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A NOTE ON OSOFSKY-SMITH THEOREM*

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A famous result of B.Osofsky says that a ring R is semisimple artinian if and only if every cyclic left R-module is injective. The crucial point of her proof was to show that such a ring has finite uniform dimension. In [7], B.Osofsky and P.F.Smith proved more generally that a cyclic module M has finite uniform dimension if every cyclic subfactor of M is an extending module. Extending modules have been studied extensively in recent years and many generalizations have been considered by many authors (see, for examples, [1-4, 6, 8, 9]). Lopez-Permouth, Oshiro and Tariq Rizvi in [6] introduced the concepts of extending modules and (quasi-)continuous modules relative a given left R-module X. Let $\mathcal{S}$ be the class of all semisimple left R-modules and all singular left R-modules. We say a left R-module N is $\mathcal{S}$-extending if N is X-extending for any $X \in \mathcal{S}$. Every extending left R-module is $\mathcal{S}$-extending but the converse is not true. Exploiting the techniques of [7] we prove the following result: Let M be a cyclic left R-module. Assume that all cyclic subfactors of M are $\mathcal{S}$-extending. Then M satisfies ACC on direct summands. As a corollary we show that if cyclic left R-module M is extending and all cyclic subfactors of M are $\mathcal{S}$-extending, then M has finite uniform dimension.

Throughout this note we write $A \leq_e B$ $(A|B)$ to denote that A is an essential submodule (a direct summand) of B.

A left R-module M is called singular if, for every $m \in M$, the annihilator $l(m)$ of m is an essential left ideal of R.

Lemma 1 ([4, 4.6]). *The following are equivalent for a left R-module M.*

(1) *M is singular.*
(2) *$M \cong L/K$ for a left R-module L and $K \leq_e L$.*

Let M, X be left R-modules. Define the family

$$\mathcal{A}(X,M) = \{A \subseteq M | \exists Y \subseteq X, \exists f \in Hom(Y,M), f(Y) \leq_e A\}.$$

Consider the properties

$\mathcal{A}(X,M)$-(C_1): For all $A \in \mathcal{A}(X,M)$, $\exists A^*|M$, such that $A \leq_e A^*$.

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$\mathcal{A}(X, M)$-(C_2): For all $A \in \mathcal{A}(X, M)$, if $B|M$ is such that $A \cong B$, then $A|M$.

$\mathcal{A}(X, M)$-(C_3): For all $A \in \mathcal{A}(X, M)$ and $B|M$, if $A|M$ and $A \cap B = 0$ then $A \oplus B|M$.

According to [6], M is said to be X-extending, X-quasi-continuous or X-continuous, respectively, if M satisfies $\mathcal{A}(X, M)$-(C_1), $\mathcal{A}(X, M)$-(C_1) and $\mathcal{A}(X, M)$-(C_3), $\mathcal{A}(X, M)$-(C_1) and $\mathcal{A}(X, M)$-(C_2).

According to [8, 1, 2], a left R-module M is called a CESS-module if every complement with essential socle is a direct summand, equivalently, every submodule with essential socle is essential in a direct summand of M. Now the following result is clear.

Proposition 2. *A left R-module M is a CESS-module if and only if M is X-extending for any semisimple left R-module X.*

Definition 3. *Let $\mathcal{S}$ be the class of all semisimple left R-modules and all singular left R-modules. A left R-module M is called $\mathcal{S}$-extending if M is X-extending for any $X \in \mathcal{S}$.*

Note that every extending left R-module is clearly $\mathcal{S}$-extending. But the following example shows that the converse is not true.

Example 4. *Let M be a free $\mathbb{Z}$-module of infinite rank. Since M is non-singular and has no socle, M is clearly $\mathcal{S}$-extending. But M is not extending by [5, Theorem 5].*

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the classes of all semisimple left R-modules, of all singular left R-modules, respectively. Then $\mathcal{S}_1 \oplus \mathcal{S}_2$ is defined to be the class of left R-modules M such that $M = A \oplus B$ is a direct sum of $A \in \mathcal{S}_1$ and $B \in \mathcal{S}_2$.

Proposition 5. *A left R-module M is $\mathcal{S}$-extending if and only if it is X-extending for any $X \in \mathcal{S}_1 \oplus \mathcal{S}_2$.*

Proof. It follows from the fact that if $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ is an exact sequence then M is X-extending if and only if it is both X'-extending and X''-extending by [6, Proposition 2.7]. □

Proposition 6. *Let M be a cyclic left R-module. Assume that all cyclic subfactors of M are $\mathcal{S}$-extending. Then M satisfies ACC on direct summands.*

Proof. We prove this by adapting the proof of [7, Theorem 1 and 4, 7.12]. Suppose that M does not satisfy ACC on direct summands and that $A_1 \subset A_2 \subset A_3 \subset \ldots\ldots$ is an infinite ascending chain of direct summands $A_i (i \geq 1)$ of M. Then there exists a submodule B_1 of M such that $M = A_1 \oplus B_1$. Thus $A_2 = A_2 \cap (A_1 \oplus B_1) = A_1 \oplus (A_2 \cap B_1)$ so that $A_2 \cap B_1$ is a direct summand of B_1. Let B_2 be a submodule of B_1 such that $B_1 = (A_2 \cap B_1) \oplus B_2$. Then $M = A_2 \oplus B_2$. Repeating this argument we can produce an infinite descending chain

$$B_1 \supset B_2 \supset B_3 \supset \dots\dots$$

of direct summands B_i of M such that $M = A_i \oplus B_i$. For each $i \geq 1$, there exists a nonzero submodule C_{i+1} of M such that $B_i = B_{i+1} \oplus C_{i+1}$. Put $C_1 = A_1$. Then

$$M = C_1 \oplus C_2 \oplus \cdots \oplus C_n \oplus B_n$$

and $\oplus_{i=n+1}^{\infty} C_i \subset B_n$ for all $n \geq 1$. Clearly C_i is cyclic since M is cyclic, and so C_i contains a maximal submodule W_i. Put

$$P = M/(\oplus_{i=1}^{\infty} W_i), \qquad Q = (\oplus_{i=1}^{\infty} C_i)/(\oplus_{i=1}^{\infty} W_i).$$

Then clearly P is a cyclic subfactor of M and Q is a semisimple submodule of P. By the hypothesis, P is X-extending for any $X \in \mathcal{S}$. Particularly P is Q-extending. It is easy to see that $Q \in \mathcal{A}(Q, P)$, and so there exists a direct summand Q^* of P such that $Q \leq_e Q^*$.

Note that $Q = \oplus_{i=1}^{\infty} S_i$ is an infinite direct sum of simple left R-modules S_i $(i \geq 1)$. Let $\{1, 2, \dots\}$ be a disjoint union of countable sets $\{I_j | j = 1, 2, \dots\}$. Set $Q_j = \oplus_{i \in I_j} S_i$, $j = 1, 2, \dots$. Then Q_j is a non-finitely generated semisimple left R-module. Clearly Q^* is a cyclic subfactor of M. By the hypothesis, Q^* is X-extending for any $X \in \mathcal{S}$. Particularly Q^* is Q_j-extending. It is easy to see that $Q_j \in \mathcal{A}(Q_j, Q^*)$, and so there exists a direct summand Q_j^* of Q^* such that $Q_j \leq_e Q_j^*$. Clearly Q_j^* is finitely generated, and thus $Q_j \neq Q_j^*$.

Let $D_j = (Q_j^* + Q)/Q$. Since $Q_j^* \cap (\oplus_{k \neq j} Q_k) = 0$ and $Q_j \neq Q_j^*$, it is easy to see that $D_j \neq 0$. Also $Q_j \leq Q \cap Q_j^* \leq Q_j^*$, so $Q \cap Q_j^* \leq_e Q_j^*$. This implies that $D_j \simeq Q_j^*/(Q_j^* \cap Q)$ is singular by Lemma 1. Hence

$$D = \sum_{j=1}^{\infty} D_j = \oplus_{j=1}^{\infty} D_j$$

is a singular submodule of Q^*/Q. Since Q^*/Q is a cyclic subfactor of M, it follows that Q^*/Q is X-extending for any $X \in \mathcal{S}$. Particularly Q^*/Q is D-extending. It is easy to see that $D \in \mathcal{A}(D, Q^*/Q)$, and so there exists a direct summand D^* of Q^*/Q such that $D \leq_e D^*$.

Since D^* is a cyclic submodule of Q^*/Q, there exists a cyclic submodule H of Q^* such that $D^* = (H + Q)/Q$. It is easy to see that $Q_j^* \cap H \neq 0$. Thus $Q_j \cap H = (Q_j^* \cap H) \cap Q_j \neq 0$. Hence there exists a non-zero simple submodule V_j of $Q_j \cap H$. Let $V = \oplus_{j=1}^{\infty} V_j$. Then $V \leq H$. Since H is a cyclic subfactor of M, it follows that H is X-extending for any $X \in \mathcal{S}$.

Particularly H is V-extending. Clearly $V \in \mathcal{A}(V, H)$, and so there exists a direct summand V^* of H such that $V \leq_e V^*$. It is easy to see that $V \neq V^*$ since V^* is cyclic. If $(V^* + Q)/Q = 0$, then $V^* \leq Q$, and thus V^* is semisimple. Hence V is a direct summand of V^*. But $V \leq_e V^*$, it follows that $V = V^*$, a contradiction. Thus $(V^* + Q)/Q \neq 0$.

For any $n \geq 1$, we have $(V^* \cap \oplus_{j=1}^n Q_j^*) \cap Q = V^* \cap (\oplus_{j=1}^n Q_j^* \cap Q) = V^* \cap (\oplus_{j=1}^n Q_j) \geq (\oplus_{j=1}^\infty V_j) \cap (\oplus_{j=1}^n Q_j) = \oplus_{j=1}^n V_j$. Since $V^* \cap (\oplus_{j=1}^n Q_j)$ is semisimple, it follows that

$$(V^* \cap \oplus_{j=1}^n Q_j^*) \cap Q = \oplus_{j=1}^n V_j.$$

Clearly, $\oplus_{j=1}^n V_j$ is a finitely generated submodule of Q. Thus there exists a finitely generated submodule N of $\oplus_{i=1}^\infty C_i$ such that $(N + \oplus_{i=1}^\infty W_i) /(\oplus_{i=1}^\infty W_i) = \oplus_{j=1}^n V_j$. Suppose that $N \leq \oplus_{i=1}^m C_i$. It is easy to see that

$$L = (\oplus_{i=1}^m C_i + \oplus_{i=1}^\infty W_i)/(\oplus_{i=1}^\infty W_i)$$

is semisimple. Thus $\oplus_{j=1}^n V_j$ is a direct summand of L. It is easy to see that L is a direct summand of P. Thus $\oplus_{j=1}^n V_j$ is a direct summand of P. Let $P = (\oplus_{j=1}^n V_j) \oplus P_1$. By modularity, $V^* \cap (\oplus_{j=1}^n Q_j^*) = (V^* \cap (\oplus_{j=1}^n Q_j^*) \cap Q) \oplus (V^* \cap (\oplus_{j=1}^n Q_j^*) \cap P_1)$. But it is easy to see that $(V^* \cap (\oplus_{j=1}^n Q_j^*)) \cap Q \leq_e V^* \cap (\oplus_{j=1}^n Q_j^*)$. Thus $(V^* \cap (\oplus_{j=1}^n Q_j^*)) \cap Q = V^* \cap (\oplus_{j=1}^n Q_j^*)$, which implies that $V^* \cap (\oplus_{j=1}^n Q_j^*) \leq Q$. This holds for each $n \geq 1$, hence it follows that $V^* \cap (\oplus_{j=1}^\infty Q_j^*) \leq Q$. But $Q \leq \oplus_{j=1}^\infty Q_j^*$, it follows that

$$(\oplus_{j=1}^\infty (Q_j^* + Q)/Q) \cap ((V^* + Q)/Q) = 0.$$

Now it follows that $(V^* + Q)/Q = 0$, which is a contradiction, because $D \leq_e D^*$. This completes the proof of the proposition. □

Now we have the main result of this paper, which generalizes Osofsky-Smith theorem ([7, Theorem 1]).

Theorem 7. *Let M be a cyclic extending left R-module. Assume that all cyclic subfactors of M are $\mathcal{S}$-extending. Then M has finite uniform dimension.*

Proof. By Proposition 6, M is a finite direct sum of indecomposable submodules. Since every direct summand of an extending module is extending, the result follows by the fact that each indecomposable extending module is uniform. □

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