j = 1,2$, *supp*$(\phi_1)$ is disjoint from $\sigma_T(x)$ and supp(ϕ_2) is contained in δ and disjoint from Γ.

By Lemmas 1.22a and 2.9, $\mathcal{E}(\phi_1 f)x = \mathcal{E}(\phi_1)y = 0$, since supp$(\phi_1)$ is disjoint from supp$(\mathcal{E} \otimes x)$ and supp$(\mathcal{E} \otimes y)$.

By Proposition 2.8, $\mathcal{E}(\phi_2 f)x = \mathcal{E}(\phi_2)y$. Thus,

$$\mathcal{E}(\phi f)x = \mathcal{E}(\phi_2 f)x = \mathcal{E}(\phi_2)y = \mathcal{E}(\phi)y,$$

so that $x \in \mathcal{D}(\mathcal{E}_1(f))$ and $\mathcal{E}_1(f)x = y$, as desired. □

Remark 2.11. Theorem 2.10 shows that, for $f \in \mathcal{M}(\mathcal{U})$, $\mathcal{E}_1$ is an extension of the Dunford-Taylor functional calculus for f analytic in a neighborhood of $\sigma(T) \cup \{\infty\}$.

III. CHARACTERIZATIONS OF DECOMPOSABILITY

Throughout this section, assume $\mathcal{U}$ is compact admissible and $\mathcal{E}$ is a $\mathcal{U}$ quasi distribution for T.

For $\mathcal{U}$ inverse closed (Definition 1.10), we characterize decomposability (Theorem 3.7) in terms of spectral mapping theorems, the support of $\mathcal{E}$ and the extended functional calculus $\mathcal{E}_1$ of the previous section. Theorem 3.4 asserts that the spectral mapping theorem for $\mathcal{E}$ holds if and only if $\sigma(T) = supp(\mathcal{E})$ (Definition 1.13). Theorem 3.7 asserts that T is decomposable if and only if an analogous local spectral mapping theorem holds; that is, $\sigma_T(x) = supp(\mathcal{E} \otimes x)$, for all $x \in X$ (see Definition 1.16). This is also equivalent to the spectral mapping theorem holding for $\mathcal{E}$ restricted to local spectral subspaces $X(T, F)$, for any closed $F \subseteq \mathbf{C}$. Decomposability is also equivalent to $\mathcal{E}_1(f)$ being in $L(X)$, whenever $f \in \mathcal{M}(\mathcal{U})$ (Definition 2.1) and f is analytic in a neighborhood of ∞.

Definition 3.1. If for every $\phi \in \mathcal{U}$,

$$\phi(\sigma(T)) = \sigma(\mathcal{E}(\phi)),$$

then we say that the *spectral mapping theorem holds for* T.

Lemma 3.2. *If $F \subseteq \mathbf{C}$ is compact, then*

$$\sigma(T|_{X(T,F)}) = supp(\mathcal{E}) \cap F.$$

Proof. By Theorem 1.7 and Proposition 1.23, $\sigma(T|_{X(T,F)}) \subseteq F$.

By Theorem 1.7, Proposition 1.23 and [6, Theorem 3.1.6],

$$\sigma(T|_{X(T,F)}) = supp(\mathcal{E}_F),$$

which equals $supp(\mathcal{E}) \cap F$. □

Lemma 3.3. *Suppose $S \in L(X)$ has the SVEP, X_1 and X_2 are invariant subspaces for S and $(\lambda - S)^{-1}$, for all $\lambda \in \rho(S)$, and $X = X_1 + X_2$. Then*

$$\sigma(S) = \sigma(S|_{X_1}) \cup \sigma(S|_{X_2}).$$

Proof. The invariance conditions imply that

$$\sigma(S|_{X_1}) \cup \sigma(S|_{X_2}) \subseteq \sigma(S).$$

For the opposite inclusion, suppose $\lambda \in \rho(S|_{X_1}) \cap \rho(S|_{X_2})$. For any $x \in X$, we may write $x = x_1 + x_2$, with $x_j \in X_j, j = 1, 2$, thus there exist y_j so that

$$(\lambda - S)y_j = x_j, \;\; y_j \in X_j, \;\; j = 1, 2.$$

Thus $(\lambda - S)(y_1 + y_2) = x$. This shows that $(\lambda - S)$ is surjective. By the SVEP, $\lambda \in \rho(S)$, as desired. □

Theorem 3.4. *If $\mathcal{U}$ is inverse closed, then the spectral mapping theorem holds for T if and only if*

$$supp(\mathcal{E}) = \sigma(T).$$

Proof. By Lemma 1.22(b), $supp(\mathcal{E}) \subseteq \sigma(T)$.

Suppose the spectral mapping theorem holds. To show that $\sigma(T) \subseteq supp(\mathcal{E})$, suppose, for the sake of contradiction, there exists $\lambda_0 \in \sigma(T) - supp(\mathcal{E})$. Choose $\phi \in \mathcal{U}$ such that $\phi(\lambda_0) = 1$ and ϕ is zero in a neighborhood of $supp(\mathcal{E})$. Since $\mathcal{E}(\phi) = 0$, $\sigma(\mathcal{E}(\phi)) = \{0\}$. But by hypothesis, $1 \in \phi(\sigma(T)) = \sigma(\mathcal{E}(\phi))$. Thus the desired inclusion holds.

Conversely, suppose $supp(\mathcal{E}) = \sigma(T)$. Fix $\phi \in \mathcal{U}$. Choose compact $F \subseteq \mathbf{C}$ whose interior F^0 contains the support of ϕ, and let $\{F^0, G\}$ be an open cover of the complex plane, so that $supp(\phi) \cap \overline{G}$ is trivial.

If $X(T, \overline{G})$ is trivial, then $X = X(T, F)$, so that the spectral mapping theorem follows from Proposition 1.26. Thus we may assume $X(T, \overline{G})$ is nontrivial.

By Lemma 3.3,

$$\sigma(\mathcal{E}(\phi)) = \sigma(\mathcal{E}(\phi)|_{X(T,F)}) \cup \sigma(\mathcal{E}(\phi)|_{X(T,\overline{G})}).$$

By Corollary 1.15, $\mathcal{E}(\phi)|_{X(T,\overline{G})} = 0$, so by Proposition 1.26 and Lemma 3.2,

$$\begin{aligned}\sigma(\mathcal{E}(\phi)) &= \sigma(\mathcal{E}(\phi)|_{X(T,F)}) \cup \{0\} = \phi(\sigma(T|_{X(T,F)}) \cup \{0\} \\ &= \phi(supp(\mathcal{E}) \cap F) \cup \{0\} = \phi(\sigma(T) \cap F) \cup \{0\} \\ &= \phi(\sigma(T)) \cup \{0\},\end{aligned}$$

since $\phi \equiv 0$ outside F. All that remains is to show that $0 \in \phi(\sigma(T))$. To see this, note first that, since $X(T, \overline{G})$ is nontrivial, $\sigma(T|_{X(T,\overline{G})})$ is nonempty. Since $supp(\phi) \cap \sigma(T|_{X(T,\overline{G})}) \subseteq supp(\phi) \cap \overline{G}$, which is empty, it follows that

$$0 \in \phi(\sigma(T|_{X(T,\overline{G})})) \subseteq \phi(\sigma(T)),$$

as desired. □

Definition 3.5. If for every closed $F \subseteq \mathbf{C}$, the spectral mapping theorem for $T|_{X(T,F)}$ holds, that is,

$$\phi(\sigma(T|_{X(T,F)})) = \sigma(\mathcal{E}(\phi)|_{X(T,F)}),$$

then we say that the *strong spectral mapping theorem holds for T*.

Definition 3.6. We will denote by $\mathcal{A}$ those functions in $\mathcal{M}(\mathcal{U})$ that are analytic in a neighborhood of ∞.

Theorem 3.7. *Suppose $\mathcal{U}$ is inverse closed and $\mathcal{E}$ is a $\mathcal{U}$ quasi distribution for T. Then the following are equivalent.*

(a) *The strong spectral mapping theorem holds for T.*
(b) *T has the SDP.*
(c) *T has the SDI.*
(d) *For every $x \in X$, $supp(\mathcal{E} \otimes x) = \sigma_T(x)$.*
(e) *T is decomposable.*

If $\mathcal{E}$ is an analytic $\mathcal{U}$ quasi distribution and $T = \mathcal{E}_1(f_1)$, then (a)–(e) *are equivalent to*

(f) *$\mathcal{E}_1$ maps $\mathcal{A}$ into $L(X)$.*

If $\mathcal{E}$ is an analytic $\mathcal{U}$ distribution, then (a)–(f) *are equivalent to*

(g) *T has property $\mathcal{B}$.*

Proof. (a) $\to$ (c). Let $\{G_0, G_1\}$ be an open cover of the complex plane, with G_0 a neighborhood of ∞, and G_1 relatively compact. Choose $\phi \in \mathcal{U}$ such that $supp(\phi) \subseteq G_1$ and $\phi = 1$ on a neighborhood of $\mathbf{C} - G_0$. Let F_0 be the closure of the complement of $\{\lambda \in \mathbf{C} \,|\, \phi(\lambda) = 1\}$, $F_1 \equiv supp(\phi)$. Apply Theorem 3.4 to $T|_{X(T,F_i)}, i = 0, 1$, to obtain

$$\sigma(T|_{X(T,F_0)}) = supp(\mathcal{E}|_{X(T,F_0)}) \subseteq G_0;$$

$$\sigma(T|_{X(T,F_1)}) = supp(\mathcal{E}|_{X(T,F_1)}) \subseteq G_1.$$

Let $P \equiv \mathcal{E}(\phi)$. For any $x \in X$, by Corollary 1.15, $Px \in X(T, F_1)$. Since for every $\psi \in \mathcal{U}$ with $supp(\psi) \cap F_0$ empty, one has

$$\mathcal{E}(\psi)(I - P)x = \mathcal{E}(\psi)x - \mathcal{E}(\psi)Px = \mathcal{E}(\psi)x - \mathcal{E}(\psi\phi)x = 0,$$

it follows by Proposition 1.23 that $(I - P)x \in X(T, F_0)$. Thus

$$X = X(T, F_0) + X(T, F_1),$$

$\mathrm{Im}(P) \subseteq X(T, F_0)$, $\mathrm{Im}(I - P) \subseteq X(T, F_1)$ and $X(T, F_1) \subseteq \mathcal{D}(T)$. This proves (c).

(c) $\to$ (b) is clear.

(b) $\to$ (d). By Lemma 1.22, it is sufficient to prove that $\sigma_T(x) \subseteq supp(\mathcal{E} \otimes x) \equiv F$.

If F is bounded, then this inclusion follows from Proposition 1.23. Thus we may assume F is unbounded.

Let $\{G_0, G_1\}$ be an open cover of the complex plane, with G_0 a neighborhood of ∞, G_1 relatively compact and $F \subseteq G_0, F \cap \overline{G_1}$ empty. By (b), we have the decomposition

$$x = x_0 + x_1, \;\; \sigma_T(x_i) \subseteq G_i, \;\; i = 0, 1.$$

Since $F \subseteq G_0$ and $\sigma_T(x_0) \subseteq G_0$, we have

$$\mathcal{E}(\phi)x = 0 = \mathcal{E}(\phi)x_0$$

whenever $\phi \in \mathcal{U}$ and $supp(\phi) \cap \overline{G_0}$ is empty. Thus $\mathcal{E}(\phi)x_1 = 0$. This implies that $x_1 \in X_{\mathcal{E}_1}(T, \overline{G_0})$. By Proposition 1.23, we therefore have

$$\sigma_T(x_1) = supp(\mathcal{E} \otimes x_1) \subseteq \overline{G_0}.$$

Thus we have

$$\sigma_T(x) \subseteq \sigma_T(x_0) \cup \sigma_T(x_1) \subseteq \overline{G_0}.$$

Since G_0 is an arbitrary open set containing F, it follows that $\sigma_T(x) \subseteq F$, as desired.
(d) $\to$ (a). Let $F \subseteq \mathbf{C}$ be closed, $S \equiv T|_{X(T,F)}$. By Theorem 3.4, it is sufficient to show that $\sigma(S) = supp(\mathcal{E}|_{X(T,F)})$. This follows from the fact that

$$\sigma(S) = \bigcup_{x \in X(T,F)} \sigma_T(x) \quad \text{and} \quad supp(\mathcal{E}|_{X(T,F)}) = \bigcup_{x \in X(T,F)} supp(\mathcal{E} \otimes x);$$

the former equality is from Proposition 0.4(b).

The equivalence of (b) and (e) follows from Corollary 1.24.

Now suppose $\mathcal{E}$ is analytic and $T = \mathcal{E}_1(f_1)$.

(d) $\to$ (f). By Proposition 2.2(2), it is sufficient to show that $X \subseteq \mathcal{D}(\mathcal{E}_1(f))$, for $f \in \mathcal{A}$.

Given $f \in \mathcal{A}$, analytic in the neighborhood, δ, of ∞, $x \in X$, decompose x as follows. Choose $\phi \in \mathcal{U}$ such that $\phi = 1$ on a neighborhood of $\mathbf{C} - \delta$, and define

$$y \equiv x - \mathcal{E}(\phi)x.$$

By Proposition 2.2, $\mathcal{E}(\phi)x \in \mathcal{D}(\mathcal{E}_1(f))$.

It remains to show that $y \in \mathcal{D}(\mathcal{E}_1(f))$. If $\psi \in \mathcal{U}$ has support outside δ, then

$$\mathcal{E}(\psi)y = \mathcal{E}(\psi)x - \mathcal{E}(\psi)\mathcal{E}(\phi)x = \mathcal{E}(\psi - \psi\phi)x = \mathcal{E}(0)x = 0.$$

Thus $supp(\mathcal{E} \otimes y) \subseteq \delta$. By (d), $\sigma_T(y) \subseteq \delta$. By Theorem 2.10, $y \in \mathcal{D}(\mathcal{E}_1(f))$, as desired.
(f) $\to$ (a). By Theorem 3.4 and Lemma 1.22(b), it is sufficient to show that

$$\sigma(T|_{X(T,F)}) \subseteq supp(\mathcal{E}|_{X(T,F)}),$$

for any closed $F \subseteq \mathbf{C}$.

For fixed closed $F \subseteq \mathbf{C}$, let $S \equiv T|_{X(T,F)}$, $\mathcal{F} \equiv \mathcal{E}|_{X(T,F)}$. Suppose $\lambda_0 \notin supp(\mathcal{F})$. Choose $\phi \in \mathcal{U}$ such that $\phi = 1$ on a neighborhood of λ_0 and $supp(\phi) \cap supp(\mathcal{F})$ is empty. Define

$$\psi(\lambda) \equiv \frac{\phi(\lambda_0) - \phi(\lambda)}{\lambda_0 - \lambda}, \quad \text{if } \lambda \neq \lambda_0,$$
$$\psi(\lambda_0) \equiv 0.$$

Then $\psi \in \mathcal{A}$ and $\psi(\infty) = 0$. By Theorem 2.3(3), $\mathrm{Im}(\mathcal{F}_1(\psi)) \subseteq \mathcal{D}(\mathcal{F}_1(f_1)) = \mathcal{D}(S)$, since $g(z) \equiv z\psi(z) \in \mathcal{A}$. Moreover

$$\begin{aligned} I &= I - \mathcal{F}(\phi) = \phi(\lambda_0)I - \mathcal{F}_1(\phi) \\ &= \mathcal{F}_1(\phi(\lambda_0) - \phi) = (\lambda_0 - S)\mathcal{F}_1(\psi). \end{aligned}$$

Thus $(\lambda_0 - S)$ is surjective. By the SVEP, $\lambda_0 \in \rho(S)$, as desired.

Now suppose $\mathcal{E}$ is an analytic $\mathcal{U}$ distribution.
(b) $\to$ (g) is well-known.
(g) $\to$ (f). By Proposition 2.2(2), to show that $\mathcal{E}_1(f) \in L(X)$, it is sufficient to show that $X \subseteq \mathcal{D}(\mathcal{E}_1(f))$.

Let $\{\phi_n\}_{n\in\mathbf{N}}$ be as in Definition 1.17(b). If $\mathcal{E}(f(\phi_n - \phi_m))x \to 0$, it follows that $x \in \mathcal{D}(\mathcal{E}_2(f)) = \mathcal{D}(\mathcal{E}_1(f))$ by Proposition 2.2.

So fix $x \in X$, and let δ be a neighborhood of ∞ in which f is analytic, $\Gamma = \partial\delta$ such that

$$f(z) = f(\infty) + \int_\Gamma f(w)(w-z)^{-1}\,\frac{dw}{2\pi i},\ \forall z \in \delta.$$

Thus by Proposition 2.8,

$$\begin{aligned} \mathcal{E}(f(\phi_n - \phi_m))x &= f(\infty)\mathcal{E}(\phi_n - \phi_m)x \\ &\quad + \int_\Gamma f(w)\mathcal{E}(z \mapsto (\phi_n(z) - \phi_m(z))(w-z)^{-1})x\,\frac{dw}{2\pi i}. \end{aligned}$$

Let

$$x_{n,m}(w) \equiv \mathcal{E}(z \mapsto (\phi_n(z) - \phi_m(z))(w-z)^{-1})x.$$

Then

$$(w - T)x_{n,m}(w) = \mathcal{E}(\phi_n - \phi_m)x,\ \forall w \in \Gamma,$$

thus $(w - T)x_{n,m}(w) \to 0$, as $n, m \to \infty$, uniformly on Γ. By Property $\mathcal{B}$, $x_{n,m}(w) \to 0$ uniformly on Γ, which implies that

$$\mathcal{E}(f(\phi_n - \phi_m))x = f(\infty)\mathcal{E}(\phi_n - \phi_m)x + \int_\Gamma f(w)x_{n,m}(w)\,\frac{dw}{2\pi i} \to 0,$$

as desired. $\square$

IV. A CLASS OF EXAMPLES

Throughout this section, X will be $BC^1(\mathbf{R})$, $h \in C^1(\mathbf{R})$ and $T \equiv M_h$, that is,

$$Tf \equiv hf \ \ (f \in X),$$

with maximal domain $\mathcal{D}(T) \equiv \{f \in X \mid hf \in X\}$.

Using the results of Section III, we will characterize those h for which T is decomposable and those h for which T has a $C_c^1(\mathbf{C})$ quasi distribution. Both may be characterized in terms of the rate of growth of h'. If we define, for any $n \in \mathbf{N}$,

$$D_{n,h} \equiv \sup\{|h'(x)| \mid x \in \mathbf{R}, |h(x)| \le n\}, \tag{4.1}$$

then T has a $C_c^1(\mathbf{C})$ quasi distribution if and only if $D_{n,h}$ is finite, for all $n \in \mathbf{N}$ (Corollary 4.3), while T is decomposable if and only if the sequence $\{\frac{D_{n,h}}{n^2}\}_{n\in\mathbf{N}}$ is bounded (Theorem 4.7(a) $\iff$ (b)). Thus, in order that T be decomposable, it is necessary, but not sufficient, that T have a $C_c^1(\mathbf{C})$ functional calculus. We also characterize decomposability of T in terms of both local and global spectra (Theorem 4.7); for example, T is decomposable if and only if $\rho(T)$ is nonempty.

Examples 4.9 give an example of an h for which T does not have a compact-admissible functional calculus, and an example of an h for which T has a $C_c^1(\mathbf{C})$ quasi distribution, but is not decomposable.

Let's consider the maximal compact-admissible algebra for T,

$$\mathcal{U}_h \equiv \{g \in C_c^1(\mathbf{C}) \mid g \circ h \in BC^1(\mathbf{R})\}.$$

Clearly T has a $\mathcal{U}_h$ functional calculus, given by

$$[\mathcal{E}(g)]\, f \equiv (g \circ h)f \ \ (f \in X, g \in \mathcal{U}_h).$$

Lemma 4.2. *For any $n \in \mathbf{N}$, $D_{n,h}$ is finite if and only if*

$$\{g \in C^1(\mathbf{C}) \mid \text{ support of } g \subseteq \{z \in \mathbf{C} \mid |z| \le n\}\} \subseteq \mathcal{U}_h.$$

Proof. This is clear from $(g \circ h)'(x) = h'(x)g'(h(x))$. □

Corollary 4.3. *The following are equivalent.*

(a) *T has a $C_c^1(\mathbf{C})$ quasi distribution.*

(b) *$D_{n,h} < \infty$, for all $n \in \mathbf{N}$.*

Before considering decomposability, we need some lemmas.

Lemma 4.4. (1) *$supp(\mathcal{E}) = \overline{h(\mathbf{R})}$.*

(2) *$supp(\mathcal{E} \otimes f) = \overline{h(supp(f))}$, for any $f \in X$.*

Proof. (1) $\mathcal{E} = 0$ on the open set G if and only if $\phi \circ h = 0$ whenever $\phi \in \mathcal{U}_h$ is supported in G if and only if $h(\mathbf{R}) \cap G$ is empty. The union over all such G is the complement of $\overline{h(\mathbf{R})}$.
(2) As with the proof of (1), $\mathcal{E} \otimes f = 0$ on the open set G if and only if $h(supp(f)) \cap G$ is empty. □

Lemma 4.5. *Suppose $f \in X$ and $\lambda \in \rho_T(f)$. Then $\lambda \notin \overline{h(supp(f))}$, and the local resolvent for f is given by*

$$[R(\lambda, f)](x) = \frac{f(x)}{(\lambda - h(x))} \quad \textit{if } f(x) \neq 0,$$

$$[R(\lambda, f)](x) = 0 \quad \textit{if } f(x) = 0.$$

Proof. We have, for $\lambda \in \rho_T(f), x \in \mathbf{R}$,

$$(*) \qquad (\lambda - h(x))\,[R(\lambda, f)](x) = f(x).$$

If $f(x) \neq 0$, then $\lambda - h(x) \neq 0$. Thus $\lambda \notin \overline{h(\{x \in \mathbf{R} \,|\, f(x) \neq 0\})}$; since $\rho_T(f)$ is open, it follows that $\lambda \notin \overline{h(supp(f))}$, and (*) implies that

$$[R(\lambda, f)](x) = \frac{f(x)}{(\lambda - h(x))} \quad \text{if } f(x) \neq 0.$$

Assertion (*) now also implies that, when $f(x) = 0$, $[R(\lambda, f)](x) = 0$. □

Now we obtain an interesting characterization of the local spectrum. It is either what one would expect, for a multiplication operator, or the entire complex plane.

Proposition 4.6. *If $f \in X$, then the following are equivalent.*

(a) $\sigma_T(f) = \overline{h(supp(f))}$.
(b) $\{\frac{|f(x)h'(x)|}{1+|h(x)|^2}\}_{x \in \mathbf{R}}$ *is bounded.*
(c) *For any $\lambda \notin \overline{h(supp(f))}$,*

$$x \mapsto \frac{f(x)}{(\lambda - h(x))} \in BC^1(\mathbf{R}).$$

Otherwise, $\sigma_T(f) = \mathbf{C}$.

Proof. A small calculation shows that (b) and (c) are equivalent. The equivalence of (c) and (a), and the fact that the alternative is $\sigma_T(f) = \mathbf{C}$, follows from Lemma 4.5. □

Theorem 4.7. *The following are equivalent.*

(a) *T is decomposable.*
(b) $\{\frac{D_{n,h}}{n^2}\}_{n=1}^{\infty}$ *is bounded.*
(c) $\{\frac{|h'(x)|}{1+|h(x)|^2}\}_{x \in \mathbf{R}}$ *is bounded.*

(d) *For any* $f \in BC^1(\mathbf{R})$, $\lambda \notin \overline{h(supp(f))}$,

$$x \mapsto \frac{f(x)}{(\lambda - h(x))} \in BC^1(\mathbf{R}).$$

(e) T *has a* $C_c^1(\mathbf{C})$ *quasi distribution and the spectral mapping theorem holds; that is,*

$$\sigma(\mathcal{E}(g)) = g(\sigma(T)), \ \forall g \in C_c^1(\mathbf{C}).$$

(f) $\sigma(T) = \overline{h(\mathbf{R})}$
(g) $\sigma(T) \neq \mathbf{C}$.
(h) T *has a* $C_c^1(\mathbf{C})$ *quasi distribution and*

$$\sigma_T(f) = \overline{h(supp(f))},$$

for all $f \in X$.
(i) T *has a* $C_c^1(\mathbf{C})$ *quasi distribution and*

$$\sigma_T(f) \neq \mathbf{C},$$

for all $f \in X$.
(j) *There exists* g *such that both* g *and* $\frac{1}{g}$ *are in* X, *and* $\sigma_T(g) \neq \mathbf{C}$.

Proof. A little calculation shows that (b), (c) and (d) are equivalent.
(d) $\to$ (f). Letting f be the constant function in (d) implies that $x \mapsto (\lambda - h(x))^{-1} \in BC^1(\mathbf{R})$, for $\lambda \notin \overline{h(\mathbf{R})}$, thus $\lambda \in \rho(T)$, with

$$(\lambda - T)^{-1} f \equiv \frac{f}{\lambda - h} \ \ (f \in X).$$

This shows that $\sigma(T) \subseteq \overline{h(\mathbf{R})}$. The converse inclusion follows as in the proof of Lemma 4.5.
(f) $\to$ (g) is clear, since $h \in C^1(\mathbf{R})$.
(g) $\to$ (c). By Proposition 4.6, if (c) fails, then $\mathbf{C} = \sigma_T(1) \subseteq \sigma(T)$.
(a) $\to$ (h). Fix $n \in \mathbf{N}$. We will show that $D_{n,h} < \infty$.
Let

$$G_1 \equiv \{z \in \mathbf{C} \,|\, |z| < n+2\}, \ \ G_2 \equiv \{z \in \mathbf{C} \,|\, |z| > n+1\}.$$

Since T is decomposable, there exist $f_1, f_2 \in X$ such that $f_1 + f_2 = 1$, and $\sigma_T(f_i) \subseteq \overline{G_i}$, for $i = 1, 2$. By Lemma 4.5, $\overline{h(supp(f_2))} \subseteq \sigma_T(f_2) \subseteq \overline{G_2}$, thus $supp(f_2) \subseteq h^{-1}(\overline{G_2})$. This implies that

$$f_1(x) = 1 \ \ \forall x \in h^{-1}\left(\{z \in \mathbf{C} \,|\, |z| \leq n\}\right).$$

By Proposition 4.6, the map

$$x \mapsto \frac{f_1(x) h'(x)}{1 + |h(x)|^2}, \text{ hence } x \mapsto \frac{h'(x)}{1 + |h(x)|^2},$$

is bounded on

$$h^{-1}\left(\{z \in \mathbf{C} \,|\, |z| \leq n\}\right) = \{x \in \mathbf{R} \,|\, |h(x)| \leq n\}.$$

This implies that $h'(x)$ is bounded on that set, which is saying that $D_{n,h} < \infty$.

By Corollary 4.3, T has a $C_c^1(\mathbf{R})$ quasi distribution. Theorem 3.7 and Lemma 4.4(2) now imply (h).
(e) $\rightarrow$ (f). By Theorem 3.4, $\sigma(T) = supp(\mathcal{E})$, which, by Lemma 4.4(1), equals $\overline{h(\mathbf{R})}$.
(d) $\rightarrow$ (a). Since (d) is equivalent to (b), T has a $C_c^1(\mathbf{C})$ quasi distribution, by Corollary 4.3. For any $f \in X$, by Proposition 4.6,

$$\sigma_T(f) = \overline{h(supp(f))},$$

thus by Lemma 4.4(2),

$$supp(\mathcal{E} \otimes f) = \sigma_T(f).$$

By Theorem 3.7, T is decomposable.
(h) $\rightarrow$ (e). By Lemma 4.4(2), $\sigma_T(f) = supp(\mathcal{E} \otimes f)$, for any $f \in X$, thus (e) follows from Theorem 3.7.
(h) $\leftrightarrow$ (i) follows from Proposition 4.6 and the fact that $h \in C^1(\mathbf{R})$.
(j) $\rightarrow$ (c). If (c) fails, then by Proposition 4.6, $\sigma_T(1) = \mathbf{C}$. Suppose both g and $\frac{1}{g}$ are in $BC^1(\mathbf{R})$. Define $B \in L(X)$ by

$$(Bf) \equiv \frac{f}{g}.$$

Then $\sigma_T(1) = \sigma_T(Bg) \subseteq \sigma_T(g)$, thus $\sigma_T(g) = \mathbf{C}$, so that (j) fails.
(c) $\rightarrow$ (j). If (j) fails, then $\sigma_T(1) = \mathbf{C}$, thus by Proposition 4.6, (c) fails. □

Corollary 4.8. *If h is a polynomial, then T is decomposable.*

Proof. This follows from (a) $\iff$ (c) of Theorem 4.7. □

Example 4.9. (1) Let $h(x) \equiv \sin(x^2)$. Then by Lemma 4.2, $\mathcal{U}_h$ is trivial; in particular, T does not have a compact-admissible functional calculus.
(2) Let $h(x) \equiv x(2+\sin(x^3))$. Then by Corollary 4.3, T has a $C_c^1(\mathbf{C})$ quasi distribution, while by Theorem 4.7(a) $\iff$ (c), T is not decomposable.

Remark 4.10. If T had a quasi-admissible functional calculus, then T would be decomposable ([20, Corollary IV.9.8]). Thus Example 4.9(2) is an example of an operator with a compact-admissible functional calculus (in fact, a $C_c^1(\mathbf{C})$ quasi distribution) that does not have a quasi-admissible functional calculus.

Remark 4.11. On another space $\mathcal{A}_k$, where "weighted derivatives"

$$(1+|x|)^{|m|}D^m f(x) \ \ (x \in \mathbf{R}^2, |m| \le k)$$

are bounded, it is shown in [2, Lemma 1.3] that the bounded operator M_h, for $h \in \mathcal{A}_k$, is generalized scalar. See also [2, Lemmas 1.4 and 1.5], and [3].

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