Math. J. Okayama Univ. **41** (1999), 21–25

HIGHER DERIVATIVES AND FINITENESS IN RINGS

HOWARD E. BELL

Abstract. Let n be a positive integer, R a prime ring, U a nonzero right ideal, and d a derivation on R. Under appropriate additional hypotheses, we prove that if $d^n(U)$ is finite, then either R is finite or d is nilpotent. We also provide an extension to semiprime rings.

In [2] it is proved that if R is a prime ring and d is a derivation on R such that $d(R)$ is finite, then either R is finite or $d = 0$. This result invites an investigation of prime rings with derivation such that $d^n(U)$ is finite for some derivation d, some $n \geq 1$, and some ideal (or right ideal) U. If U is a nonzero ideal, or if U is a nonzero right ideal and R is suitably-restricted, we can show that either R is finite or d is nilpotent on R.

1. Preliminaries

Let R be a ring and S a nonempty subset of R, and let f be a mapping from R to R. We say that f is nilpotent on S if $f^n(S) = \{0\}$ for some positive integer n; more generally, we call f periodic on S if there exist distinct positive integers m, n such that $f^n(x) = f^m(x)$ for all $x \in S$. We denote the right annihilator of S by $A_r(S)$.

We begin by stating and proving a lemma from [1].

Lemma 1.1. *An infinite prime ring contains no nonzero finite right ideal.*

Proof. Let R be infinite and prime, and suppose H is a nonzero finite right ideal. Let $H \setminus \{0\} = \{x_1, x_2, ..., x_n\}$. For each $i = 1, 2, ..., n$, define $f_i : R \to H$ by $f_i(r) = x_i r$ for all $r \in R$. Then $f_i(R)$ is finite, hence $\ker f_i = A_r(x_i)$ is a right ideal of R having finite index in R. Thus $K = \bigcap_{i=1}^{n} \ker f_i$ is a right ideal of finite index, necessarily nonzero, such that $HK = \{0\}$. But this cannot happen in a prime ring. $\square$

It is well-known that if R is a ring of prime characteristic p and d is a derivation on R, then d^p is also a derivation. This observation is the key to the following lemma, which we shall use several times.

Supported by the Natural Sciences and Engineering Research Council of Canada, Grant No. 3961.

Lemma 1.2. *Let n be a fixed positive integer, and let $\mathcal{R}$ be a class of prime rings with the following property:*

(*) If $R \in \mathcal{R}$ admits a nonzero derivation d such that $d(U)$ is finite for some nonzero ideal (resp. right ideal) U, then R is finite.

Then for any $R \in \mathcal{R}$ and any derivation d such that $d^n(U)$ is finite for some nonzero ideal (resp. right ideal) U, either R is finite or d is nilpotent on R.

Proof. It will suffice to prove the right ideal version. Let $R \in \mathcal{R}$ and U a nonzero right ideal of R, and let d be a derivation on R such that $d^n(U)$ is finite. If $char R = 0$, then $d^n(U) = \{0\}$; and by a result of Chung and Luh [4], d is nilpotent on R. Thus, we assume that R has prime characteristic p. Let P be the smallest power of p which is at least n, and let $\delta = d^P$. Since δ is a derivation and $\delta(U)$ is finite, it follows from $(*)$ that either R is finite or $\delta = 0$; and the latter possibility implies that d is nilpotent on R. □

2. The case of U an ideal

If U is assumed to be an ideal, then we can show that $d^n(U)$ can be finite only in the obvious ways.

Theorem 2.1. *Let n be a fixed positive integer. Let R be a prime ring and d a derivation on R such that $d^n(U)$ is finite for some nonzero ideal U. Then either R is finite or d is nilpotent on R.*

Proof. Let R be any prime ring, U any nonzero ideal and d a derivation on R such that $d(U)$ is finite. Consider the map $\Phi : U \to d(U)$ given by $\Phi(x) = d(x)$ for all $x \in U$. Then $\ker \Phi = \{x \in U \mid d(x) = 0\}$ is a subring of U of finite index in U, so by a result of Lewin[5], $\ker \Phi$ contains an ideal H of U which has finite index in U. If $H = \{0\}$, then U is finite; and by Lemma 1.1, R is finite. Suppose, then, that $H \neq \{0\}$. For all $x \in U$ and $y \in H$, we have $0 = d(yx) = yd(x) + d(y)x = yd(x)$; and therefore $yUd(U) = \{0\}$. But for $y \in H \setminus \{0\}, yU$ is a nonzero right ideal of R, hence $A_r(yU) = \{0\}$. Thus $d(U) = \{0\}$, and it follows easily that $d = 0$. Our result now follows by Lemma 1.2. □

3. The case of U a right ideal

Most of the proof of Theorem 2.1 works if U is assumed to be only a right ideal; the hypothesis that U is a two-sided ideal is used only in showing that $y \in H \setminus \{0\}$ implies $yU \neq \{0\}$. Of course, if R is a domain, the same implication holds; hence, we have

Theorem 3.1. *Let R be a ring with no nonzero divisors of zero, and U a nonzero right ideal of R. If d is a derivation on R and $d^n(U)$ is finite for some positive integer n, then either R is finite or d is nilpotent on R.*

By combining Theorem 2.1 and a result in [3], we obtain

Theorem 3.2. *Let R be a prime ring and U a nonzero right ideal of R. If d is a nonzero derivation and there exists a positive integer n for which $d^n(U)$ is finite and central, then d is nilpotent on R.*

Proof. Assume d is not nilpotent. Then by the final result in [3], R is commutative and hence U is an ideal. By Theorem 2.1, R is finite, hence a finite commutative domain - i.e. a finite field. But it is known that finite fields admit no nonzero derivations. □

Whether we can always replace U in Theorem 2.1 by a right ideal is an open question; however, we do have an affirmative answer for PI-rings.

Theorem 3.3. *Let R be a prime PI-ring, and let d be a derivation on R such that $d^n(U)$ is finite for some nonzero right ideal U and some positive integer n. Then either R is finite or d is nilpotent on R.*

Proof. In view of Lemma 1.2 and its proof, we may assume that $d(U)$ is finite and R has prime characteristic p. It is well known that a prime PI-ring has nonzero center Z; and if $z \in Z \setminus \{0\}$, then $d(z^p) = pz^{p-1}d(z) = 0$, so R has nonzero central constants.

Suppose that $d(U) \neq \{0\}$, and let $|d(U)| = k$. Then for any nonconstant $u \in U$ and nonzero central constant z, there exist distinct m, $n \in \{1, 2, ..., k+1\}$ such that $d(z^m u) = d(z^n u)$ - i.e. $(z^m - z^n)d(u) = 0$; and since Z has no elements which are zero divisors in R, we get $z^m = z^n$. It follows easily that there exist distinct integers M, N such that $z^M = z^N$ for all central constants z, hence Z satisfies the identity $z^{Mp} = z^{Np}$ and therefore Z is a finite field.

Since R is a prime PI-ring, its central localization R_Z is a primitive PI-ring [6, Theorem 6.1.30]. Moreover, since Z is a field, $R \cong R_Z$ and hence R is primitive. By a classical result of Kaplansky, R is therefore finite-dimensional over Z; hence R is finite. □

In the proof of this theorem, the right ideal property of U is used only twice: in the proof of Lemma 1.2, to show that d nilpotent on U implies d nilpotent on R, and in the argument above to guarantee that $ZU \subseteq U$. Thus, our methods yield

Theorem 3.4. *Let R be a prime PI-ring and S an additive subgroup such that $ZS \subseteq S$. If R admits a derivation d such that $d^n(S)$ is finite, then either R is finite or d is nilpotent on S.*

4. A theorem on semiprime rings

We conclude the paper with a theorem which replaces "nilpotent" by "periodic", and which is available in the setting of semiprime rings.

Theorem 4.1. *Let R be a semiprime ring having no nonzero finite right ideals. If U is a nonzero right ideal of R and d is a derivation on R such that $d^n(U)$ is finite for some positive integer n, then U contains a nonzero right ideal U_1 of R such that d is periodic on U_1.*

The proof uses a rather general lemma.

Lemma 4.2. *Let R be an arbitrary ring and S a nonempty subset of R. If $f : R \to R$ is a mapping such that $f(S) \subseteq S$ and $f^n(S)$ is finite for some positive integer n, then f is periodic on S.*

Proof. Since $f(S) \subseteq S$, for each positive integer k we have $f^{k+1}(S) = f^k(f(S)) \subseteq f^k(S)$. Thus, if $f^n(S)$ is finite, the chain $f^n(S) \supseteq f^{n+1}(S) \supseteq f^{n+2}(S) \supseteq \ldots$ must become stationary at some point, say at $f^N(S) = \{x_1, x_2, \ldots, x_m\}$. Then for each $u \geq 1$, the ordered m-tuple $(f^u(x_1), f^u(x_2), ..., f^u(x_m))$ is a permutation of $(x_1, x_2, ..., x_m)$. Therefore there exist distinct u, $v \geq 1$ such that $f^u(x_i) = f^v(x_i)$ for all $i = 1, 2, ..., m$. Now for each $x \in S$, $f^N(x) = x_i$ for some $i = 1, 2, ..., m$; therefore $f^{N+u}(x) = f^{N+v}(x)$ for all $x \in S$. □

Proof of Theorem 4.1. Let U be a nonzero right ideal with $d^n(U)$ finite. Let T be the torsion ideal of R; and for each prime p, let T_p be the p-primary component of T. If $T = \{0\}$, then $d^n(U) = 0$, so clearly d is periodic on U. If $T \neq \{0\}$ and $U \cap T = \{0\}$, then $UT = \{0\}$; and it follows easily by semiprimeness that $TU = 0$ as well. It follows that $Ud^m(U) = \{0\} = d^m(U)U$ for all $m \geq n$. By applying d to these equations repeatedly, we see that $d^i(U)d^j(U) = \{0\}$ for all nonnegative i, j with $i \geq n$ or $j \geq n$. By Leibniz' formula, we obtain $d^{2n-1}(U^2) = \{0\}$, hence d is periodic on U^2.

The remaining case is that of $U \cap T \neq \{0\}$, in which case $U \cap T_p \neq \{0\}$ for some prime p. Now by semiprimeness of R, $pT_p = \{0\}$; thus, $V = U \cap T_p$ is a nonzero right ideal of R with $pV = \{0\}$. Moreover, $d^P(V)$ is finite, where P is the smallest power of p which is at least n.

It remains only to prove that if V is any nonzero right ideal with $pV = \{0\}$ and $d^{p^\alpha}(V)$ finite for some α, then d is periodic on some nonzero right ideal contained in V. We use induction on $\left|d^{p^\alpha}(V)\right|$. A crucial observation is that, by Leibniz' formula,

(1)
$$d^{p^\alpha}(xy) = d^{p^\alpha}(x)y + xd^{p^\alpha}(y) \text{ for } x, y \in R \text{ with at least one of } x, y \text{ in } V.$$

If $\left|d^{p^\alpha}(V)\right| = 1$, then $d^{p^\alpha}(V) = \{0\} = d^{p^{\alpha+1}}(V)$, so d is obviously periodic on V. Now assume the result holds for nonzero right ideals $\widehat{V}$ with $p\widehat{V} = \{0\}$ and $\left|d^{p^\alpha}(\widehat{V})\right| < k$, and let V be a nonzero right ideal with $pV = \{0\}$ and $\left|d^{p^\alpha}(V)\right| = k$. If V contains a nonzero right ideal I of R with $\left|d^{p^\alpha}(I)\right| < k$, the desired conclusion is immediate from the inductive hypothesis; hence we assume that for every nonzero right ideal I contained in V, $d^{p^\alpha}(I) = d^{p^\alpha}(V)$. Now since V is infinite and $d^{p^\alpha}(V)$ is finite, V contains a nonzero subset S such that $d^{p^\alpha}(S) = \{0\}$; and since R is semiprime, for $s \in S \setminus \{0\}$, sR is a nonzero right ideal contained in V. Therefore, by (1) we get $d^{p^\alpha}(V) = d^{p^\alpha}(sR) = sd^{p^\alpha}(R) \subseteq V$; hence d^{p^α} is periodic on V by Lemma 4.2. Thus, d is periodic on V. □

References

[1] H. E. Bell and A. A. Klein, On finiteness, commutativity, and periodicity in rings, Math J. Okayama Univ. 35 (1993), 181-188.

[2] H. E. Bell and A. A. Klein, Ideals contained in subrings, Houston J. Math. 24 (1998), 1-8.

[3] H. E. Bell, A. A. Klein, and J. Lucier, Nilpotent derivations and commutativity, Math. J. Okayama Univ. 40 (1998), 1-6.

[4] L. O. Chung and J. Luh, Nilpotency of derivatives on an ideal, Proc. Amer. Math. Soc. 90 (1984), 211-214.

[5] J. Lewin, Subrings of finite index in finitely-generated rings, J. Algebra 5 (1967), 84-88.

[6] L. H. Rowen, Ring Theory I and II, Academic Press 1988.

Howard E. Bell
Department of Mathematics
Brock University
St. Catharines, Ontario
Canada, L2S 3A1

(*Received November 26, 1999*)