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A CHARACTERIZATION OF ONE DIMENSIONAL N-GRADED GORENSTEIN RINGS OF FINITE COHEN-MACAULAY REPRESENTATION TYPE

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1. Introduction

Let $R = \bigoplus R_n$ be an $\mathbf{N}$-graded Cohen-Macaulay ring where $R_0 = k$ is a field. We denote by $\mathrm{mod}R$ the category of finitely generated graded R-modules whose morphisms are graded R-homomorphisms that preserve degrees. We also denote by $\mathrm{CM}R$ the full subcategory of $\mathrm{mod}R$ consisting of all graded maximal Cohen-Macaulay modules. In the paper [1] , we have shown that if k is an algebraically closed field of characteristic 0 and if R is a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type, then there exists an MCM generating exceptional sequence. In that work, we had to compute the dimension of $\mathrm{Ext}^n_R(X, Y)$ as k-vector space, for all indecomposable graded maximal Cohen-Macaulay modules X and Y and for all $n \in \mathbf{N}$. Through this computation, we noticed the importance of the invariants $\mathrm{d}(R)$ and $\mathrm{d}_n(R)$ of R that are defined as follows:

$$\mathrm{d}(R) := \sup\{\sum_{n\geq 0} \dim_k \mathrm{Ext}^n_R(X, Y) \mid X, Y \in \mathrm{CM}R \text{ are indecomposable}\},$$
$$\mathrm{d}_n(R) := \sup\{\dim_k \mathrm{Ext}^n_R(X, Y) \mid X, Y \in \mathrm{CM}R \text{ are indecomposable}\}.$$

In the present paper, we will give a characterization of one dimensional $\mathbf{N}$-graded Gorenstein rings of finite Cohen-Macaulay representation type utilizing $\mathrm{d}(R)$ and $\mathrm{d}_0(R)$. More precisely, let R be a positively dimensional $\mathbf{N}$-graded Gorenstein ring with isolated singularity where $R_0 = k$ is an algebraically closed field of characteristic 0. Then the invariant $\mathrm{d}(R)$ can take only 7 values in $\{1, 2, 3, 4, 6, 9, \infty\}$. Moreover, if $\mathrm{d}(R) < \infty$, then $\dim R = 1$ and R is isomorphic to one of the rings in the list (1) below and in each case we are able to compute $\mathrm{d}(R)$ and $\mathrm{d}_n(R)$.

	A_{2m+1}	A_{2m}	D_{2m}	D_{2m+1}	E_6	E_7	E_8
$\mathrm{d}(R) = \mathrm{d}_0(R)$	1	2	3	4	4	6	9
$\mathrm{d}_n(R)$ $(n \geq 1)$	1	2	1	2	3	4	6

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2. Preliminaries

In this section, we assume $R = \bigoplus R_n$ is a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type and assume that $R_0 = k$ is an algebraically closed field of characteristic 0. In this case, it is known that R is isomorphic to one of the following rings (c.f.[4]).

$$
(1) \qquad \begin{array}{lll}
(A_n) & R = k[x,y]/(y^2 - x^n) & (n \geq 2) \\
(D_n) & R = k[x,y]/(xy^2 - x^n) & (n \geq 3) \\
(E_6) & R = k[x,y]/(x^3 + y^4) & \\
(E_7) & R = k[x,y]/(x^3 + xy^3) & \\
(E_8) & R = k[x,y]/(x^3 + y^5) &
\end{array}
$$

Moreover the Auslander-Reiten quiver of CMR for each type can be described as they are shown in [1, Figures (1) – (7)] . We denote by Γ the Auslander-Reiten quiver of CMR.

For indecomposable graded maximal Cohen-Macaulay modules X and Y, we write $X \preceq Y$ if $X \cong Y$ or if there exists a finite path from X to Y in Γ.

Lemma 2.1. [1, Lemma 3.3.] *The following hold for indecomposable graded maximal Cohen-Macaulay modules X and Y.*

(i) *There are no cyclic paths in Γ.*
(ii) *If* $\mathrm{Hom}(X,Y) \neq 0$, *then* $X \preceq Y$.
(iii) *If* $\mathrm{Ext}^1_R(X,Y) \neq 0$, *then* $Y \preceq \tau X$. *Here, τX denotes the Auslander-Reiten translation of X.*

It follows from lemma 2.1.(iii) that, for a fixed X, τX is the right bound of the set $\{Y \in \mathrm{CM}R \mid \text{indecomposable, } \mathrm{Ext}^1_R(X,Y) \neq 0\}$ in Γ. Now, we are giving the left bound of this set.

Lemma 2.2. *For indecomposable graded maximal Cohen-Macaulay modules X and Y, if* $\mathrm{Ext}^1_R(X,Y) \neq 0$ *then we have* $\Omega X \preceq Y \preceq \tau X$. *Here, ΩX denotes the first syzygy module of X.*

Proof. Let $0 \to Y \to Z \xrightarrow{\pi} X \to 0$ be a non-split exact sequence. Taking the first syzygy of X; $0 \to \Omega X \to F \to X \to 0$ where F is free, we have the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega X & \longrightarrow & F & \longrightarrow & X & \longrightarrow & 0 \\ & & {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Z & \xrightarrow{\pi} & X & \longrightarrow & 0 \end{array}$$

Suppose $f = 0$ in this diagram. Then the morphism g will induce a morphism $X \to Z$ which contradicts the fact that π is not a split epimorphism. Therefore $f \neq 0$, and we get $\Omega X \preceq Y$. □

Lemma 2.3. *For any indecomposable graded maximal Cohen-Macaulay modules X and Y, we have $\sharp\{n \in \mathbf{N} \mid \operatorname{Ext}^n_R(X, Y) \neq 0\} \leq 1$.*

Proof. If X is free, then the lemma is obviously true. Thus we may assume that X is non-free, and hence $\Omega^i X$ $(i > 0)$ are also non-free and $\tau\Omega^i X$ $(i \geq 0)$ are well-defined. Now assume that $\operatorname{Ext}^n_R(X, Y) \neq 0$ for some $n > 0$. Since $\operatorname{Ext}^1_R(\Omega^{n-1}X, Y) \cong \operatorname{Ext}^n_R(X, Y) \neq 0$, we have $\Omega^n X \preceq Y \preceq \tau\Omega^{n-1}X$ by lemma 2.2. On the other hand, since there exists a sequence $\cdots \prec \Omega^{i+1}X \preceq \tau\Omega^i X \prec \Omega^i X \preceq \tau\Omega^{i-1}X \prec \Omega^{i-1}X \preceq \tau\Omega^{i-2}X \prec \cdots \preceq \tau\Omega X \prec \Omega X \preceq \tau X \prec X$ and since there is no cyclic path in Γ, one sees that $Y \not\preceq \tau\Omega^m X$ for all $m \geq n$ and $\Omega^m X \not\preceq Y$ for all $0 \leq m < n$. Therefore we have $\operatorname{Ext}^m_R(X, Y) = \operatorname{Ext}^1_R(\Omega^{m-1}X, Y) = 0$ for all $m \neq n$ by lemma 2.2. □

3. Main theorem

In this section, we define the invariants $\mathrm{d}(R)$ and $\mathrm{d}_0(R)$ by which we will give a characterization of one dimensional $\mathbf{N}$-graded Gorenstein rings of finite Cohen-Macaulay representation type.

Definition 3.1. For an $\mathbf{N}$-graded Cohen-Macaulay ring R (not necessarily of dimension one) with $R_0 = k$ being a field, we define $\mathrm{d}(R)$ and $\mathrm{d}_n(R)$ as follow:

$$\mathrm{d}(R) := \sup\{\sum_{n\geq 0} \dim_k \operatorname{Ext}^n_R(X, Y) \mid X, Y \in \mathrm{CM}R \text{ are indecomposable}\},$$

$$\mathrm{d}_n(R) := \sup\{\dim_k \operatorname{Ext}^n_R(X, Y) \mid X, Y \in \mathrm{CM}R \text{ are indecomposable}\}.$$

Now we are ready to state our main theorem of this paper.

Theorem 3.2. *Let k be an algebraically closed field of characteristic 0 and let R be a positively dimensional $\mathbf{N}$-graded Gorenstein ring with isolated singularity where $R_0 = k$. Then the following conditions are equivalent.*

(i) *R is a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type.*
(ii) $\mathrm{d}(R) < \infty$
(ii') $\mathrm{d}_0(R) < \infty$
(iii) $\mathrm{d}(R) \leq 9$
(iii') $\mathrm{d}_0(R) \leq 9$

To show this theorem, we need the graded version of Brauer-Thrall 1 theorem for graded maximal Cohen-Macaulay modules, due to [4], [3] and [2].

Theorem 3.3 (graded version of Brauer-Thrall 1 theorem)**.** *Let R be an $\mathbf{N}$-graded Cohen-Macaulay ring with isolated singularity where $R_0 = k$ is a perfect field. If* $\sup\{\mathrm{e}(X) \mid X \in \mathrm{CM}R$ is indecomposable$\} < \infty$*, then R is of finite Cohen-Macaulay representation type. Here* $\mathrm{e}(X)$ *denotes the multiplicity of the irrelevant maximal ideal along X.*

Proof of 3.2. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (ii') and (iii) $\Rightarrow$ (iii') $\Rightarrow$ (ii') are trivial. First, we show (ii') $\Rightarrow$ (i). Since $\mathrm{d}_0(R) < \infty$, we see that $\dim_k R_n = \dim_k \mathrm{Hom}(R, R(n)) \leq \mathrm{d}_0(R) < \infty$ for all n. Therefore the Hilbert polynomial of R is constant. Hence $\dim R = 1$. For any indecomposable graded maximal Cohen-Macaulay module X, $\dim_k X_n = \dim_k \mathrm{Hom}(R, X(n)) \leq \mathrm{d}_0(R)$ for all n. Therefore the multiplicity $\mathrm{e}(X)$ of X is bounded by $\mathrm{d}_0(R)$. Hence R is of finite Cohen-Macaulay representation type by theorem 3.3.

To prove (i) $\Rightarrow$ (iii'), it is enough to compute $\sup\{\dim_k \mathrm{Hom}(R, Y)$, $\dim_k \mathrm{Hom}(Y, R)$, $\dim_k \mathrm{Hom}(X_i, Y)$, $\dim_k \mathrm{Hom}(Y_i, Y) \mid Y \in \mathrm{CM}R$ is indecomposable $\}$ where X_i and Y_i are in [1, Figures (1) – (7)] . For an indecomposable graded maximal Cohen-Macaulay module X, we denote by X^+ (resp. X^-) the smallest additive full subcategory of $\mathrm{CM}R$ containing all indecomposable graded maximal Cohen-Macaulay modules Y with $X \preceq Y$ (resp. $Y \preceq X$). Then, by induction on the length of the path from X to Y (resp. from Y to X), one can easily check that $\dim_k \mathrm{Hom}(X, Y) = 1$ (resp. $\dim_k \mathrm{Hom}(Y, X) = 1$) for all indecomposable $Y \in X^+$ (resp. $Y \in X^-$) with Y is not free and $\tau Y \notin X^+$ (resp. $\tau^- Y \notin X^-$). Since R is a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type, we may assume that R is one of the rings given in (1) . Thus we are able to compute $\dim_k \mathrm{Hom}(R, R(n)) = \dim_k \mathrm{Hom}(R(-n), R)$ for all n by Hilbert function. Since the functor $\mathrm{Hom}(R, -)$ (resp. $\mathrm{Hom}(-, R)$) is an exact functor on R^+ (resp. R^-), it is possible to compute $\dim_k \mathrm{Hom}(R, Y)$ (resp. $\dim_k \mathrm{Hom}(Y, R)$) for all $Y \in R^+$ (resp. $Y \in R^-$) by using Auslander-Reiten quiver. Since $\mathrm{Hom}(R, Y) = 0$ (resp. $\mathrm{Hom}(Y, R) = 0$) for all $Y \notin R^+$ (resp. $Y \notin R^-$) by lemma 2.1, it is possible to compute $\dim_k \mathrm{Hom}(R, Y)$

and $\dim_k \operatorname{Hom}(Y, R)$ for all $Y \in \mathrm{CM}R$. For any $X \in \{X_i, Y_i\}_i$, since we have already computed $\dim_k \operatorname{Hom}(X, R(n)) = \dim_k \operatorname{Hom}(X(-n), R)$ and since $\operatorname{Hom}(X, -)$ is an exact functor on X^+, it is also possible to compute $\dim_k \operatorname{Hom}(X, Y)$ for all $Y \in X^+$ by using Auslander-Reiten quiver. In this way we can accomplish the computation of $\dim_k \operatorname{Hom}(X, Y)$ for any indecomposable $X, Y \in \mathrm{CM}R$ and get the invariant $\mathrm{d}_0(R)$. The result is shown in table 1. Looking at this table we have $\mathrm{d}_0(R) \leq 9$.

Table 1

type	A_{2m+1}	A_{2m}	D_{2m}	D_{2m+1}	E_6	E_7	E_8
$\mathrm{d}_0(R)$	1	2	3	4	4	6	9

Finally, we prove (iii') $\Rightarrow$ (iii). Because we have already proved (iii') $\Rightarrow$ (i), we may assume that R is one given in (1). Since $\operatorname{Ext}^n_R(X, Y) \cong \operatorname{Ext}^1_R(\Omega^{n-1}X, Y)$ for all $n > 0$ and by lemma 2.3, it is enough to show $\mathrm{d}_0(R) \geq d_1(R)$. For any indecomposable graded maximal Cohen-Macaulay module X, the first syzygy ΩX of X is also an indecomposable graded maximal Cohen-Macaulay module. Since there exists a natural epimorphism $\operatorname{Hom}(\Omega X, Y) \twoheadrightarrow \operatorname{Ext}^1_R(X, Y)$, one can see $\mathrm{d}_0(R) \geq \mathrm{d}_1(R)$ and get $\mathrm{d}(R) \leq 9$. $\square$

Remark 3.4. Let R be a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type with $R_0 = k$ being algebraically closed field of characteristic 0 (i.e. R is isomorphic to one of the rings given in (1)). In the above proof, we showed how to compute the invariant $\mathrm{d}_0(R)$. Remark that we can also compute the invariant $\mathrm{d}_n(R)$ ($n \geq 1$) by using Auslander-Reiten quiver in a similar way to this. Since $\operatorname{Ext}^n_R(X, Y) \cong \operatorname{Ext}^1_R(\Omega^{n-1}X, Y)$, we have $\mathrm{d}_n(R) = \mathrm{d}_1(R)$ for $n \geq 1$. We will show how to compute $\mathrm{d}_1(R)$. For an indecomposable graded maximal Cohen-Macaulay module X, we denote by $X^{(1)}$ the smallest additive full subcategory of $\mathrm{CM}R$ containing all indecomposable graded non-free maximal Cohen-Macaulay modules Y with $\Omega X \preceq Y \preceq \tau X$. We also denote by $X^{(1)'}$ the smallest additive full subcategory of $\mathrm{CM}R$ containing all indecomposable graded non-free maximal Cohen-Macaulay modules Y with $\tau X \prec Y$ and $X \npreceq Y$. It turns out from lemma 2.1 and lemma 2.2 that $\operatorname{Ext}^1_R(X, Y) = 0$ for all $Y \notin X^{(1)}$ and $\operatorname{Ext}^n_R(X, Y) = 0$ for all $Y \in X^{(1)'}$ and for all n. And it follows from lemma 2.3 that $\operatorname{Ext}^1_R(X, -)$ is an exact functor on $X^{(1)} \cup X^{(1)'}$. Hence it is possible to compute $\mathrm{d}_1(R)$ (and therefore $\mathrm{d}_n(R)$ for all $n \geq 1$) by using Auslander-Reiten quiver. The results are given in following table.

Table 2

	A_{2m+1}	A_{2m}	D_{2m}	D_{2m+1}	E_6	E_7	E_8
$\mathrm{d}(R) = \mathrm{d}_0(R)$	1	2	3	4	4	6	9
$\mathrm{d}_n(R)\ (n \geq 1)$	1	2	1	2	3	4	6

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