Math. J. Okayama Univ. **41** (1999), 1–14

TENSOR PRODUCTS AND QUOTIENT RINGS WHICH ARE FINITE COMMUTATIVE PRINCIPAL IDEAL RINGS

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ABSTRACT. We give structure theorems for tensor products $R \otimes S$, and quotient rings Q/I to be finite commutative principal ideal rings with identity, where Q is a polynomial ring and I is an ideal of Q generated by univariate polynomials. We also show when Q/I is a direct product of finite fields or Galois rings.

Finite commutative rings with identity are nice examples of Artinian rings, [5], and they have applications in combinatorics. A ring R is called a principal ideal ring (abbreviated PIR) if, for any ideal I of R, there exists $x \in I$ such that $I = Rx = xR$, [6]. We consider when a finite commutative ring with identity is a PIR. These PIRs are useful to define as error-correcting codes, [2], [3] and [10].

We give structure theorems for tensor products and quotient rings, and all rings considered are commutative with identity. Theorem 1.11 gives a necessary condition for a tensor product $R \otimes S$ to be a finite PIR, where R and S are not assumed to be PIRs. Let $Q = R[x_1, \ldots, x_n]$, where R is a finite principal ideal ring and I is an ideal of Q generated by univariate polynomials. Theorem 2.1 gives conditions for Q/I to be a finite principal ideal ring. Theorem 2.11 shows when Q/I is a direct product of finite fields or Galois rings.

This paper is a continuation of the results given in [3] and [4].

1. Tensor products of rings

The tensor product over $\mathbb{Z}$ is written as $\otimes$. For any ring R and prime p, the $p-$component of R is defined by

$$R_p = \{r \in R \mid p^k r = 0 \text{ for some positive integer } k\}.$$

1991 *Mathematics Subject Classification.* Primary: 13F10; Secondary: 13F20, 13M10, 16P10.

Key words and phrases. finite commutative rings, principal ideal rings, tensor products.

If the ideals of a ring form a chain, then it is called a *chain ring* (see [8, p.184]). By Lemma 1.3, every finite local PIR and every field is a chain ring. The radical of a finite ring R is the largest nilpotent ideal $\mathcal{N}(R)$.

Lemma 1.1 ([4, Lemma 3]). *A finite ring is a PIR if and only if its radical is a principal ideal.*

Let R be an arbitrary ring, p a prime, and let $f \in R[x]$. Denote by $\overline{f}$ the image of f in $R[x]/pR[x]$. We say that f is *squarefree (irreducible) modulo p* if $\overline{f}$ is squarefree (respectively, irreducible). A *Galois ring $GR(p^m, r)$* is a ring of the form $(\mathbb{Z}/(p^m))[x]/(f(x))$, where p is a prime, m an integer, and $f(x) \in \mathbb{Z}/(p^m)[x]$ is a monic polynomial of degree r which is irreducible modulo p. If $R = GR(p^m, r) = (\mathbb{Z}/(p^m))[y]/(g(y)) \neq 0$ is a Galois ring which is not a field, then $m > 1$, because $(\mathbb{Z}/(p))[y]/(g(y))$ is a field, given that $g(y)$ is irreducible modulo p.

The ring $GR(p^n, r)$ is well defined independently of the monic polynomial of degree r (see [12, §16]).

Notice that $GR(p^m, 1) \cong \mathbb{Z}/(p^m)$ and $GR(p, r) \cong GF(p^r)$, the finite field of order p^r. Lemma 1.2, first proved in [14], shows that a tensor product of Galois rings is a PIR.

Lemma 1.2 ([12, Theorem 16.8]). *Let p be a prime, k_1, k_2, r_1, r_2 positive integers, and let $k = \min\{k_1, k_2\}$, $d = \gcd(r_1, r_2)$, $m = \operatorname{lcm}(r_1, r_2)$. Then*

$$GR(p^{k_1}, r_1) \otimes GR(p^{k_2}, r_2) \cong \prod_1^d GR(p^k, m).$$

In particular,

$$GF(p^{r_1}) \otimes GF(p^{r_2}) \cong \prod_1^d GF(p^m).$$

Lemma 1.3 ([12, Theorem 17.5]). *Let R be a finite commutative ring which is not a field. Then the following conditions are equivalent:*

1. *R is a chain ring;*
2. *R is a local PIR;*
3. *there exist a prime p and integers m, r, n, s, t such that*

$$R \cong GR(p^m, r)[x]/(g(x), p^{m-1}x^t),$$

where n is the index of nilpotency of the radical of R, $t = n - (m - 1)s > 0$, $g(x) = x^s + ph(x)$, $\deg(h) < s$, and the constant term of $h(x)$ is a unit in $GR(p^m, r)$.

Let R be a chain ring as defined in Lemma 1.3(3). The characteristic of R is p^m and its residue field is $R/\mathcal{N}(R) \cong GF(p^r)$. The polynomial $g(x)$ is called an *Eisenstein polynomial.* Since $GR(p^m, r)/pGR(p^m, r) \cong GF(p^r)$, we get $R/pR \cong GF(p^r)[x]/(x^s)$. By Lemma 1.4, R is a Galois ring if and only if $s = 1$.

Lemma 1.4 ([12, Exercise 16.9])**.** *A chain ring of characteristic p^m is a Galois ring if and only if its radical is generated by p. A PIR of characteristic p^m is a direct product of Galois rings if and only if its radical is generated by p.*

Lemma 1.5 ([4, Lemma 9])**.** *If R is a Galois ring, and S is a chain ring, then $R \otimes S$ is a PIR.*

Lemma 1.6 ([4, Lemma 10])**.** *Let R and S be chain rings which are not Galois rings, and let* $\mathrm{char}\,(R) = p^m$, $\mathrm{char}\,(S) = p^n$, *for a prime p and positive integers m, n. Then $R \otimes S$ is not a PIR.*

Theorem 1.7 ([4, Theorem 1])**.** *A tensor product $R \otimes S$ of two finite commutative PIRs is a PIR if and only if, for each prime p, at least one of the rings R_p or S_p is a direct product of Galois rings.*

For rings R_p and S_p, which are p components , it is false that $R_p \otimes S_p \neq 0$ being a PIR implies that both R_p and S_p are PIRs. For example, let $R_p = \mathbb{Z}/(p)$ and $S_p = GR(p^m, r)[x]/(x^s)$ then by Lemma 1.2,

$$\begin{aligned} R_p \otimes S_p &= \mathbb{Z}/(p) \otimes (GR(p^m, r)[x]/(x^s)) \cong (\mathbb{Z}/(p) \otimes GR(p^m, r))[x]/(x^s) \\ &\cong GF(p^r)[x]/(x^s) \cong S_p/pS_p. \end{aligned}$$

By Lemma 1.3, S_p cannot be a PIR when $m \geq 2$ and $s \geq 2$, yet $R_p \otimes S_p \cong GF(p^r)[x]/(x^s)$ is a PIR since $GF(p^r)[x]$ is a PIR for all integers $r, s \geq 1$. This provides motivation to prove Lemma 1.9, which relies on Lemma 1.8.

Lemma 1.8 ([12, Theorem 17.1, p.337-338])**.** *Let R be a finite local ring satisfying* $\mathrm{char}\,(R) = p^m$ *for a prime p and positive integer m. If $\mathcal{N}(R)$ has a minimum of k generators then $R \cong GR(p^m, q)[x_1, \ldots, x_k]/J$ for some primary ideal J, $GR(p^m, q)$ is the largest Galois extension of $\mathbb{Z}/(p^m)$ in R, and $R/\mathcal{N}(R) \cong GF(p^q)$.*

Lemma 1.9. *Let R and S be finite local rings satisfying* $\mathrm{char}\,(R) = p^m$, $\mathrm{char}\,(S) = p^n$, *for a prime p and positive integers $m, n \geq 1$. If S/pS is not a PIR then $R \otimes S$ is not a PIR.*

Proof. If $\mathcal{N}(S)$ has a minimum of k generators then by Lemma 1.8, $S \cong \mathbb{Z}/(p^n)[x_1, \ldots, x_k]/J$ for some primary ideal J. Since S is not a PIR, $k \geq 2$. Let $R = \mathbb{Z}/(p^m)$ and consider the following sequence of homomorphic images, with $J' \cong J/pJ$. $(R \otimes S)/p(R \otimes S) \to (R/pR) \otimes (S/pS) =$

$\mathbb{Z}/(p) \otimes (\mathbb{Z}/(p)[x_1, \dots, x_k]/J') \cong \mathbb{Z}/(p)[x_1, \dots, x_k]/J' = S/pS$. Since a homomorphic image of a PIR is a PIR and S/pS is not a PIR, $\mathbb{Z}/(p^m) \otimes S$ is not a PIR. Now let $\mathcal{N}(R)$ have a minimum of l generators. By Lemma 1.8, $R \cong \mathbb{Z}/(p^m)[x_1, \dots, x_l]/I$ for some primary ideal I and $l \geq 1$. Let $R \to \mathbb{Z}/(p^m)$ be the canonical homomorphism. This induces the homomorphism $R \otimes S \to (\mathbb{Z}/(p^m)) \otimes S$. Since $(\mathbb{Z}/(p^m)) \otimes S$ is not a PIR, $R \otimes S$ is not a PIR. □

Lemma 1.10. *Let R and S be finite local rings which are not both PIRs, satisfying* $\operatorname{char}(R) = p^m$, $\operatorname{char}(S) = q^n$, *for primes p, q and positive integers m, n. If $R \otimes S$ is a PIR then* 1. *or* 2. *is satisfied.*

1. *$p \neq q$ or $R = 0$ or $S = 0$, in which case $R \otimes S = 0$;*
2. *$p = q$, $R \neq 0 \neq S$, R is a Galois ring and S/pS is a finite chain ring which is not a Galois ring, or R and S may be interchanged.*

Proof. Condition (2). Let $R \otimes S$ and R be PIRs and S be a ring which is not a PIR. By Lemma 1.9, S/pS is a PIR. By Lemma 1.3, $S/pS \cong GF(p^r)[x]/(x^s)$ for some integers $r, s \geq 1$. Assume that S/pS is a Galois ring. Then $S/pS \cong GF(p^r)$. Since S is a local ring, $(p) = \mathcal{N}(S)$ is a maximal ideal of S. However, by Lemma 1.4, S is a Galois ring, which is false since S is not a PIR. Therefore S/pS is a chain ring which is not a Galois ring.

Assume that R is not a Galois ring. It follows that both R and S/pS are chain rings which are not Galois rings. By Lemma 1.6, $R \otimes (S/pS)$ is a not PIR. Since $R \otimes (S/pS)$ is a homomorphic image of $R \otimes S$, $R \otimes S$ is not a PIR. Hence R is a Galois ring, so (2) is satisfied. □

The converse of Lemma 1.10 is false. For example, let $R = \mathbb{Z}/(p^m)$ and $S = GR(p^m, r)[x]/(x^s)$ where $s \geq 2$. Then $R \otimes S = \mathbb{Z}/(p^m) \otimes GR(p^m, r)[x]/(x^s) \cong (\mathbb{Z}/(p^m) \otimes GR(p^m, r))[x]/(x^s) \cong GR(p^m, r)[x]/(x^s) = S$ is not a PIR by Lemma 1.3, yet $S/pS \cong GF(p^r)[x]/(x^s)$ is a PIR which is not a Galois ring. Therefore as proved in Theorem 1.11, only the necessary condition of Theorem 1.7 is true when R and S are not both PIRs.

Theorem 1.11. *If a tensor product $R \otimes S$ of two finite commutative rings is a PIR, then, for each prime p, at least one of the rings R_p or S_p is a direct product of Galois rings.*

Proof. If R and S are both PIRs, then the theorem follows from Theorem 1.7. Assume that R and S are not both PIRs. Since $R \otimes S$ is a PIR, for each prime p, $R_p \otimes S_p$ is a PIR. Consider the case when R_p and S_p are local rings. If R_p and S_p are both PIRs, then by Lemmas 1.5 and 1.6, R_p or S_p must be a Galois ring. If R_p and S_p are not both PIRs, then by Lemma 1.10, R_p or S_p must be a Galois ring. Now consider the case when R_p and S_p decompose into direct products of local rings. Since tensor product distributes over direct products, if both decompositions contain rings which are not Galois rings, then $R_p \otimes S_p$ will contain a factor in its representation as a direct product, which is a tensor product of two rings, where neither ring is a Galois ring. Such a factor is not a PIR by Lemma 1.6. Thus at least one of the rings R_p or S_p is a direct product of Galois rings. $\square$

Theorem 1.11 could only provide a necessary condition for $R \otimes S$ to be a finite commutative PIR. We give necessary and sufficient conditions for this to be true in Lemmas 1.13 and 1.14 in the special case when $R \otimes S$ is a direct product of either Galois rings or finite fields. Lemma 1.12 is required for Lemmas 1.13, 1.14 and Corollary 2.8. Lemma 1.5 follows from Lemma 1.12.

Lemma 1.12. *Let R be a direct product of Galois rings and S be a PIR. Then $R \otimes S$ is a PIR. If $\mathcal{N}(S) = gS$ for some generator $g \in S$, then $\mathcal{N}(R \otimes S) = g(R \otimes S)$, the ideal generated by g in $R \otimes S$.*

Proof. Let $\mathrm{char}\,(R) = p^m$, $\mathrm{char}\,(S) = q^n$, for primes p, q and positive integers m, n. If $p \neq q$, then $R \otimes S = 0$ is a PIR.

Suppose that $p = q$. Let $(g) = g(R \otimes S)$. Since (g) is nilpotent, $(g) \subseteq \mathcal{N}(R \otimes S)$. If R is not a finite field, it follows from Lemma 7 of [4] that $p \in gS$, and so $p \in (g)$. Since S/gS and R/pR are direct products of finite fields, by Lemma 1.2, so is $(R \otimes S)/(g)$. Therefore $(g) = \mathcal{N}(R \otimes S)$. By Lemma 1.1, $R \otimes S$ is a PIR. $\square$

Lemma 1.13. *Let R and S be finite rings satisfying* $\mathrm{char}\,(R) = p^m$, $\mathrm{char}\,(S) = p^n$, *for a prime p and positive integers $m, n \geq 1$. The ring $R \otimes S$ is a direct product of Galois rings if and only if so too are R and S.*

Proof. The 'if' part. This is immediate by Lemma 1.2, since tensor product distributes over direct products.

The 'only if' part. Since $R \otimes S$ is a PIR, by Theorem 1.11, either R or S is a direct product of Galois rings. Assume that R is a direct product of Galois rings. If S/pS is not a PIR, then by Lemma 1.9, neither is $R \otimes S$, so S/pS must be a PIR.

Assume that S is a PIR. Since $R \otimes S$ is a direct product of Galois rings, by Lemma 1.4, $\mathcal{N}(R \otimes S) = p(R \otimes S)$ and $\mathcal{N}(R) = pR$. If $\mathcal{N}(S) = gS$ for some generator $g \in S$ then by Lemma 1.12, $\mathcal{N}(R \otimes S) = g(R \otimes S)$. By Lemma 1.4, $g = p$, so S must be a direct product of Galois rings.

Now assume that S is not a PIR. By Lemma 1.10, S/pS is a PIR such that, as a direct product of local rings, no factor of S/pS is a Galois ring. By Lemma 1.3, each factor of S/pS is of the form $GF(p^r)[x]/(x^{s_i})$

for some integers $r \geq 1, s_i \geq 2$. Since R/pR is a direct product of finite fields, $(R/pR) \otimes (S/pS)$ must contain a factor of the form $T = GF(p^t) \otimes (GF(p^r)[x]/(x^{s_1})) \cong GF(p^{lcm(t,r)})[x]/(x^{s_1})$ by Lemma 1.2.

The class of finite direct products of Galois rings is closed for homomorphic images by Lemma 1.4. The same is true for a finite direct product of finite fields such as $(R \otimes S)/p(R \otimes S)$. Therefore since $(R/pR) \otimes (S/pS)$ is a homomorphic image of $(R \otimes S)/p(R \otimes S)$, it must be a finite direct product of finite fields. Since T is not a direct product of finite fields this contradiction implies that S must be a PIR. Therefore S must be a direct product of Galois rings. $\square$

Lemma 1.14. *Let R and S be finite rings satisfying* $\operatorname{char}(R) = p^m$, $\operatorname{char}(S) = p^n$, *for a prime p and positive integers $m, n \geq 1$. The ring $R \otimes S$ is a direct product of finite fields if and only if so too are R and S.*

Proof. The 'if' part. This is immediate by Lemma 1.2, since tensor product distributes over direct products.

The 'only if' part. By Lemma 1.13, R and S are direct products of Galois rings. By Lemma 1.12, $\mathcal{N}(R \otimes S) = g(R \otimes S)$, and $\mathcal{N}(S) = gS$ for some generator $g \in S$. Since $R \otimes S$ is a direct product of finite fields, $\mathcal{N}(R \otimes S) = 0 = \mathcal{N}(S)$, so S is a direct product of finite fields. If R is a direct product of Galois rings which are not all finite fields, then so too must be $R \otimes S$, by Lemma 1.2. This contradiction implies that R is a direct product of finite fields. $\square$

We now give a more general version of Lemma 1.1 for a local ring.

Lemma 1.15. *If R is a local ring with maximal ideal $\mathfrak{m}$, which is not necessarily Noetherian but satisfies $\cap_n \mathfrak{m}^n = 0$, then the following conditions on R are equivalent:*

1. $\mathfrak{m}$ *is principal;*
2. *R is a PIR;*
3. *R is a chain ring, hence R is Noetherian.*

Proof. (3)$\Longrightarrow$(2) Let $\pi \in \mathfrak{m} \backslash \mathfrak{m}^2$. Since R is a chain ring, $\pi \notin \mathfrak{m}^e$ for $e > 1$. So $(\pi) \neq \mathfrak{m}^e$ for $e > 1$, and $(\pi) = \mathfrak{m}$. Now since all ideals are of the form $\mathfrak{m}^e = (\pi^e)$, R is a PIR.
(2)$\Longrightarrow$(1) is immediate.
(1)$\Longrightarrow$(3) This is similar to the proof of Theorem 31.5 in [13]. Let $\mathfrak{m} = (\pi)$. Then $\mathfrak{m}^e = (\pi^e)$ for all $e \geq 1$. Since $\cap_n \mathfrak{m}^n = 0$ and every ideal $\mathfrak{a}$ satisfies $\mathfrak{a} \subseteq \mathfrak{m}$, for some $e \geq 1$, $\mathfrak{a} \subseteq \mathfrak{m}^e$ and $\mathfrak{a} \not\subset \mathfrak{m}^{e+1}$. For ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ of a ring R, $\mathfrak{a} \subseteq \mathfrak{c} \iff \mathfrak{a} : \mathfrak{b} \subseteq \mathfrak{c} : \mathfrak{b}$, $\mathfrak{a} \not\subset (\pi^{e+1})$ implies $\mathfrak{a} : (\pi^e) \not\subset (\pi^{e+1}) : (\pi^e) = (\pi)$, hence $\mathfrak{a} : (\pi^e) = R$. Since $(\mathfrak{a} : \mathfrak{b}) = R$ implies $\mathfrak{b} \subseteq \mathfrak{a}$, we see $(\pi^e) \subseteq \mathfrak{a}$, and

hence $\mathfrak{a} = (\pi^e) = \mathfrak{m}^e$. As every ideal of R is a power of $\mathfrak{m}$, R is a chain ring. □

2. Quotient rings of polynomial rings

For a finite commutative ring R, $Q = R[x_1, \dots, x_n]$ is a polynomial ring over R. The following theorem describes rings of the form

$$R[x_1, \dots, x_n]/(f_1(x_1), \dots, f_n(x_n))$$

which are finite PIRs. This gives a generalization of the main result of [9]. Theorem 1.7 is used in the proof of Theorem 2.1. Ideals of the form $(f_1(x_1), \dots, f_n(x_n))$ are called *elementary ideals* (see [11, Definition 1.14]). Some definitions are needed before we can state these results.

When $\mathbb{F}$ is a field, and $f = g_1^{m_1} \cdots g_k^{m_k}$, where $f \in \mathbb{F}[x]$ and $g_1, \dots, g_k$ are irreducible polynomials over $\mathbb{F}$, by $\mathrm{SP}(f)$ we denote the squarefree part $g_1 \cdots g_k$ of f. We assume that $\mathrm{SP}(0) = 0$.

Let $R = GR(p^m, r) = (\mathbb{Z}/(p^m))[y]/(g(y)) \neq 0$ be a Galois ring which is not a field ($m \geq 2$). We say that a polynomial $f(x) \in R[x]$ is *basic* if all nonzero coefficients of $f(x)$ belong to the subset

$$\mathcal{B} = \{ay^b \mid \text{where } 0 < a < p \text{ and } 0 \leq b < r\}$$

of the Galois ring R, where r is the degree of $g(y)$. Clearly, for every $f \in R[x]$, there exist unique basic polynomials

$$f', f'' \in \mathcal{B}[x] \subseteq R[x] \text{ such that } f - f' - pf'' \in p^2R[x].$$

Recall the definition of $\overline{f}$ which follows Lemma 1.1. For any $f \in R[x]$, there exists a unique basic polynomial $\mathrm{SP}(f) \in R[x]$ such that $\overline{\mathrm{SP}(f)} = \mathrm{SP}(\overline{f})$. Therefore there exists a unique basic polynomial $\mathrm{UP}(f) \in R[x]$ such that $\overline{f} = \overline{\mathrm{SP}(f)\,\mathrm{UP}(f)}$ or, equivalently, $f' - \mathrm{SP}(f)\,\mathrm{UP}(f) \in pR[x]$. Since f' is basic, $(f')'' = 0$ for any f, and so $(f' - \mathrm{SP}(f)\,\mathrm{UP}(f))'' = -(\mathrm{SP}(f)\,\mathrm{UP}(f))''$. So we introduce the following notation

$$\widehat{f} = \overline{f'' + (f' - \mathrm{SP}(f)\,\mathrm{UP}(f))''} = \overline{f'' - (\mathrm{SP}(f)\,\mathrm{UP}(f))''}.$$

For any $f, g \in GR(p^n, r)[x]$, it is clear that $\overline{f} = \overline{g}$ if and only if $f' = g'$.

Let R be a finite commutative local ring. A polynomial $f(x) \in R[x]$ is *regular* if it is not a zero divisor. By [12, Theorem 13.6], if $f(x)$ is regular, then there exists a unit $u \in R$ and monic polynomial $e(x) \in R[x]$ such that $f = ue$. All our theorems hold for regular polynomials $f(x)$. However, for simplicity, we assume that these polynomials are monic.

A finite direct product of rings is a PIR if and only if all its components are PIRs (see [15, Theorem 33]). Every finite PIR is a direct product of chain rings (see [12, §6]). The main case of describing all polynomial rings

$$Q = R[x_1, \dots, x_n]/(f_1(x_1), \dots, f_n(x_n))$$

which are finite PIRs is the case where R is a finite chain ring. From [12, Theorem 13.2(c)] , Q is finite if and only if all the $f_i(x_i)$ are regular. Theorem 2.1 gives necessary and sufficient conditions for Q to be a PIR. The sufficient conditions were proved in [4, Theorem 2].

Theorem 2.1. *Let R be a finite commutative chain ring, and let $f_1, \dots, f_n$ be univariate monic polynomials over R. Then*

$$Q = R[x_1, \dots, x_n]/(f_1(x_1), \dots, f_n(x_n))$$

is a PIR and all rings $R_i = R[x_i]/(f_i(x_i))$ for $1 \le i \le n$ are PIRs, if and only if one of the following conditions is satisfied:

1. *R is a field and the number of polynomials f_i which are not squarefree does not exceed one;*
2. *R is a Galois ring of characteristic p^m, for a prime p and a positive integer $m \ge 2$, the number of polynomials $f_1, \dots, f_n$ which are not squarefree modulo p does not exceed one, and, if $f = f_i$ is not squarefree modulo p, then $\widehat{f}$ is coprime with $\overline{\mathrm{UP}(f)}$;*
3. *R is a chain ring, which is not a Galois ring, R has characteristic p^m for a prime p, $n = 1$, and f_1 is squarefree modulo p.*

Lemma 2.2 ([4, Lemma 11])**.** *Let R be a Galois ring of characteristic p^m, $f(x)$ a monic polynomial over R, and let $Q = R[x]/(f(x))$. Then Q is a direct product of Galois rings if and only if $f(x)$ is squarefree modulo p.*

Lemma 2.3. *Let $R = GR(p^m, r)$ be a Galois ring, where $m \ge 2$, let $f(x) \in R[x]$ be a monic polynomial which is not squarefree modulo p, and let $Q = R[x]/(f(x))$. Then Q is a PIR if and only if $\overline{\mathrm{UP}(f)}$ is coprime with $\widehat{f}$.*

Proof. When $\overline{f}$ is not squarefree, we get $\mathrm{UP}(f) \neq 0$ and $\mathrm{SP}(f) \neq 0$.

Suppose that $\widehat{f}$ is coprime with $\overline{\mathrm{UP}(f)}$. Denote by h a basic polynomial in $R[x]$ such that $\overline{h}$ is the product of all irreducible divisors of $\overline{f}$ which do not divide $\widehat{f}$. Let $g = \mathrm{SP}(f) + ph \in R[x]$. It is proved in [4, Lemma 12] that the radical $\mathcal{N}(Q)$ is equal to the ideal I generated in Q by g.

Conversely, suppose that the radical is a principal ideal generated by some polynomial $g \in R[x]$.

Since $(\overline{g}) = (\,\mathrm{SP}(\overline{f})) = \mathcal{N}(\mathbb{Z}/(q)[x]/(\overline{f}))$, we get $\overline{g} = \overline{t}\,\mathrm{SP}(\overline{f}) + \overline{e}\overline{f}$ for some $t = t' \in R$ and $e(x) \in R[x]$. There exists an integer $s = s' \in R$ such that $ts \equiv 1(\mathrm{mod}\ p)$. Since $\overline{s(g - ef)} = \overline{st\,\mathrm{SP}(f)} = \overline{\mathrm{SP}(f)} = \mathrm{SP}(\overline{f})$ and

$(\overline{g}) = (\overline{\mathrm{SP}(f)})$, g generates the same ideal as $s(g-ef)$ in $Q = R[x]/(f)$, so we can replace g by $s(g-ef)$. To simplify the notation, we assume that $\overline{g} = \overline{\mathrm{SP}(f)}$, and so $g' = \mathrm{SP}(f)$.

Given $p \in \mathcal{N}(Q)$, we get $p = vf + wg$ for some $v, w \in R[x]$. Since $(vf+wg)' = (v'f' + w'g')' = 0$, it follows that $\overline{v'f'} + \overline{w'g'} = 0$. Therefore $\overline{w'} = -\overline{v'}\,\overline{\mathrm{UP}(f)}$, whence $w' = -v'\,\mathrm{UP}(f) + pz$ for some $z = z' \in R[x]$.

Further, $p = (v'+pv'')(f'+pf'')+(w'+pw'')(g'+pg'')+p^2u$, for some $u \in R[x]$. Notice that $f' = (\mathrm{UP}(f)g')'$, as $\overline{f'} = \overline{f} = \overline{\mathrm{UP}(f)g'}$. Since $\mathrm{UP}(f)$ and g' are basic, $\mathrm{UP}(f)g' = (\mathrm{UP}(f)g')'+p(\mathrm{UP}(f)g')'' = f'+p(\mathrm{UP}(f)g')''$. It follows that $f' - \mathrm{UP}(f)g' = -p(\mathrm{UP}(f)g')''$. Therefore we get

$$\begin{aligned} p^{m-1} &= p^{m-2}[(v'+pv'')(f'+pf'') + (-v'\,\mathrm{UP}(f) + pz + pw'')(g'+pg'')] \\ &= p^{m-2}[v'(f' - \mathrm{UP}(f)g' + pf'') - v'\,\mathrm{UP}(f)pg'' + pv''f' + pg'(z+w'')] \\ &= p^{m-1}[v'(-(\mathrm{UP}(f)g')'' + f'') - \mathrm{UP}(f)v'g'' + v''(\mathrm{UP}(f)g')' + g'(z+w'')]. \end{aligned}$$

When $p^m = 0$, $p^{m-1}A = p^{m-1}B$ if and only if $\overline{A} = \overline{B}$ where $A, B \in R[x]$. Hence

$$\begin{aligned} \overline{1} &= \overline{v'}(\overline{-(\mathrm{UP}(f)g')'' + f''}) - \overline{\mathrm{UP}(f)}(\overline{v'g''}) + \overline{v''}(\overline{(\mathrm{UP}(f)g')'}) + \overline{g'}(\overline{z+w''}) \\ &= \overline{v'}\widehat{f} - \overline{\mathrm{UP}(f)}(\overline{v'g''}) + \overline{v''}\,\overline{\mathrm{UP}(f)g'} + \overline{g'}(\overline{z+w''}). \end{aligned}$$

Since all irreducible factors of $\overline{\mathrm{UP}(f)}$ divide $\overline{g'} = \overline{\mathrm{SP}(f)}$, they also divide the polynomial $\overline{\mathrm{UP}(f)}(\overline{v'g''}) + \overline{v''}\,\overline{\mathrm{UP}(f)g'} + \overline{g'}(\overline{z+w''})$. So we see that $\overline{\mathrm{UP}(f)}$ must be coprime with $\widehat{f}$. This completes the proof. $\square$

Example 2.4. *We demonstrate Lemma 2.3 in the case Q is a finite local ring. Let $R = GR(p^m, r)$. Then $R/(\mathcal{N}(R)) \cong GF(p^r)$. For $c \in GF(p^r)[x]$, define $c_b \in R[x]$ as the unique basic polynomial satisfying $\overline{c_b} = c$. Then c_b and c have the same coefficients identified under the canonical injective mapping of sets $\mathcal{B} \to GF(p^r)$. Notice that $\mathcal{B}$ is not the isomorphic copy of $GF(p^r)$ contained in R. For example, if $R = \mathbb{Z}/(3^2)$, then $\mathcal{B} = \{0,1,2\} \subset \{0,1,2,\ldots,8\} = R$, $R/(\mathcal{N}(R)) \cong GF(3) = \{0,1,2\}$, yet $F = \{0,3,6\}$ is the isomorphic copy of $GF(3)$ contained in R.*

Let $R = GR(p^m, r)$ and let $e \in R[x]$ be a monic irreducible polynomial ([12, p.254]). Let $f = e^n$ for some integer $n \geq 1$ and $Q = R[x]/(f)$. By [12, Theorem 13.7(b)] , $\overline{e} = c^\ell$ for some monic irreducible $c \in GF(p^r)[x]$ and an integer $\ell \geq 1$. Therefore $\overline{\mathrm{SP}(f)} = \mathrm{SP}(\overline{f}) = c$ and $\mathrm{SP}(f) = c_b$. Now as $c^{\ell n} = \overline{f} = \overline{\mathrm{SP}(f)\,\mathrm{UP}(f)} = c\,\overline{\mathrm{UP}(f)}$, $\overline{\mathrm{UP}(f)} = c^{\ell n - 1}$ and $\mathrm{UP}(f) = (c^{\ell n-1})_b$. Evidently $\widehat{f} = \overline{(e^n)'' - (c_b(c^{\ell n-1})_b)''}$. It follows from Lemma 2.6 that $\mathcal{N}(Q) = (p, c_b)$. Since $(\overline{f}) = (c^{\ell n}) \subseteq (d) \subset F_{p^r}[x]$, $Q/\mathcal{N}(Q) = (GR(p^m, r)[x]/(f))/(p, c_b) \cong GF(p^r)[x]/(c) \cong GF(p^{r\ degree(c)})$. Hence Q is a finite local ring. Therefore, by the Chinese Remainder Theorem for *ideals ([7, Exercise 2.6, p.80]), for an arbitrary monic polynomial f, the ring $R[x]/(f)$ is a finite local ring if and only if $f = e^n$ where e is a monic irreducible polynomial and $n \geq 1$. By [12, Theorem 13.6] , this is also true when f and hence e are regular but not monic. We see that, for such a local ring Q which is not a Galois ring, it is a PIR if and only if c does not divide $\widehat{f}$.*

Lemma 2.5 ([4, Lemma 13])**.** *Let R be a chain ring of characteristic p^m which is not a Galois ring, let $f(x)$ be a monic polynomial over R, and let $Q = R[x]/(f(x))$. Then Q is a PIR if and only if f is squarefree modulo p.*

Lemma 2.6 ([4, Lemma 4])**.** *Let F be a finite field, $P = F[x_1, \dots , x_n]$, and let I be the ideal generated by $f_1(x_1), \dots , f_n(x_n)$ in P. Then the radical of P/I is equal to the ideal generated by the squarefree parts of all polynomials $f_1, \dots , f_n$.*

Proof of Theorem 2.1. The ring Q is isomorphic to the tensor product of the rings $R_i = R[x_i]/(f_i(x_i))$, for $i = 1, \dots , n$. Since $\mathrm{char}\,(R) = p^m$ where $m = 1$, if R is a field, $R_i = (R_i)_p$ for $i = 1, \dots , n$ and $Q = Q_p$.

(1): Suppose that R is a field of characteristic p. Then all the R_i are PIRs. Theorem 1.7 tells us that Q is a PIR if and only if at least $n - 1$ of the rings R_i are direct products of Galois rings. By Lemma 2.2, this is equivalent to the fact that at most one of the polynomials $f_i(x_i)$ is not squarefree.

(2): Suppose that R is a Galois ring. By Lemma 2.3, all R_i are PIRs if and only if, for each polynomial $f_i(x_i)$ which is not squarefree modulo p, $\overline{\mathrm{UP}(f_i)}$ is coprime with $\widehat{f_i}$. Further, suppose that this condition is satisfied. As in case (1), we see that Q is a PIR if and only if at most one of the polynomials $f_i(x_i)$ is not squarefree modulo p.

(3): Suppose that R is a chain ring which is not a Galois ring. Since the class of finite direct products of Galois rings is closed for homomorphic images by Lemma 1.4, we see that each R_i is not a direct product of Galois rings. Theorem 1.7 shows that $n = 1$. By Lemma 2.5, Q is a PIR if and only if $f_1(x_1)$ is squarefree modulo p. □

Our Theorem 2.1 immediately gives the following Theorem 1 of [9] for finite rings.

Corollary 2.7 ([9, Theorem 1])**.** *Let F be a field of characteristic $p > 0$, $a_1, \dots , a_n$ nonnegative integers, $b_1, \dots , b_n$ positive integers, and let*

$$R = F[x_1, \dots , x_n]/(x_1^{a_1}(1 - x_1^{b_1}), \dots , x_n^{a_n}(1 - x_n^{b_n})).$$

Then R is a PIR if and only if one of the following conditions is satisfied:

1. *$a_1, \ldots, a_n \leq 1$ and p divides at most one number among $b_1, \ldots, b_n$;*
2. *exactly one of $a_1, \ldots, a_n$, say a_1, is greater than 1 and p does not divide each of $b_2, \ldots, b_n$.*

Proof. Consider the polynomial $f = x^a(1 - x^b)$. By [1, Lemma 2.85], a polynomial is squarefree if and only if it is coprime with its derivative. Since $\operatorname{char}(F) = p > 0$, then f is squarefree if and only if $a = 1$ and p does not divide b. Thus Theorem 2.1 completes the proof. $\square$

In our second Corollary to Theorem 2.1, we give an explicit generator g for the radical of Q when Q is a PIR.

Corollary 2.8. *Let $R = GR(p^m, r)$ be a Galois ring, where $m \geq 1$, let $f_1, \ldots, f_n$ be univariate monic polynomials over R with $f_1(x_1)$ not squarefree modulo p and let*

$$Q = R[x_1, \ldots, x_n]/(f_1(x_1), \ldots, f_n(x_n))$$

be a PIR. Let $\mathcal{N}(S_1) = gS_1$ where $S_1 = R[x_1]/(f_1(x_1))$. Then $\mathcal{N}(Q) = gQ$ where gQ is the ideal generated by $g = g(x_1)$ in Q.

Proof. By Theorem 2.1, Q is a PIR and $f_1(x_1)$ is not squarefree modulo p, so the rings $R_i = R[x_i]/(f_i)$ for $2 \leq i \leq n$ are Galois rings. By Lemma 1.12, $S_2 \cong R_2 \otimes S_1$ is a PIR and $\mathcal{N}(S_2) = gS_2$. Repeating this argument with $S_{i+1} \cong R_{i+1} \otimes S_i$ for $2 \leq i \leq n-1$, we get $\mathcal{N}(Q) = gQ$. $\square$

Let Q be the PIR defined in Corollary 2.8. Let R be a Galois ring which is not a finite field. From the proof of Lemma 2.3, using the ring $S_1 = R[x_1]/(f(x_1))$, one may choose $g = \mathrm{SP}(f(x_1)) + ph(x_1)$. Also, if $f_i(x_i)$ for $1 \leq i \leq n$ are squarefree modulo p, then either by Lemma 1.2 and Lemma 1.4, or by the same proof as Corollary 2.8, $\mathcal{N}(Q) = pQ$. If R is a finite field, then $g = \mathrm{sp}(f_1)$, the squarefree part of f_1, generates $\mathcal{N}(Q)$.

Theorem 2.1 provides conditions for the ring Q to be a PIR. Theorem 2.11 provides similar conditions for Q to be a special type of PIR. To prove it, Lemmas 1.13, 1.14 and the following two lemmas are required.

Lemma 2.9. *Let us assume that $S = R[x]/(f(x))$ is a direct product of Galois rings, where R is a chain ring and f is monic. Then R is a Galois ring and f is squarefree modulo p.*

Proof. By Lemmas 2.2 and 2.5, f is squarefree modulo p. Assume that R is not a Galois ring. By Lemma 1.3, $R \cong GR(p^m, r)[y]/(y^s + ph(y), p^{m-1}y^t)$ for suitable $h(y)$ and integers m, r, t where $s \geq 2$. It follows that $S/pS \cong GF(p^r)[x, y]/(\overline{f(x)}, y^s) \cong GF(p^r)[x]/(\overline{f(x)}) \otimes GF(p^r)[y]/(y^s)$. Since $s \geq 2$, $GF(p^r)[y]/(y^s)$ is a finite chain ring which is not a finite field, yet $GF(p^r)[x]/(\overline{f(x)})$ is a direct product of finite fields since $\overline{f(x)}$ is squarefree. Consider the following ring. For some integer $q \geq 2$, by Lemma 1.2, $GF(p^q) \otimes (GF(p^r)[y]/(y^s)) \cong \prod_1^d (GF(p^l)[y]/(y^s))$, where $d = gcd(q, r)$ and $l = lcm(q, r)$. Since this ring is not a direct product of finite fields, neither is $(GF(p^r)[x]/\ (\overline{f(x)})\ \otimes GF(p^r)[y]/(y^s)) = S/pS$. This is a contradiction, by Lemma 1.4, since S is a direct product of Galois rings. Therefore R must be a Galois ring. □

Lemma 2.10. *Let us assume that $S = R[x]/(f(x))$ is a direct product of finite fields, where R is a chain ring and f is monic. Then R is a finite field and f is squarefree.*

Proof. By Lemma 2.9, f is squarefree modulo p, and $R \cong GR(p^m, r)$ where $m, r \geq 1$ are integers. Assume that R is not a finite field ($m \geq 2$). Since f is squarefree modulo p, $S = R[x]/(f(x))$ is a direct product of Galois rings of characteristic $p^m > p$, which is a contradiction. Therefore $m = 1$. So R is a finite field and f is squarefree. □

Theorem 2.11. *Let R be a finite commutative chain ring satisfying $\operatorname{char}(R) = p^m$, and $Q = R[x_1, \dots, x_n]/(f_1(x_1), \dots, f_n(x_n))$ where $f_1 \dots, f_n$ are monic polynomials. Then*

1. *Q is a direct product of finite fields if and only if R is a finite field and all the f_i are squarefree;*
2. *Q is a direct product of Galois rings if and only if R is a Galois ring and all the f_i are squarefree modulo p.*

Proof. Define $R_i = R[x_i]/(f_i(x_i))$ for $i = 1, \dots, n$. Then $Q \cong \otimes_{i=1}^n R_i$. Since $R = R_p$, $Q = Q_p$, where R_p is the $p-$component of R.

(1) The 'if' part. If R is a finite field and f is squarefree, then by the chinese remainder theorem for ideals ([7, Exercise.2.6, p.80]), $R[x]/(f(x))$ is a direct product of finite fields. By Lemma 1.2, a tensor product of finite fields is a direct product of finite fields, so tensor product distributes over direct products. Then Q is a direct product of finite fields.

The 'only if' part. By Lemma 1.14, if $R_1 \otimes R_2$ is a direct product of finite fields, then so too are R_1 and R_2. By iterating this argument, if $Q \cong \otimes_{i=1}^n R_i$ is a direct product of finite fields, then so is each R_i. By Lemma 2.10, R is a finite field and all the f_i are squarefree.

(2) The 'if' part. If R is a Galois ring and f is squarefree modulo p, then by Lemma 2.2, $R[x]/(f(x))$ is a direct product of Galois rings. The proof is now identical to (1) replacing 'finite field' by 'Galois ring', 'squarefree' by 'squarefree modulo p' and using Lemmas 1.13 and 2.9. □

Finally, let us consider the case when the ideal $I \triangleleft R[x]$ contains several univariate polynomials $I = (f_1(x), \dots, f_r(x))$. Let R be a finite local ring.

We say that $g \in R[x]$ is *primary* if (g) is a primary ideal in $R[x]$ (see [12, p.254]). Lemma 2.12 follows from [12], Theorem 13.11.

Lemma 2.12. *Let R be a finite local ring. Let $f \in R[x]$ be a monic polynomial, then $f = \prod_{i=1}^{s} g_i$, where the g_i are monic primary coprime polynomials, for some integer $s \geq 1$. This factorization of f is unique up to associates. That is, if $f = \prod_{i=1}^{t} h_i$, then $s = t$ and after renumbering, $(g_i) = (h_i) \triangleleft R[x]$.*

For a finite local ring R , we may now define a *greatest common divisor* of two monic polynomials $f_1, f_2 \in R[x]$. For $j = 1, 2$, let $f_j = \prod_{i=1}^{s(j)} g_i^{(j)}$, where the $g_i^{(j)}$ are monic primary coprime polynomials. Define $gcd(f_1, f_2) = \prod_{i=1}^{s} g_i^{(j)}$, where $g_i^{(j)}$ divides both f_1 and f_2, for some integer $s \geq 1$. Then by Lemma 2.12, $gcd(f_1, f_2)$ is well-defined and is unique up to associates. Similarly $gcd(f_1, \ldots, f_r)$ is defined for $f_1, \ldots, f_r \in R[x]$. Then we see that $(gcd(f_1, \ldots, f_r)) = (f_1, \ldots, f_r)$. Therefore, the theorems in this paper which are stated for rings of the form $Q = R[x]/(f_1(x))$ hold for rings of the form $Q = R[x]/(f_1(x), \ldots, f_r(x))$, where the f_i are monic, or more generally, are regular polynomials.

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(*Received March 18, 1999*)
(*Revised January 24, 2000*)