Math. J. Okayama Univ. **41** (1999), 81–102

EXCEPTIONAL SEQUENCES OVER GRADED COHEN-MACAULAY RINGS

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1. INTRODUCTION

The concept of exceptional sequences was developed by Gorodentsev and Rudakov in [3] and generalized by Bondal in [1]. It is a crucial tool to classify exceptional modules over hereditary algebras and exceptional sheaves over weighted projective lines. In fact, let $\mathcal{A}$ be the category of finitely generated modules over a hereditary algebras or the category of coherent sheaves over a weighted projective line and let $\mathcal{E}$ be an exceptional sequence in $\mathcal{A}$. Then the length of $\mathcal{E}$ is smaller than or equal to the rank n of the Grotherndieck group of $\mathcal{A}$ and $\mathcal{E}$ is called a *complete* exceptional sequence if the length of $\mathcal{E}$ is equal to n. It is known that any exceptional sequence can be enlarged to a complete exceptional sequence ([2, Lemma 1], [5, Lemma 3.1.3]) and the braid group B_n on n strings acts transitively on the set of complete exceptional sequences ([2], [5, Theorem 3.3.1]). Fixing a complete exceptional sequence $\mathcal{E}$, for any exceptional object $X \in \mathcal{A}$ we can find an element $\sigma \in B_n$ such that X belongs to $\sigma\mathcal{E}$. Hence we are able to classify exceptional objects in $\mathcal{A}$ by a fixed complete exceptional sequence $\mathcal{E}$ with the action of the braid group B_n.

In this paper, we shall consider exceptional sequences in the category of finitely generated graded modules over a graded ring. To compare with above cases, exceptional sequences for graded modules may have infinite length. Moreover the braid group B on infinite strings acts on the set of exceptional sequences of infinite length, but the action may not be transitive. Therefore we have to consider the following condition for the exceptional sequence $\mathcal{E}$ to classify exceptional modules;

(1) For any exceptional module E, there exists an element $\sigma \in G = \mathbf{Z} \ltimes (B \ltimes \mathbf{Z}^{\infty})$ such that E belongs to the sequence $\sigma\mathcal{E}$.

If an exceptional sequence $\mathcal{E}$ satisfies the condition (1), we call it a *generating* exceptional sequence, and if there exists a generating exceptional sequence $\mathcal{E}$, then we can classify exceptional modules by $\mathcal{E}$ with the action of G. But it is not easy to show the existence of generating exceptional sequences. So we consider the following conditions;

(2) We assume that any indecomposable maximal Cohen-Macaulay module is an exceptional module. The exceptional sequence $\mathcal{E}$ of infinite length is called an *MCM generating* exceptional sequence if, for any indecomposable maximal Cohen-Macaulay module E, there exists an element $\sigma \in G$ such that E belongs to the sequence $\sigma\mathcal{E}$.

(3) An exceptional sequence $\mathcal{E}$ of infinite length is called a *maximal* exceptional sequence if there is no exceptional module that can be added into $\mathcal{E}$ as an exceptional sequence, i.e. put $\mathcal{E} = (\cdots, E_{i-1}, E_i, E_{i+1}, \cdots)$, and for any exceptional module E and integer n, the sequence of exceptional modules $(\cdots, E_{n-1}, E_n, E, E_{n+1}, E_{n+2}, \cdots)$ is not an exceptional sequence.

If there exists an MCM generating exceptional sequence $\mathcal{E} = (\cdots, E_{i-1}, E_i, E_{i+1}, \cdots)$, one can see that $\mathcal{E}$ generates $\mathfrak{D}^{\mathrm{b}}(\mathrm{mod\,R})$ as a triangulated category, i.e.$\mathfrak{D}^{\mathrm{b}}(\mathrm{mod\,R})$ is the smallest triangulated full subcategory containing all E_i. In this sense, to investigate $\mathfrak{D}^{\mathrm{b}}(\mathrm{mod\,R})$, one needs to find an MCM generating exceptional sequence. In this paper, we shall show the following theorems as main results.

Theorem 3.1. *If R is a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type, then there exists an MCM generating exceptional sequence.*

Theorem 4.1. *Any MCM generating exceptional sequence is maximal.*

Corollary 4.2. *Any generating exceptional sequence is maximal.*

Acknowledgenemt. The author thanks to Professor Yuji Yoshino for his great help in writing this paper.

2. Notations and Definitions

Let $R = \oplus R_n$ be an $\mathbf{N}$-graded Cohen-Macaulay ring and assume that $R_0 = k$ be a field. We denote by $\mathrm{mod}\,R$ the category of finitely generated graded R-modules whose morphisms are graded R-homomorphisms that preserve degrees. We also denote by $\mathrm{CM}\,R$ the full subcategory of $\mathrm{mod}\,R$ consisting of all maximal Cohen-Macaulay modules and by $\mathfrak{D}^{\mathrm{b}}(\mathrm{mod\,R})$ the derived category of $\mathrm{mod}\,R$ consisting of all bounded complexes.

For a complex $E^\cdot \in \mathfrak{D}^{\mathrm{b}}(\mathrm{mod}\,\mathrm{R})$, we write as E^m the graded R-module in the m-th position in the complex. Hence $E^\cdot$ is of the form;

$$\cdots \to E^{m-1} \to E^m \to E^{m+1} \to \cdots .$$

On the other hand, the degree n part of $E^\cdot$ as graded R-modules is denoted by $(E^\cdot)_n$ that is a complex of k-vector spaces;

$$\cdots \to (E^{m-1})_n \to (E^m)_n \to (E^{m+1})_n \to \cdots .$$

In such a meaning we have two kind of shifted complexes of $E^\cdot$ by $r \in \mathbf{Z}$. One is $E^\cdot[r]$ that is the complex whose m-th position $E^\cdot[r]^m$ is E^{r+m}, and the other is $E^\cdot(r)$ that is the complex whose degree n part $(E^\cdot(r))_n$ as graded R-modules is $(E^\cdot)_{r+n}$.

Since $\mathrm{End}(X^\cdot) = \mathrm{H}^0(\mathrm{RHom}(X^\cdot, X^\cdot))$ is finite-dimensional over k for any $X^\cdot \in \mathfrak{D}^{\mathrm{b}}(\mathrm{mod}\,\mathrm{R})$, it is easy to see that the category $\mathfrak{D}^{\mathrm{b}}(\mathrm{mod}\,\mathrm{R})$ is a Krull-Schmidt category, i.e. any complexes in $\mathfrak{D}^{\mathrm{b}}(\mathrm{mod}\,\mathrm{R})$ are uniquely decomposed into direct sums of indecomposable complexes.

In this paper, we are mostly interested in exceptional complexes that are defined in the following definition. Compare with the definitions in [1] and [3].

Definition 2.1. For a complex $E^\cdot \in \mathfrak{D}^{\mathrm{b}}(\mathrm{mod}\,\mathrm{R})$, $E^\cdot$ is called *exceptional* if $\mathrm{RHom}(E^\cdot, E^\cdot)$ is isomorphic to k. A sequence of exceptional complexes $\mathcal{E} = (\cdots, E^\cdot{}_{i-1}, E^\cdot{}_i, E^\cdot{}_{i+1}, \cdots)$ (which may be of infinite length) is called an *exceptional sequence* if $\mathrm{RHom}(E^\cdot{}_i, E^\cdot{}_j) = 0$ for $i > j$.

Note that every exceptional complex $E^\cdot$ is indecomposable, since $\mathrm{End}(E^\cdot) = k$. However, there are lots of indecomposable complexes which are not exceptional.

Example 2.2.
1. Let $R = k[x]$ be a polynomial ring in one variable. Then it is easy to verify that any indecomposable module is exceptional, and $\mathcal{E} = (\cdots, R(-1), R, R(1), R(2), \cdots)$ is an exceptional sequence.
2. Let R be a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type, then one can check that any indecomposable maximal Cohen-Macaulay module is exceptional, because one knows the structures of such modules (c.f.[6], or see proposition 3.2).
3. To contrast the above, let $R = k[x, y]/(y^2)$ with the degree of x (resp. y) is one. Then the graded maximal Cohen-Macaulay module $M = R/(y)$ is indecomposable, but not exceptional.

As in this example, one sees that there are numbers of exceptional modules.

Definition 2.3. For complexes $E^\cdot$ and $F^\cdot \in \mathfrak{D}^b(\mathrm{mod}\,R)$, note that $\mathrm{RHom}(E^\cdot, F^\cdot)$ is a complex of k-vector spaces and there are canonical morphisms $f_{EF} : \mathrm{RHom}(E^\cdot, F^\cdot) \otimes_k E^\cdot \to F^\cdot$ and $g_{EF} : E^\cdot \to \mathrm{D}\,\mathrm{RHom}(E^\cdot, F^\cdot) \otimes_k F^\cdot$, where D denotes the k-dual of complexes of k-vector spaces. The *left mutation functor* $\mathfrak{L}_{E^\cdot} : \mathfrak{D}^b(\mathrm{mod}\,R) \to \mathfrak{D}^b(\mathrm{mod}\,R)$ defined by $E^\cdot$ is given in such a way that, for an object $F^\cdot \in \mathfrak{D}^b(\mathrm{mod}\,R)$, $\mathfrak{L}_{E^\cdot} F^\cdot$ is defined by the triangle;

$$\mathrm{RHom}(E^\cdot, F^\cdot) \otimes_k E^\cdot \xrightarrow{f_{EF}} F^\cdot \longrightarrow \mathfrak{L}_{E^\cdot} F^\cdot \longrightarrow \mathrm{RHom}(E^\cdot, F^\cdot) \otimes_k E^\cdot[1].$$

Dually, the *right mutation functor* $\mathfrak{R}_{F^\cdot} : \mathfrak{D}^b(\mathrm{mod}\,R) \to \mathfrak{D}^b(\mathrm{mod}\,R)$ defined by $F^\cdot$ is given by the triangle;

$$\mathrm{D}\,\mathrm{RHom}(E^\cdot, F^\cdot) \otimes_k F^\cdot[-1] \longrightarrow \mathfrak{R}_{F^\cdot} E^\cdot \longrightarrow E^\cdot \xrightarrow{g_{EF}} \mathrm{D}\,\mathrm{RHom}(E^\cdot, F^\cdot) \otimes_k F^\cdot .$$

Let $\mathcal{E} = (\cdots, E^\cdot_{i-1}, E^\cdot_i, E^\cdot_{i+1}, \cdots)$ be an exceptional sequence of infinite length. Then note that the sequences $(\cdots, E^\cdot_{i-1}, E^\cdot_{i+1}, \mathfrak{R}_{E^\cdot_{i+1}} E^\cdot_i, E^\cdot_{i+2}, \cdots)$ and $(\cdots, E^\cdot_{i-1}, \mathfrak{L}_{E^\cdot_i} E^\cdot_{i+1}, E^\cdot_i, E^\cdot_{i+2}, \cdots)$ are again exceptional sequences. And it is a routine work to show the following equalities.

$$\tag{2.1} \begin{cases} \mathfrak{R}_{E^\cdot_{i+2}}(\mathfrak{R}_{E^\cdot_{i+1}} E^\cdot_i) \cong \mathfrak{R}_{(\mathfrak{R}_{E^\cdot_{i+2}} E^\cdot_{i+1})}(\mathfrak{R}_{E^\cdot_{i+2}} E^\cdot_i), \\ \mathfrak{L}_{E^\cdot_i}(\mathfrak{L}_{E^\cdot_{i+1}} E^\cdot_{i+2}) \cong \mathfrak{L}_{(\mathfrak{L}_{E^\cdot_i} E^\cdot_{i+1})}(\mathfrak{L}_{E^\cdot_i} E^\cdot_{i+2}). \end{cases}$$

See also [1] and [3].

Corresponding to such exceptional sequences of infinite length, we take the braid group B on infinite strings, i.e. B is a group generated by σ_i $(i \in \mathbf{Z})$, with relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all i and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|j - i| \geq 2$. Then one can show that B acts on the set of exceptional sequences of infinite length by

$$\tag{2.2} \begin{cases} \sigma_i\mathcal{E} = (\cdots, E^\cdot_{i-1}, E^\cdot_{i+1}, \mathfrak{R}_{E^\cdot_{i+1}} E^\cdot_i, E^\cdot_{i+2} \cdots), \\ \sigma_i^{-1}\mathcal{E} = (\cdots, E^\cdot_{i-1}, \mathfrak{L}_{E^\cdot_i} E^\cdot_{i+1}, E^\cdot_i, E^\cdot_{i+2} \cdots). \end{cases}$$

for any exceptional sequence $\mathcal{E} = (\cdots, E^\cdot_{i-1}, E^\cdot_i, E^\cdot_{i+1}, \cdots)$. By virtue of the equations (2.1) this action is well-defined.

On the other hand, the abelian groups $\mathbf{Z}$ and $\mathbf{Z}^\infty = \oplus_{i=-\infty}^{\infty} \mathbf{Z}e_i$ that is a direct sum of $\mathbf{Z}$ also act on the set of exceptional sequences of infinite length by

$$\tag{2.3} \begin{cases} n \cdot \mathcal{E} = (\cdots, E^\cdot_{i-1+n}, E^\cdot_{i+n}, E^\cdot_{i+1+n}, \cdots), \\ e_i\mathcal{E} = (\cdots, E^\cdot_{i-1}, E^\cdot_i[1], E^\cdot_{i+1}, \cdots). \end{cases}$$

Now let G be the semi direct product $\mathbf{Z} \ltimes (B \ltimes \mathbf{Z}^\infty)$. Then the above argument shows that G acts on the set of exceptional sequences of infinite length.

Definition 2.4. Let $\mathcal{E}$ be an exceptional sequence of infinite length.

1. We say that an exceptional complex $\mathrm{E}^{\cdot}$ is generated by $\mathcal{E}$ if there exists an element σ in G such that $\mathrm{E}^{\cdot}$ appears in the sequence $\sigma\mathcal{E}$.
2. $\mathcal{E}$ is called a *generating* exceptional sequence if any exceptional module is generated by $\mathcal{E}$.
3. $\mathcal{E}$ is called a *maximal* exceptional sequence if there is no exceptional complex that can be added into $\mathcal{E}$ as an exceptional sequence, i.e. put $\mathcal{E} = (\cdots, \mathrm{E}^{\cdot}{}_{i-1}, \mathrm{E}^{\cdot}{}_{i}, \mathrm{E}^{\cdot}{}_{i+1}, \cdots)$, and for any exceptional complex $\mathrm{E}^{\cdot}$ and integer n, the sequence of exceptional complexes $(\cdots, \mathrm{E}^{\cdot}{}_{n-1}, \mathrm{E}^{\cdot}{}_{n}, \mathrm{E}^{\cdot}, \mathrm{E}^{\cdot}{}_{n+1}, \mathrm{E}^{\cdot}{}_{n+2}, \cdots)$ is not an exceptional sequence.
4. $\mathcal{E}$ is called an *MCM generating* exceptional sequence if any indecomposable maximal Cohen-Macaulay module is generated by $\mathcal{E}$.

Note that if an MCM generating exceptional sequence exists, then every indecomposable maximal Cohen-Macaulay module must be exceptional.

Example 2.5. Let $R = k[x]$ be a polynomial ring in one variable. If M is an indecomposable non-free R-module, then M is of the form $(R/x^nR)(m)$ for some $n \in \mathbf{N}$ and $m \in \mathbf{Z}$, and it has a free resolution of the form

$$0 \to R(m-n) \xrightarrow{x^n} R(m) \to M \to 0.$$

Let $\mathcal{E} = (\cdots, E_{i-1}, E_i, E_{i+1}, \cdots)$ be an exceptional sequence with $E_i = R(i)$ $(\forall i \in \mathbf{Z})$. Note that $\mathcal{E}$ already contains all indecomposable free modules. For a non-free indecomposable module M having the a free resolution of the above form, one can easily check that $\sigma_{m-n}^{-1}\sigma_{m-n+1}\sigma_{m-n+2}\cdots\sigma_{m-1}\mathcal{E}$ contains M. As a consequence, $\mathcal{E}$ is a generating exceptional sequence.

If there is an MCM generating exceptional sequence $\mathcal{E} = (\cdots, \mathrm{E}^{\cdot}{}_{i-1}, \mathrm{E}^{\cdot}{}_{i}, \mathrm{E}^{\cdot}{}_{i+1}, \cdots)$, then one can show that the smallest triangulated full subcategory $\mathcal{D}$ that contains all complexes in $\mathcal{E}$ contains all modules (and therefore all complexes). Indeed for any module M, the n-th syzygy $\Omega^n M$ is a maximal Cohen-Macaulay module for $n \gg 0$. Hence $\mathcal{D} = \mathfrak{D}^{\mathrm{b}}(\mathrm{mod}\,\mathrm{R})$. It is also easy to see that $\{[\mathrm{E}^{\cdot}{}_{i}]\}_{i\in\mathbf{Z}}$ is a basis of the Grotherndieck group of $\mathfrak{D}^{\mathrm{b}}(\mathrm{mod}\,\mathrm{R})$. Hence to investigate $\mathfrak{D}^{\mathrm{b}}(\mathrm{mod}\,\mathrm{R})$, we need to study exceptional complexes. In the next section, we shall show that an MCM generating exceptional sequence can be found for a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type.

3. ONE DIMENSIONAL $\mathbf{N}$-GRADED GORENSTEIN RINGS OF FINITE COHEN-MACAULAY REPRESENTATION TYPE

In this section $R = \oplus R_n$ always denotes a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type with $R_0 = k$ being an algebraically closed field of characteristic 0. The aim of this section is to give a complete proof of the following theorem.

Theorem 3.1. *If R is a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type, then there exists an MCM generating exceptional sequence.*

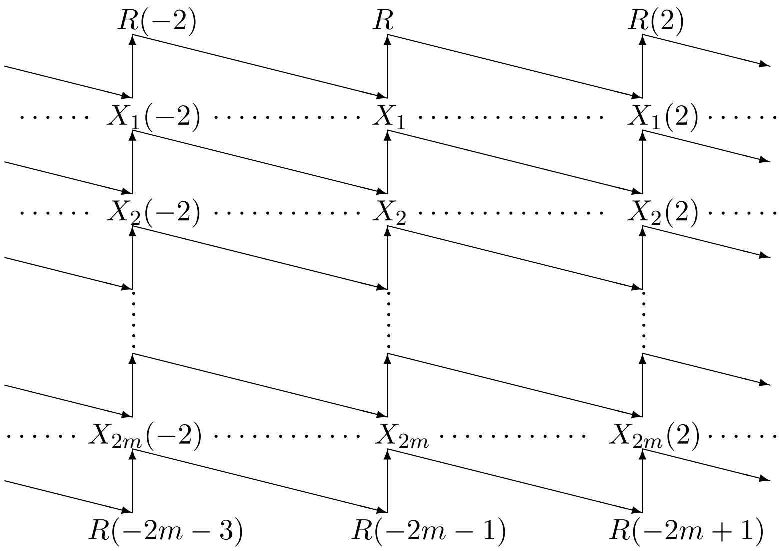
To prove this theorem, since R is a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type, we may assume R is isomorphic to one of following types of rings.

$$
\begin{array}{llll}
(A_n) & R = k[x,y]/(y^2 - x^n) & (n \geq 2) \\
(D_n) & R = k[x,y]/(xy^2 - x^n) & (n \geq 3) \\
(E_6) & R = k[x,y]/(x^3 + y^4) & \\
(E_7) & R = k[x,y]/(x^3 + xy^3) & \\
(E_8) & R = k[x,y]/(x^3 + y^5) &
\end{array}
\tag{3.1}
$$

Moreover the Auslander-Reiten quiver of $\mathrm{CM}\,R$ for each type can be described as they are shown in Figures (1) – (7) below.In fact, the Auslander-Reiten quivers of ungraded maximal Cohen-Macaulay modules over R are shown in [6, page 41-43, 75-83]. However, since we are considering the category of graded modules, we have to discriminate a module M in $\mathrm{CM}\,R$ from the shifted module $M(n)$ for $n \neq 0$, and hence the Auslander-Reiten quiver of $\mathrm{CM}\,R$ is obtained by expanding the Auslander-Reiten quiver of ungraded maximal Cohen-Macaulay modules.

Now we exhibit the Auslander-Reiten quiver of $\mathrm{CM}\,R$ in each case.

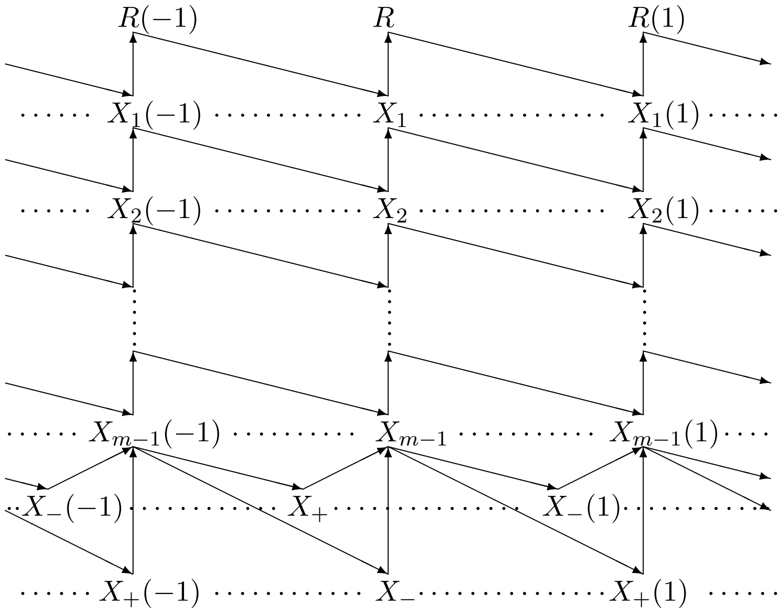
Figure(1) : the type of (A_n) with $n = 2m + 1$.



Here in Figure (1),

$$X_i = \begin{cases} (x^i, y)R, & 1 \le i \le m, \\ X_{2m-i+1}(2m - 2i + 1), & m + 1 \le i \le 2m. \end{cases}$$

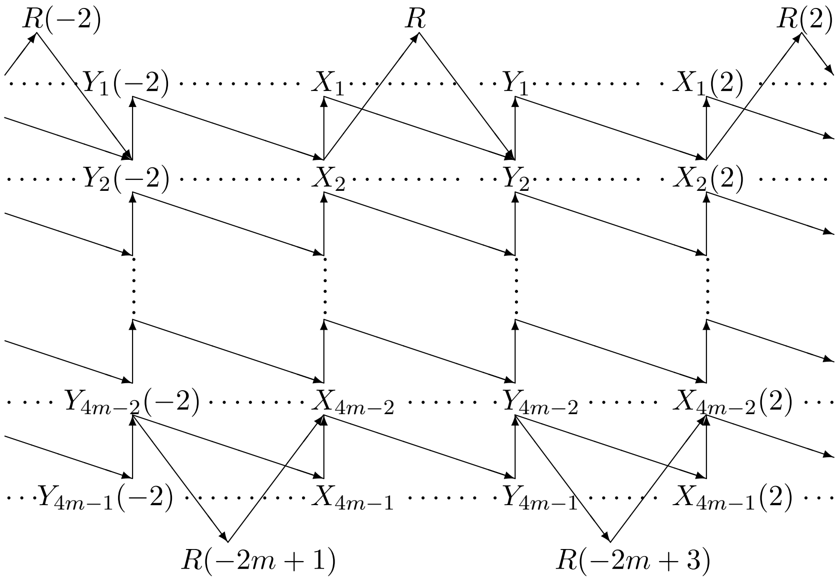
Figure(2) : the type of (A_n) with $n = 2m$.



Here in Figure (2),

$$X_i = \mathrm{Coker}\begin{pmatrix} y & x^j \\ -x^{2m-j} & -y \end{pmatrix}, X_\pm = \mathrm{Coker}(y \pm x^m).$$

Figure(3) : the type of (D_n) with $n = 2m$.



Here in Figure (3),

$$X_1 = \mathrm{Coker}(y^2 + x^{2m-1}),\ Y_1 = \mathrm{Coker}(x),$$

For $2 \le i \le 2m$,

$$X_i = \begin{cases} \mathrm{Coker}\begin{pmatrix} xy & x^j \\ x^{2m-j} & -y \end{pmatrix}, & i = 2j, \\ \mathrm{Coker}\begin{pmatrix} y & x^j \\ x^{2m-1-j} & -y \end{pmatrix}, & i = 2j+1, \end{cases}$$

$$Y_i = \begin{cases} \mathrm{Coker}\begin{pmatrix} y & x^j \\ x^{2m-j} & -xy \end{pmatrix}, & i = 2j, \\ \mathrm{Coker}\begin{pmatrix} xy & x^{j+1} \\ x^{2m-j} & -xy \end{pmatrix}, & i = 2j+1, \end{cases}$$

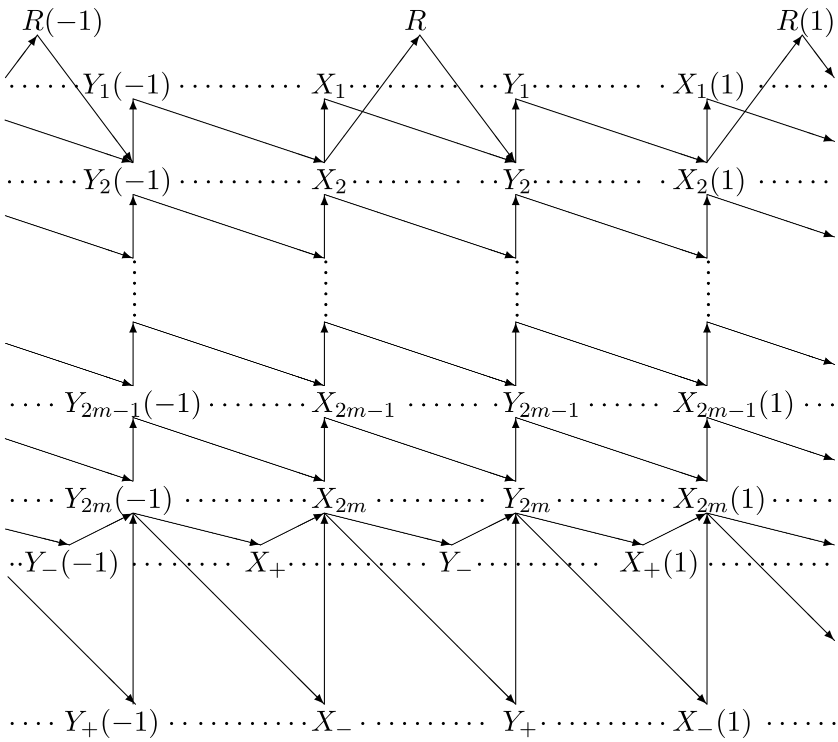
For $2m+1 \le i \le 4m-2$,

$$X_i = \begin{cases} X_{2m-2j-1}(-2j-1), & i = 2m+2j+1, \\ Y_{2m-2j}(-2j-1), & i = 2m+2j, \end{cases}$$

$$Y_i = \begin{cases} Y_{2m-2j-1}(-2j-1), & i = 2m+2j+1, \\ X_{2m-2j}(-2j+1), & i = 2m+2j, \end{cases}$$

$$X_{4m-1} = X_1(-2m+1),\ Y_{4m-1} = Y_1(-2m+1).$$

Figure(4) : the type of (D_n) with $n = 2m+1$.



Here in Figure (4),

$$X_1 = \mathrm{Coker}(y^2 - x^{2m}),\ Y_1 = \mathrm{Coker}(x),$$

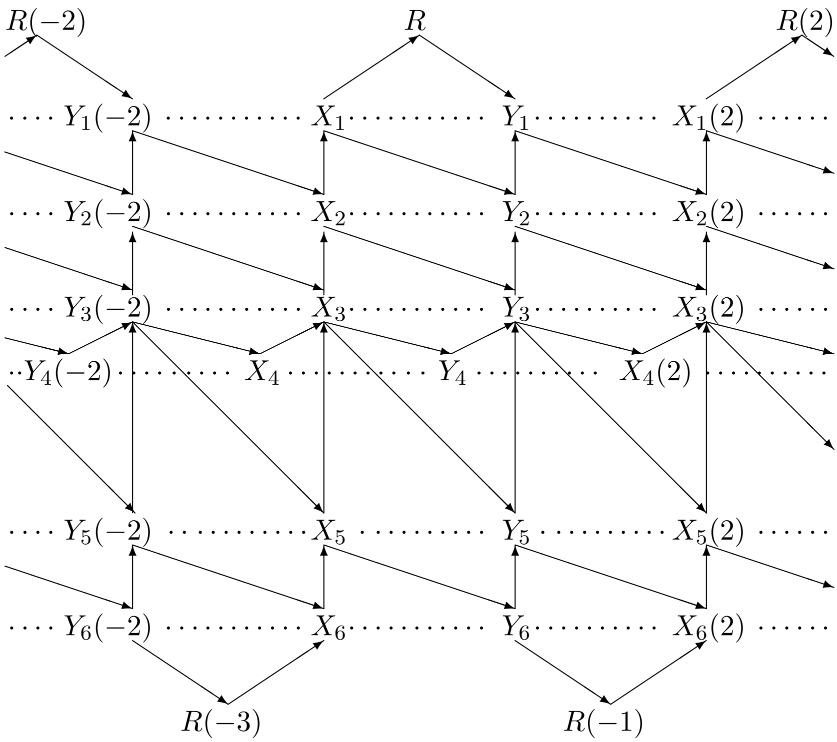
For $2 \le i \le 2m$,

$$X_i = \begin{cases} \mathrm{Coker}\begin{pmatrix} xy & x^j \\ x^{2m+1-j} & -y \end{pmatrix}, & i = 2j, \\ \mathrm{Coker}\begin{pmatrix} y & x^j \\ x^{2m-j} & -y \end{pmatrix}, & i = 2j+1, \end{cases}$$

$$Y_i = \begin{cases} \mathrm{Coker}\begin{pmatrix} y & x^j \\ x^{2m+1-j} & -xy \end{pmatrix}, & i = 2j, \\ \mathrm{Coker}\begin{pmatrix} xy & x^{j+1} \\ x^{2m+1-j} & -xy \end{pmatrix}, & i = 2j+1, \end{cases}$$

$$X_\pm = \mathrm{Coker}(y \pm x^m),\ Y_\pm = \mathrm{Coker}(xy \pm x^{m+1}).$$

Figure(5) : the type of (E_6).



Here in Figure (5),

$X_1 = \text{Coker}\begin{pmatrix} x & y \\ y^3 & -x^2 \end{pmatrix}$, $\quad Y_1 = \text{Coker}\begin{pmatrix} x^2 & y \\ y^3 & -x \end{pmatrix}$,

$X_2 = \text{Coker}\begin{pmatrix} y & 0 & x \\ x & -y^2 & 0 \\ 0 & x & -y \end{pmatrix}$, $\quad Y_2 = \text{Coker}\begin{pmatrix} y^3 & x^2 & xy^2 \\ xy & -y^2 & x^2 \\ x^2 & -xy & -y^3 \end{pmatrix}$,

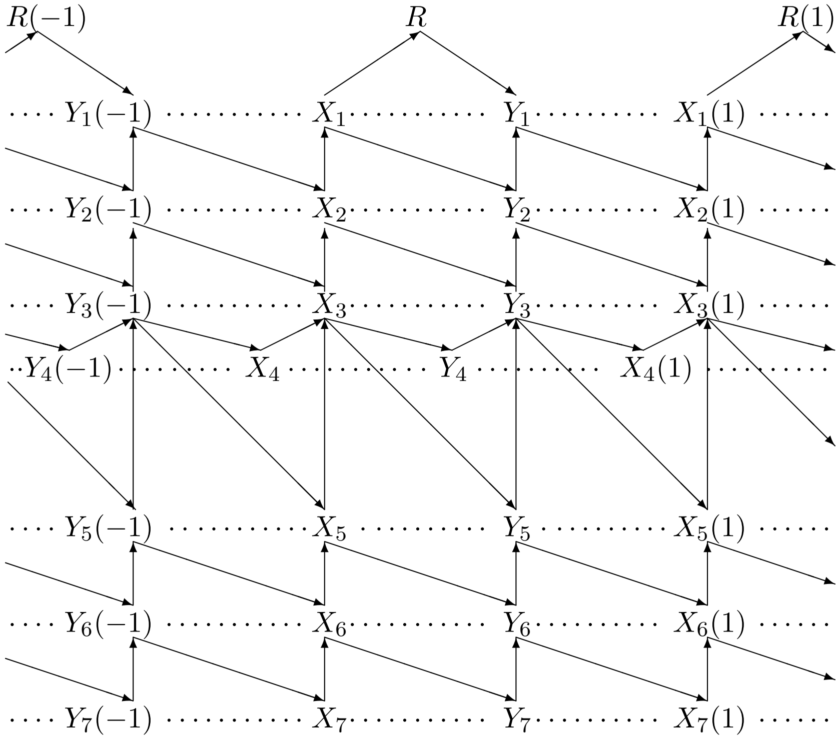
$X_3 = \text{Coker}\begin{pmatrix} x & y^2 & 0 & y \\ y^2 & -x^2 & -xy & 0 \\ 0 & 0 & x^2 & y^2 \\ 0 & 0 & y^2 & -x \end{pmatrix}$, $\quad Y_3 = \text{Coker}\begin{pmatrix} x^2 & y^2 & 0 & xy \\ y^2 & -x & -y & 0 \\ 0 & 0 & x & y^2 \\ 0 & 0 & y^2 & -x^2 \end{pmatrix}$
$= X_3(1)$,

$X_4 = \text{Coker}\begin{pmatrix} x & y^2 \\ y^2 & -x^2 \end{pmatrix}$, $\quad Y_4 = \text{Coker}\begin{pmatrix} x^2 & y^2 \\ y^2 & -x \end{pmatrix} = X_4(1)$,

$X_5 = X_2(-1)$, $\quad Y_5 = Y_2(-1)$,

$X_6 = Y_1(-3)$, $\quad Y_6 = X_1(-1)$.

Figure(6) : the type of (E_7).



Here in Figure (6),

$$X_1 = \mathrm{Coker}\begin{pmatrix} x^2 & y \\ xy^2 & -x \end{pmatrix}, \qquad Y_1 = \mathrm{Coker}\begin{pmatrix} x & y \\ xy^2 & -x^2 \end{pmatrix},$$

$$X_2 = \mathrm{Coker}\begin{pmatrix} xy^2 & -x^2 & -x^2y \\ xy & y^2 & -x^2 \\ x^2 & xy & xy^2 \end{pmatrix}, \quad Y_2 = \mathrm{Coker}\begin{pmatrix} y & 0 & x \\ -x & xy & 0 \\ 0 & -x & y \end{pmatrix},$$

$$X_3 = \mathrm{Coker}\begin{pmatrix} x^2 & xy & y & 0 \\ xy^2 & -x^2 & 0 & y \\ 0 & 0 & x & y \\ 0 & 0 & y^2 & -x \end{pmatrix}, \; Y_3 = \mathrm{Coker}\begin{pmatrix} x & y & -y & 0 \\ y^2 & -x & 0 & -y \\ 0 & 0 & x^2 & xy \\ 0 & 0 & xy^2 & -x^2 \end{pmatrix},$$

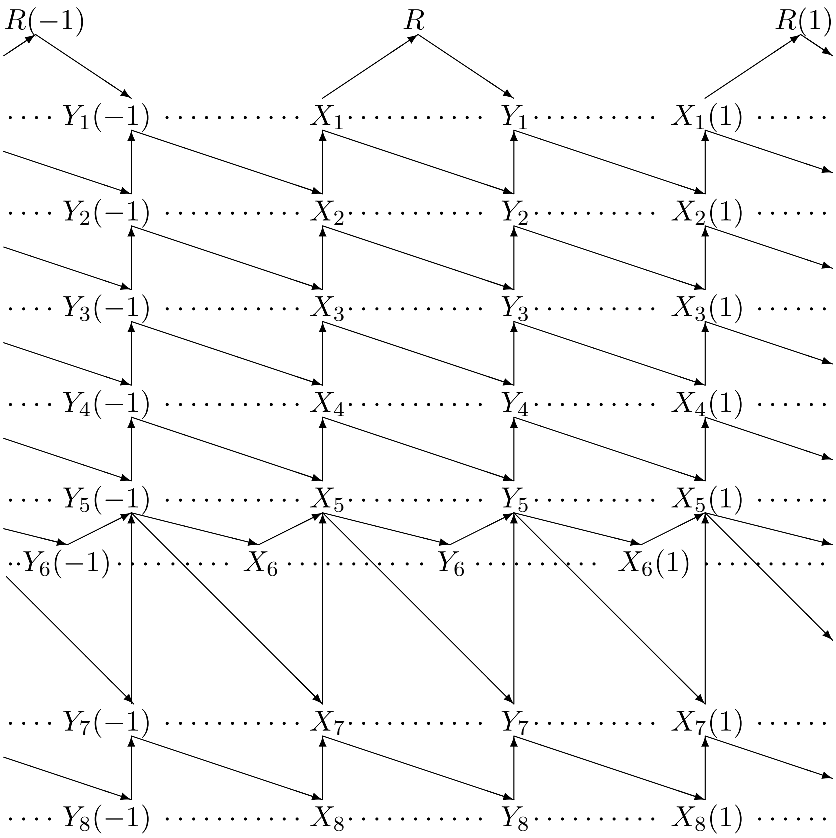
$$X_4 = \mathrm{Coker}\begin{pmatrix} x^2 & xy \\ xy^2 & -x^2 \end{pmatrix}, \qquad Y_4 = \mathrm{Coker}\begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix},$$

$$X_5 = \mathrm{Coker}\begin{pmatrix} x^2 & -y^2 & -xy \\ xy & x & -y^2 \\ xy^2 & xy & x^2 \end{pmatrix}, \quad Y_5 = \mathrm{Coker}\begin{pmatrix} x & 0 & y \\ -xy & x^2 & 0 \\ 0 & -xy & x \end{pmatrix},$$

$$X_6 = \mathrm{Coker}\begin{pmatrix} x^2 & y^2 \\ xy & -x \end{pmatrix}, \qquad Y_6 = \mathrm{Coker}\begin{pmatrix} x & y^2 \\ xy & -x^2 \end{pmatrix},$$

$$X_7 = \mathrm{Coker}(x^2 + y^3), \qquad Y_7 = \mathrm{Coker}(x).$$

Figure(7) : the type of (E_8).



Here in Figure (7),

$$X_1 = \operatorname{Coker}\begin{pmatrix} x^2 & y \\ y^4 & -x \end{pmatrix}, \qquad Y_1 = \operatorname{Coker}\begin{pmatrix} x & y \\ y^4 & -x^2 \end{pmatrix},$$

$$X_2 = \operatorname{Coker}\begin{pmatrix} y^4 & xy^3 & x^2 \\ -x^2 & y^4 & xy \\ -xy & -x^2 & y^2 \end{pmatrix}, \quad Y_2 = \operatorname{Coker}\begin{pmatrix} y & -y & 0 \\ 0 & x & -y \\ x & 0 & y^3 \end{pmatrix},$$

$$X_3 = \operatorname{Coker}\begin{pmatrix} y & -x & 0 & y^3 \\ x & 0 & -y^3 & 0 \\ -y^2 & 0 & -x^2 & 0 \\ 0 & -y^2 & -xy & -x^2 \end{pmatrix},$$

$$Y_3 = \operatorname{Coker}\begin{pmatrix} 0 & x^2 & -y^3 & 0 \\ -x^2 & xy & 0 & -y^3 \\ 0 & -y^2 & -x & 0 \\ y^2 & 0 & y & -x \end{pmatrix},$$

$$X_4 = \operatorname{Coker}\begin{pmatrix} y^4 & x^2 & 0 & -xy^2 & 0 \\ -x^2 & xy & 0 & -y^3 & 0 \\ 0 & -y^2 & -x & 0 & y^3 \\ -xy^2 & y^3 & 0 & x^2 & 0 \\ -y^3 & 0 & -y^2 & xy & -x^2 \end{pmatrix},$$

$$Y_4 = \operatorname{Coker}\begin{pmatrix} y & -x & 0 & 0 & 0 \\ x & 0 & 0 & y^2 & 0 \\ -y^2 & 0 & -x^2 & 0 & -y^3 \\ 0 & -y^2 & 0 & x & 0 \\ 0 & 0 & y^2 & y & -x \end{pmatrix},$$

$$X_5 = \operatorname{Coker}\begin{pmatrix} y^4 & xy^2 & x^2 & 0 & 0 & xy \\ -x^2 & y^3 & xy & -x & 0 & 0 \\ -xy^2 & -x^2 & y^3 & 0 & -xy & 0 \\ 0 & 0 & 0 & y & -x & 0 \\ 0 & 0 & 0 & 0 & y^2 & -x \\ 0 & 0 & 0 & x & 0 & y^2 \end{pmatrix},$$

$$Y_5 = \operatorname{Coker}\begin{pmatrix} y & -x & 0 & 0 & 0 & -x \\ 0 & y^2 & -x & xy & 0 & 0 \\ x & 0 & y^2 & 0 & xy & 0 \\ 0 & 0 & 0 & y^4 & xy^2 & x^2 \\ 0 & 0 & 0 & -x^2 & y^3 & xy \\ 0 & 0 & 0 & -xy^2 & -x^2 & y^3 \end{pmatrix},$$

$$X_6 = \operatorname{Coker}\begin{pmatrix} y^4 & xy^2 & x^2 \\ -x^2 & y^3 & xy \\ -xy^2 & -x^2 & y^3 \end{pmatrix}, \qquad Y_6 = \operatorname{Coker}\begin{pmatrix} y & -x & 0 \\ 0 & y^2 & -x \\ x & 0 & y^2 \end{pmatrix},$$

$$X_7 = \operatorname{Coker}\begin{pmatrix} x^2 & y^2 & 0 & xy \\ y^3 & -x & -y^2 & 0 \\ 0 & 0 & x & y^2 \\ 0 & 0 & y^3 & -x^2 \end{pmatrix},$$

$$Y_7 = \operatorname{Coker}\begin{pmatrix} x & y^2 & 0 & y \\ y^3 & -x^2 & -xy^2 & 0 \\ 0 & 0 & x^2 & y^2 \\ 0 & 0 & y^3 & -x \end{pmatrix},$$

$$X_8 = \operatorname{Coker}\begin{pmatrix} x^2 & y^2 \\ y^3 & -x \end{pmatrix}, \qquad Y_8 = \operatorname{Coker}\begin{pmatrix} x & y^2 \\ y^3 & -x^2 \end{pmatrix}.$$

In the rest of this section, R is one of the rings given in (3.1). First, to prove theorem 3.1 , we need the following proposition.

Proposition 3.2. *Every indecomposable maximal Cohen-Macaulay module over R is exceptional.*

To show this proposition, the following lemma is necessary.

Lemma 3.3. *Let Γ be the Auslander-Reiten quiver of* $\mathrm{CM}\,R$*. Then, for indecomposable maximal Cohen-Macaulay modules X and Y, the following hold.*

1. *If there is a non-trivial morphism from X to Y that is not an isomorphism, then there is a sequence of irreducible morphisms whose composite morphism from X to Y is non-trivial. In particular, if there is no path from $[X]$ to $[Y]$ in Γ, then we have* $\mathrm{Hom}(X, Y) = 0$.
2. *There is no cyclic path in Γ*
3. *If $\ell \geq 0$, there is no path from $[X]$ to $[\tau X(-\ell)]$ in Γ. If $\ell > 0$, there is no path from $[X]$ to $[X(-\ell)]$ in Γ.*
4. *Let τ be the Auslander-Reiten translation and let Ω^n be the n-th syzygy functor. If X is non-free, then we have isomorphisms,*
$$\tau X \cong \Omega^1 X(a),\ \tau^2 X \cong X(-b),$$
where a and b are non-negative integers that are given as in the following table 1 in each case.

Table 1

type	A_{2m+1}	A_{2m}	D_{2m}	D_{2m+1}	E_6	E_7	E_8
a	$2m-1$	$m-1$	$2m-1$	m	5	4	7
b	4	2	2	1	2	1	1

5. *If there is no path from $[X]$ to $[\tau Y]$, then* $\mathrm{Ext}^1(Y, X) = 0$.

Proof. Just observe the Auslander-Reiten quivers (Figures (1)-(7)) for the proof (2) and (3).

(1) Suppose that there is an $f(\neq 0) \in \mathrm{Hom}(X, Y)$ that is not an isomorphism, and that there is no finite sequence of irreducible morphisms from X to Y whose composition is non-trivial. We want to have a contradiction from these assumptions.

Considering the right almost split map $\oplus_i Z_i \to Y$, we can lift f to a morphism $g : X \to \oplus_i Z_i$. Since $f \neq 0$, we have a non-trivial morphism $X \to Z_i \to Y$ for some i, where $Z_i \to Y$ is an irreducible morphism and $X \to Z_i$ is not an isomorphism by the assumption. Continuing this procedure, for any integer n, we have a sequence of irreducible morphisms

$$W_n \xrightarrow{g_n} W_{n-1} \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_3} W_2 \xrightarrow{g_2} W_1 \xrightarrow{g_1} Y,$$

and a non-isomorphism $f_n : X \to W_n$ such that the composition $g_1 \cdot g_2 \cdots g_n \cdot f_n \neq 0$. Since there are only a finite number of isomorphism classes of indecomposable maximal Cohen-Macaulay modules up to degree shifting, there is a $[Z] \in \Gamma$ such that, among the set $\{W_n | n \in \mathbf{N}\}$, there are infinitely many modules of the form $Z(\ell)(\ell \in \mathbf{Z})$. Thus we have a sequence of morphisms

$$Z(\ell_n) \xrightarrow{h_n} Z(\ell_{n-1}) \xrightarrow{h_{n-1}} \cdots \xrightarrow{h_3} Z(\ell_2) \xrightarrow{h_2} Z(\ell_1) \xrightarrow{h_1} Y,$$

(each h_i is a composition of irreducible morphisms) and a morphism $q_n : X \to Z(\ell_n)$ such that the composition $h_1 \cdot h_2 \cdots h_n \cdot q_n \neq 0$. By (3), we must have $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_n \geq \cdots$.

Suppose $\inf\{\ell_n\} = -\infty$. In this case we find a strictly decreasing subsequence ℓ_{n_i}, and $\mathrm{Hom}(X, Z(\ell_{n_i})) \neq 0$. But this contradicts the fact that $\oplus_{n \in \mathbf{Z}} \mathrm{Hom}(X, Z(n))$ is a finitely generated graded R-module.

Therefore $\inf\{\ell_n\} > -\infty$, hence $\ell_n = \ell_{n+1} = \ell_{n+2} \cdots$ for large n. In this case, since $\mathrm{Hom}(Z(\ell_{n+1}), Z_n) = \mathrm{End}(Z(\ell_n))$ is a finite dimensional k-algebra that is local, if we take an enough large integer ν, then any composition of ν non-isomorphisms in $\mathrm{End}(Z(\ell_n))$ is trivial. Thus $h_{\ell_n} \cdot h_{\ell_{n+1}} \cdots h_{\ell_{n+\nu}} = 0$, which is also a contradiction.

(4) Let ω_R be the canonical module of R. Then, since R is a Gorenstein ring, we have $\omega_R \cong R(a)$ with an integer a which is called the a-invariant of R. The a-invariants for R are listed in Table 1 . On the other hand, since τ is defined as $\tau X = \mathrm{Hom}(\Omega^1(\mathrm{tr}\, X), \omega_R)$ (c.f.[6]), it can be seen that $\tau X \cong (\Omega^1 X)(a)$ as desired. For the second isomorphism in (4), observe Figure (1)-(7).

(5) Let $0 \to X \to Z \to Y \to 0$ be an exact sequence. We have only to show that this is a split exact sequence. Let $0 \to \tau Y \to E \to Y \to 0$ be the Auslander-Reiten sequence ending in Y. From the property of the Auslander-Reiten sequences we have the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & Y & \longrightarrow & 0 \\
 & & f \downarrow & & g \downarrow & & \| & & \\
0 & \longrightarrow & \tau Y & \longrightarrow & E & \xrightarrow{\pi} & Y & \longrightarrow & 0
\end{array}$$

Suppose $f = 0$ in this diagram. Then the morphism g will induce a morphism $Y \to E$ which contradicts the fact that π is not a split epimorphism. Therefore $f \neq 0$, hence $\mathrm{Hom}(X, \tau Y) \neq 0$. It thus follows from (1) that there must be a path from $[X]$ to $[\tau Y]$. □

proof of 3.2. Let X be an indecomposable maximal Cohen-Macaulay module over R. We have to show $\mathrm{RHom}(X, X) \cong k$, that is $\mathrm{Hom}(X, X) \cong k$

and $\mathrm{Ext}^n(X,X)=0$ for $n\geq 1$. First, consider a non-trivial homomorphism $f(\neq 0)\in \mathrm{End}(X)$. If f is a non-isomorphism, then by Lemma 3.3 (1) there is a path from $[X]$ to itself, that contradicts Lemma 3.3 (2). Therefore every non-zero element of $\mathrm{End}(X)$ is an isomorphism, and hence $\mathrm{End}(X)$ is a division ring that is finite over k. Since k is an algebraically closed field, we have $\mathrm{End}(X)\cong k$. Second, we prove that $\mathrm{Ext}^n(X,X)=0$ for $n\geq 1$. Note from Lemma 3.3 (4),

$$\tau(\Omega^{n-1}X)=\tau^n X(-(n-1)a)=\begin{cases}\tau X(-\frac{n-1}{2}b-(n-1)a) & \text{if } n \text{ is odd},\\ X(-\frac{n}{2}b-(n-1)a) & \text{if } n \text{ is even}.\end{cases}$$

In any case, $\tau(\Omega^{n-1}X)$ is isomorphic to either $\tau X(-\ell)$ for some $\ell\geq 0$ or $X(-\ell)$ for some $\ell>0$. Since there is no path from $[X]$ to such a module in Γ, it follows from Lemma 3.3 (5) that $\mathrm{Ext}^n(X,X)=\mathrm{Ext}^1(\Omega^{n-1}X,X)=0$.

□

We prepare a lemma to show theorem 3.1.

Lemma 3.4. *Under the assumption of theorem 3.1, suppose* $0\to L\to\oplus_{i=1}^m M_i^{\oplus r_i}\to N\to 0$ *is an Auslander-Reiten sequence in* $\mathrm{CM}\,R$ *where each* M_i *is an indecomposable maximal Cohen-Macaulay module. Then we have* $\mathfrak{L}_{M_m}\mathfrak{L}_{M_{m-1}}\cdots\mathfrak{L}_{M_1}N[-1]\cong L$ *and* $\mathfrak{R}_{M_m}\mathfrak{R}_{M_{m-1}}\cdots\mathfrak{R}_{M_1}L[1]\cong N$.

Proof. We only prove that $\mathfrak{L}_{M_m}\mathfrak{L}_{M_{m-1}}\cdots\mathfrak{L}_{M_1}N[-1]$ is isomorphic to L. (For $\mathfrak{R}_{M_m}\mathfrak{R}_{M_{m-1}}\cdots\mathfrak{R}_{M_1}L[1]$, the proof is completely similar.) We claim that we can take a triangle $\oplus_{i=1}^l M_i^{\oplus r_i}\to N\to\mathfrak{L}_{M_l}\mathfrak{L}_{M_{l-1}}\cdots\mathfrak{L}_{M_1}N\to\oplus_{i=1}^l M_i^{\oplus r_i}[1]$ for each l. We prove it by induction on l. Since there is no path from $[M_i]$ to $[M_j(-\ell)]$ and $[\tau M_j(-\ell)]$ $(i\neq j,\ell\geq 0)$ in the Auslander-Reiten quiver Γ, we have $\mathrm{RHom}(M_i,M_j)=0$ for $i\neq j$ by Lemma 3.3.Similarly, we have $\mathrm{RHom}(M_i,N)\cong\mathrm{Hom}(M_i,N)$. We note that $\dim_k\mathrm{Hom}(M_i,N)=r_i$ since apply $\mathrm{RHom}(M_i,-)$ to $0\to L\to\oplus_{i=1}^m M_i^{\oplus r_i}\to N\to 0$. If $l=1$, the above claim follows from the definition of the left mutation. So we may assume $l\geq 2$ and we can take the triangle $\oplus_{i=1}^{l-1}M_i^{\oplus r_i}\to N\to\mathfrak{L}_{M_{l-1}}\mathfrak{L}_{M_{l-2}}\cdots\mathfrak{L}_{M_1}N\to\oplus_{i=1}^{l-1}M_i^{\oplus r_i}[1]$. Applying the functor $\mathfrak{L}_{M_l}$ to this triangle, we get the new triangle;

(3.2)
$$\mathfrak{L}_{M_l}\oplus_{i=1}^{l-1}M_i^{\oplus r_i}\to\mathfrak{L}_{M_l}N\to\mathfrak{L}_{M_l}\mathfrak{L}_{M_{l-1}}\mathfrak{L}_{M_{l-2}}\cdots\mathfrak{L}_{M_1}N\to\mathfrak{L}_{M_l}\oplus_{i=1}^{l-1}M_i^{\oplus r_i}[1].$$

Since $\mathrm{RHom}(M_i,M_j)=0$ for $i\neq j$, one sees that $\mathfrak{L}_{M_l}\oplus_{i=1}^{l-1}M_i^{\oplus r_i}\cong\oplus_{i=1}^{l-1}M_i^{\oplus r_i}$. On the other hand, it follows from the definition of $\mathfrak{L}_{M_l}N$ that we can take the following triangle;

$$M_l^{\oplus r_l} \to N \to \mathfrak{L}_{M_l} N \to M_l^{\oplus r_l}[1]. \tag{3.3}$$

Applying the octahedral axiom for the triangles (3.2) and (3.3), we have the diagram:

$$\begin{tikzcd}
 & V \arrow[ddl] \arrow[dr] & \\
M_l^{\oplus r_l} \arrow[ur] \arrow[d] & & \oplus_{i=1}^{l-1} M_i^{\oplus r_i} \arrow[ll] \arrow[ddl] \\
N \arrow[rr] \arrow[dr] & & \mathfrak{L}_{M_l}\mathfrak{L}_{M_{l-1}} \cdots \mathfrak{L}_{M_2}\mathfrak{L}_{M_1} N \arrow[u] \arrow[uul] \\
 & \mathfrak{L}_{M_l} N \arrow[uul] \arrow[ur] &
\end{tikzcd}$$

Thus we obtain two triangles;

$$\begin{aligned}
&M_l^{\oplus r_l} \to V \to \oplus_{i=1}^{l-1} M_i^{\oplus r_i} \to M_l^{\oplus r_l}[1], \\
&V \to N \to \mathfrak{L}_{M_l}\mathfrak{L}_{M_{l-1}} \cdots \mathfrak{L}_{M_2}\mathfrak{L}_{M_1} N \to V[1].
\end{aligned}$$

And one sees that V is isomorphic to $\oplus_{i=1}^{l} M_i^{\oplus r_i}$ because $\mathrm{RHom}(M_i, M_j) = 0$ for $i \neq j$ and hence the first triangle splits. Since $0 \to L \to \oplus_{i=1}^{m} M_i^{\oplus r_i} \to N \to 0$ is an Auslander-Reiten sequence, it follows from the construction of the triangle $\oplus_{i=1}^{m} M_i^{\oplus r_i} \to N \to \mathfrak{L}_{M_m}\mathfrak{L}_{M_{m-1}} \cdots \mathfrak{L}_{M_1} N \to \oplus_{i=1}^{m} M_i^{\oplus r_i}[1]$ that L must be isomorphic to $\mathfrak{L}_{M_m}\mathfrak{L}_{M_{m-1}} \cdots \mathfrak{L}_{M_1} N[-1]$.

□

TABLE 2

type	$\mathcal{E}_0$
A_{2m+1}	$(\cdots, R(-2m-3), R(-4), R(-2m-1), R(-2), X_{2m}, X_{2m-1}, \cdots, X_2, X_1, R(-2m+1), R, R(-2m+3), R(2), \cdots)$
A_{2m}	$(\cdots, R(-2), R(-1), X_+, X_-, X_{m-1}, X_{m-2}, \cdots, X_1, R, R(1), \cdots)$
D_{2m}	$(\cdots, R(-2m-1), R(-2), R(-2m+1), X_{4m-1}, X_{4m-2}, \cdots, X_2, X_1, R, R(-2m+3), R(2), \cdots)$
D_{2m+1}	$(\cdots, R(-2), R(-1), X_+, X_-, X_{2m}, X_{2m-1}, \cdots, X_1, R, R(1), R(2), \cdots)$
E_6	$(\cdots, R(-5), R(-2), R(-3), X_6, X_5, X_4, X_3, X_2, X_1, R, R(-1), R(2), \cdots)$
E_7	$(\cdots, R(-2), R(-1), X_7, X_6, X_5, X_4, X_3, X_2, X_1, R, R(1), R(2), \cdots)$
E_8	$(\cdots, R(-2), R(-1), X_8, X_7, X_6, X_5, X_4, X_3, X_2, X_1, R, R(1), R(2), \cdots)$

proof of 3.1. We take the sequence $\mathcal{E}_0 = (\cdots, \dot{E}_{i-1}, E_i, E)i+1, \cdots)$ in each case as in the above table 2 in which we consider the module R is set in the 0-th position in $\mathcal{E}_0$, i.e. $E_0 = R$. We shall prove that $\mathcal{E}_0$ is actually an MCM generating exceptional sequence in each case. It follows from Lemma 3.3 and Proposition 3.2, one can see that the sequence $\mathcal{E}_0$ is an exceptional sequence. So we prove that $\mathcal{E}_0$ is MCM generating. The proof is actually a matter of computation. First of all we put $\sigma, \sigma' \in G$ as follows:

The type of (A_{2m+1}); $\sigma = 2 \cdot (\sigma_{-2m}\sigma_{-2m+1}\cdots\sigma_{-2})(e_0\sigma_{-1}\sigma_{-2})(e_{-1}\sigma_{-2}\sigma_{-3})\cdots (e_{-2m+1}\sigma_{-2m}\sigma_{-2m-1})(\sigma_{-2m}\sigma_{-2m+1}\cdots\sigma_{-2})$

The type of (A_{2m}); $\sigma = 1 \cdot (\sigma_{-m-1}\sigma_{-m}\cdots\sigma_{-2})(e_0\sigma_{-1}\sigma_{-2})(e_{-1}\sigma_{-2}\sigma_{-3})\cdots (e_{-m+3}\sigma_{-m+2}\sigma_{-m+1})(e_{-m+2}\sigma_{-m+1}\sigma_{-m}\sigma_{-m-1})(e_{-m+1}\sigma_{-m}\sigma_{-m-1})(e_{-m+1}\sigma_{-m})$

The type of (D_{2m}); $\sigma' = 1 \cdot (\sigma_{-4m+1}\sigma_{-4m+2}\cdots\sigma_{-3})(e_0\sigma_{-1}\sigma_{-2})(e_0\sigma_{-1}\sigma_{-2}\sigma_{-3}) (e_{-2}\sigma_{-3}\sigma_{-4})(e_{-3}\sigma_{-4}\sigma_{-5})\cdots(e_{-4m+3}\sigma_{-4m+2}\sigma_{-4m+1})(e_{-4m+2}\sigma_{-4m+1})$
$\sigma = 1 \cdot (e_0\sigma_{-1})(e_0\sigma_{-1}\sigma_{-2})(e_{-1}\sigma_{-2}\sigma_{-3})\cdots(e_{-4m+5}\sigma_{-4m+4}\sigma_{-4m+3})(e_{-4m+4} \sigma_{-4m+3}\sigma_{-4m+2}\sigma_{-4m+1})(e_{-4m+3}\sigma_{-4m+2}\sigma_{-4m+1})(\sigma_{-4m+3}\sigma_{-4m+4}\cdots\sigma_{-1})\sigma'$

The type of (D_{2m+1}); $\sigma' = 1 \cdot (\sigma_{-2m-2}\sigma_{-2m-1}\cdots\sigma_{-3})(e_0\sigma_{-1}\sigma_{-2})(e_0\sigma_{-1}\sigma_{-2}\sigma_{-3}) (e_{-2}\sigma_{-3}\sigma_{-4})(e_{-3}\sigma_{-4}\sigma_{-5})\cdots(e_{-2m+2}\sigma_{-2m+1}\sigma_{-2m})(e_{-2m+1}\sigma_{-2m}\sigma_{-2m-1}\sigma_{-2m-2}) (e_{-2m}\sigma_{-2m-1}\sigma_{-2m-2})(e_{-2m}\sigma_{-2m-1})$
$\sigma = (e_{-1}\sigma_{-2})(e_{-1}\sigma_{-2}\sigma_{-3})(e_{-2}\sigma_{-3}\sigma_{-4})\cdots(e_{-2m+2}\sigma_{-2m+1}\sigma_{-2m})(e_{-2m+1}\sigma_{-2m} \sigma_{-2m-1}\sigma_{-2m-2})(e_{-2m}\sigma_{-2m-1}\sigma_{-2m-2})(e_{-2m}\sigma_{-2m-1})\sigma'$

The type of (E_6); $\sigma' = 1 \cdot (\sigma_{-6}\sigma_{-5}\sigma_{-4}\sigma_{-3}\sigma_{-2})(e_0\sigma_{-1}\sigma_{-2})(e_{-1}\sigma_{-2}\sigma_{-3})(e_{-2}\sigma_{-3} \sigma_{-4}\sigma_{-5})(e_{-4}\sigma_{-5}\sigma_{-6})(e_{-5}\sigma_{-6})(e_{-3}\sigma_{-4})$
$\sigma = 1 \cdot (e_0\sigma_{-1})(e_0\sigma_{-1}\sigma_{-2})(e_{-1}\sigma_{-2}\sigma_{-3}\sigma_{-4})(e_{-3}\sigma_{-4}\sigma_{-5})(e_{-4}\sigma_{-5}\sigma_{-6})(e_{-2}\sigma_{-3}) (\sigma_{-5}\sigma_{-4}\sigma_{-3}\sigma_{-2}\sigma_{-1})\sigma'$

The type of (E_7); $\sigma' = 1 \cdot (\sigma_{-7}\sigma_{-6}\sigma_{-5}\sigma_{-4}\sigma_{-3}\sigma_{-2})(e_0\sigma_{-1}\sigma_{-2})(e_{-1}\sigma_{-2}\sigma_{-3})(e_{-2} \sigma_{-3}\sigma_{-4}\sigma_{-5})(e_{-4}\sigma_{-5}\sigma_{-6})(e_{-5}\sigma_{-6}\sigma_{-7})(e_{-6}\sigma_{-7})(e_{-3}\sigma_{-4})$
$\sigma = (e_{-1}\sigma_{-2})(e_{-1}\sigma_{-2}\sigma_{-3})(e_{-2}\sigma_{-3}\sigma_{-4}\sigma_{-5})(e_{-4}\sigma_{-5}\sigma_{-6})(e_{-5}\sigma_{-6}\sigma_{-7})(e_{-6}\sigma_{-7}) (e_{-3}\sigma_{-4})\sigma'$

The type of (E_8); $\sigma' = 1 \cdot (\sigma_{-8}\sigma_{-7}\sigma_{-6}\sigma_{-5}\sigma_{-4}\sigma_{-3}\sigma_{-2})(e_0\sigma_{-1}\sigma_{-2})(e_{-1}\sigma_{-2}\sigma_{-3}) (e_{-2}\sigma_{-3}\sigma_{-4})(e_{-3}\sigma_{-4}\sigma_{-5})(e_{-4}\sigma_{-5}\sigma_{-6}\sigma_{-7})(e_{-6}\sigma_{-7}\sigma_{-8})(e_{-7}\sigma_{-8})(e_{-5}\sigma_{-6})$
$\sigma = (e_{-1}\sigma_{-2})(e_{-1}\sigma_{-2}\sigma_{-3})(e_{-2}\sigma_{-3}\sigma_{-4})(e_{-3}\sigma_{-4}\sigma_{-5})(e_{-4}\sigma_{-5}\sigma_{-6}\sigma_{-7})(e_{-6}\sigma_{-7} \sigma_{-8})(e_{-7}\sigma_{-8})(e_{-5}\sigma_{-6})\sigma'$

If R is either a type of (A_{2m+1}),(D_{2m}) or (E_6), then any non-free indecomposable maximal Cohen-Macaulay module is of the form $X_s(2t)$

or $Y_s(2t)$ in Figures (1),(3) or (5). Here putting $\rho = \sigma^t$ (resp. $\rho' = \sigma^t\sigma'$), we can show by computation using Lemma 3.4 that $\rho\mathcal{E}_0$ (resp. $\rho'\mathcal{E}_0$) contains $X_s(2t)$ (resp. $Y_s(2t)$) and all free modules. If R is either a type of (A_{2m}),(D_{2m+1}) (E_7) or (E_8), then any non-free indecomposable maximal Cohen-Macaulay module is of the form $X_s(t)$ or $Y_s(t)$ in Figures (2),(4),(6) or (7). In this case, put $\rho = \sigma^t$ (resp. $\rho' = \sigma^t\sigma'$) and $\rho\mathcal{E}_0$ (resp. $\rho'\mathcal{E}_0$) contains $X_s(t)$ (resp. $Y_s(t)$) and all free modules by Lemma 3.4. Hence in any case, we have shown that any indecomposable maximal Cohen-Macaulay module is generated by $\mathcal{E}_0$.

□

We remark that an indecomposable module is not always exceptional even if there exists an MCM generating exceptional sequence.

Example 3.5. Let $R = k[x,y]/(y^2 - x^3)$ be an $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type (A_3) and we denote by $\mathfrak{m}$ the unique maximal graded ideal of R. We put $M = \mathrm{Coker}(\mathfrak{m}(-5) \xrightarrow{xy} \mathfrak{m})$. Then one can easily check that $\mathrm{End}(M) \cong \mathrm{Ext}^1(M,M) \cong k$. It says that M is indecomposable but not exceptional. (Of cause, M is not a maximal Cohen-Macaulay module.)

4. Maximality of MCM generating exceptional sequence

In this section R is general, and not necessarily of finite Cohen-Macaulay representation type. As a final result of this paper, we shall prove that the maximality of MCM generating exceptional sequences.

Theorem 4.1. *Any MCM generating exceptional sequence is maximal.*

Proof. Let $\mathcal{E} = (\cdots, \mathrm{E}^{\cdot}_{i-1}, \mathrm{E}^{\cdot}_{i}, \mathrm{E}^{\cdot}_{i+1}, \cdots)$ be an MCM generating exceptional sequenceand assume that there exist an exceptional complex $\mathrm{E}^{\cdot}$ and integer ℓ such that $(\cdots, \mathrm{E}^{\cdot}_{\ell-1}, \mathrm{E}^{\cdot}_{\ell}, \mathrm{E}^{\cdot}, \mathrm{E}^{\cdot}_{\ell+1}, \mathrm{E}^{\cdot}_{\ell+2}, \cdots)$ is an exceptional sequence. Since $\mathcal{E}$ is MCM generating, there exist integers $i_1, i_2, \cdots, i_t$ such that $\mathrm{E}^{\cdot}$ belongs to the smallest triangulated full subcategory Δ that contains all $\mathrm{E}^{\cdot}_{i_j}$ $(j = 1, 2, \cdots, t)$. On the other hand, it follows that $(\mathrm{E}^{\cdot}_{i_1}, \mathrm{E}^{\cdot}_{i_2}, \cdots, \mathrm{E}^{\cdot}_{i_s}, \mathrm{E}^{\cdot}, \mathrm{E}^{\cdot}_{i_{s+1}}, \cdots, \mathrm{E}^{\cdot}_{i_t})$ is also an exceptional sequence for some s, and therefore $\mathcal{E}' = (\mathrm{E}^{\cdot}_{i_1}, \mathrm{E}^{\cdot}_{i_2}, \cdots, \mathrm{E}^{\cdot}_{i_t}, \mathrm{F}^{\cdot})$ is an exceptional sequence where $F = \mathfrak{R}_{\mathrm{E}^{\cdot}_{i_t}} \mathfrak{R}_{\mathrm{E}^{\cdot}_{i_{t-1}}} \cdots \mathfrak{R}_{\mathrm{E}^{\cdot}_{i_{s+1}}} \mathrm{E}^{\cdot}$. Since $\mathrm{F}^{\cdot} \in \Delta$, there are integers $r_1, r_2, \cdots, r_t \in \mathbf{Z}$ such that $[\mathrm{F}^{\cdot}] = \sum_{j=1}^{t} r_j [\mathrm{E}^{\cdot}_{i_j}]$ in the Grotherndieck group $K_0(\Delta)$ of Δ. In such a situation, we claim that $r_j = 0$ for any $j = 1, 2, \cdots, t$. We prove this claim by induction on j, and we assume $r_m = 0$ for $m < j$. We compute the Euler characteristic $\chi([\mathrm{F}^{\cdot}], [\mathrm{E}^{\cdot}{}_{i_j}]) = \sum(-1)^n \dim_k \mathrm{H}^n(\mathrm{RHom}(\mathrm{F}^{\cdot}, \mathrm{E}^{\cdot}{}_{i_j})) = 0$ $(j = 1, 2, \cdots, t)$, since $\mathcal{E}'$ is an exceptional sequence.

On the other hand, since $[\mathrm{F}^{\cdot}] = \sum_{m=1}^{t} r_j [\mathrm{E}^{\cdot}{}_{i_m}]$, and since $\chi([\mathrm{E}^{\cdot}{}_{i_j}], [\mathrm{E}^{\cdot}{}_{i_j}]) = 1$ and $\chi([\mathrm{E}^{\cdot}{}_{i_m}], [\mathrm{E}^{\cdot}{}_{i_j}]) = 0$ for $m > j$ because of exceptionality of $\mathcal{E}'$, we have

$$
\begin{aligned}
\chi([\mathrm{F}^{\cdot}], [\mathrm{E}^{\cdot}{}_{i_j}]) &= \chi(\sum_{m=1}^{t} r_m [\mathrm{E}^{\cdot}{}_{i_m}], [\mathrm{E}^{\cdot}{}_{i_j}]) \\
&= \sum_{m=1}^{t} r_m \chi([\mathrm{E}^{\cdot}{}_{i_m}], [\mathrm{E}^{\cdot}{}_{i_j}]) \\
&= \sum_{m=1}^{j-1} r_m \chi([\mathrm{E}^{\cdot}{}_{i_m}], [\mathrm{E}^{\cdot}{}_{i_j}]) + r_j \chi([\mathrm{E}^{\cdot}{}_{i_j}], [\mathrm{E}^{\cdot}{}_{i_j}]) \\
&\qquad\qquad + \sum_{m=j+1}^{t} \chi([\mathrm{E}^{\cdot}{}_{i_m}], [\mathrm{E}^{\cdot}{}_{i_j}]) \\
&= r_j.
\end{aligned}
$$

Therefore we show $r_j = 0$ $(j = 1, 2. \cdots, t)$, i.e. $[\mathrm{F}^{\cdot}] = 0$ in $K_0(\Delta)$. It follows that $\chi([\mathrm{F}^{\cdot}], [\mathrm{F}^{\cdot}]) = 0$. But it contradicts the fact that $\mathrm{RHom}(\mathrm{F}^{\cdot}, \mathrm{F}^{\cdot}) \cong k$. □

We can also show the following corollary, whose proof is the same as above.

Corollary 4.2. *Any generating exceptional sequence is maximal.*

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(*Received July 10, 2000*)