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ON A 4-SPACE WITH CERTAIN GENERAL CONNECTION RELATED WITH A MINKOWSKI-TYPE METRIC ON R^4_+

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ABSTRACT. From a Minkowski-type metric on R^4_+ satisfying the Einstein condition, we derive a nonlinear partial differential equation. We obtained a solution for it under certain condition in the 4-dimensional case. Using this solution we shall make a model space on R^4 with certain general connection which admits an interesting exposition for geodesics.

1. INTRODUCTION

On $R^n_+ = R^{N-1} \times R_+$ with the canonical coordinates $(x_1, \dots, x_{n-1}, x_n)$, $x_n > 0$ for $n > 3$, we consider a Minkowski-type pseudo-Riemannian metric:

$$ds^2 = \frac{1}{(x_n)^2}\left(\frac{1}{Q}drdr + r^2 \sum_{\alpha,\beta=2}^{n-1} h_{\alpha\beta}du^\alpha du^\beta - Pdx_n dx_n\right), \tag{1.1}$$

where $r = (x_1^2 + x_2^2 + \cdots + x_{n-1}^2)^{1/2}$ and $\sum_{\alpha,\beta=2}^{n-1} h_{\alpha,\beta}du^\alpha du^\beta$ is the standard-metric of the unit sphere S^{n-2} : $r^2 = 1$ in R^{n-1}.
If this metric satisfies the Einstein condition :

$$R_{ij} = \frac{R}{n}g_{ij}.$$

where R_{ij},g_{ij} and R are the components of the Ricci tensor, the metric tensor and scalar curvature of ds^2, respectively, then under the rerstriction:

$$\frac{\partial Q}{\partial u_\alpha} = \frac{\partial P}{\partial u_\beta} = 0,$$

Q as function of $x = r/x_n$ and $t = x_n$, satisfies the partial differential equation:

$$\begin{aligned}(1.2)\quad & (2Q-\varphi)x^2\frac{\partial^2 Q}{\partial x^2} - (3Q-2\varphi)xt\frac{\partial^2 Q}{\partial x\partial t} + (Q-\varphi)t^2\frac{\partial^2 Q}{\partial t^2} \\ & + ((2n-4)Q - n\varphi)x\frac{\partial Q}{\partial x} - ((n-4)Q-(n-2)\varphi)t\frac{\partial Q}{\partial t} \\ & -\frac{1}{Q}(x\frac{\partial Q}{\partial x} - t\frac{\partial Q}{\partial t})\left(2(Q-\varphi)x\frac{\partial Q}{\partial x} - (Q-2\varphi)t\frac{\partial Q}{\partial t}\right) + 2(n-3)Q(1-Q) = 0\end{aligned}$$

and

$$P = \frac{x^2}{Q-\varphi}, \tag{1.3}$$

where φ is an auxiliary integral free function, and the converse holds by Theorem 1 in [10].
When $n = 4$, for the Minkowski manifold MI^4 with the metric :

$$ds^2 = \frac{1}{(x_4)^2}\left(\sum_{a=1}^{3} dx_a dx_a - dx_4 dx_4\right), \tag{1.4}$$

the above function $\varphi(x)$ becomes $1 - x^2$. For $n = 4$ and $\varphi = 1 - x^2$, (1.2) becomes

$$\begin{aligned}
&(2Q - 1 + x^2)x^2\frac{\partial^2 Q}{\partial x^2} - (3Q - 2 + 2x^2)xt\frac{\partial^2 Q}{\partial x \partial t} + (Q - 1 + x^2)t^2\frac{\partial^2 Q}{\partial t^2} \\
&+ 4(Q - 1 + x^2)x\frac{\partial Q}{\partial x} + 2(1 - x^2)t\frac{\partial Q}{\partial t} - \frac{1}{Q}(x\frac{\partial Q}{\partial x} - t\frac{\partial Q}{\partial t}) \times \\
&\left(2(Q - 1 + x^2)x\frac{\partial Q}{\partial x} - (Q - 2 + 2x^2)t\frac{\partial Q}{\partial t}\right) + 2Q(1 - Q) = 0.
\end{aligned} \tag{1.5}$$

By Theorem 1 and Theorem 2 in [11], we have two kinds of solutions of (1.5) as follows :

Type 1. $Q = 1 + ax^2t^2$, a =constant ;
Type 2. Q depends only on x.

For $Q = 1 + ax^2t^2$ and $\varphi = 1 - x^2$, we obtain easily

$$Q = 1 + ar^2, \quad P = \frac{1}{1 + at^2}. \tag{1.6}$$

When $a = 0$, the metric (1.1) becomes the one of MI^4. In this paper, we shall investigate the properties of geodesics of the space with these Q and P.

2. A Related 4-Space with the metric (1.1) with (1.6)

Using the canonical coordinates (x_1, x_2, x_3, x_4) of R^4, the metric (1.1) with Q, P by (1.6) can be written as

$$ds^2 = \sum_{i,j=1}^{4} g_{ij} dx_i dx_j \tag{2.1}$$

where

$$g_{bc} = \frac{1}{x_4 x_4}\left(\delta_{bc} - \frac{a x_b x_c}{1 + ar^2}\right), \quad g_{b4} = 0, \qquad b, c = 1, 2, 3,$$
$$g_{44} = -\frac{1}{x_4 x_4(1 + a x_4 x_4)},$$

from which $(g^{ij}) = (g_{ij})^{-1}$ is given by

$$g^{bc} = x_4 x_4 \left(\delta^{bc} + a x_b x_c\right), \quad g^{b4} = 0, \quad g^{44} = -x_4 x_4(1 + a x_4 x_4).$$

Since we have

$$\frac{\partial g_{bc}}{\partial x_d} = \frac{a}{x_4x_4(1+ar^2)}\left(-\delta_{bd}x_c - \delta_{dc}x_b + \frac{2ax_bx_cx_d}{1+ar^2}\right),$$
$$\frac{\partial g_{bc}}{\partial x_4} = -\frac{2}{x_4}g_{bc}, \quad \frac{\partial g_{44}}{\partial x_b} = 0, \quad \frac{\partial g_{44}}{\partial x_4} = \frac{2(1+2ax_4x_4)}{(x_4)^3(1+ax_4x_4)^2},$$

the Christoffel symbols computed by (2.1) are given by

$$\left\{{}^{i}_{jh}\right\} = \frac{1}{2}\sum_k g^{ik}\left(\frac{\partial g_{jk}}{\partial x_h} + \frac{\partial g_{kh}}{\partial x_j} - \frac{\partial g_{jh}}{\partial x_k}\right)$$

as

$$\left\{{}^{e}_{bc}\right\} = \frac{1}{2}x_4x_4\left(\delta^{ed} + ax_ex_d\right)\frac{2a}{x_4x_4(1+ar^2)}\left(-\delta_{bc}x_d + \frac{ax_bx_cx_d}{1+ar^2}\right)$$
$$= -ax_e\left(\delta_{bc} - \frac{ax_bx_c}{1+ar^2}\right), \quad \left\{{}^{4}_{bc}\right\} = -\frac{1+ax_4x_4}{x_4}\left(\delta_{bc} - \frac{ax_bx_c}{1+ar^2}\right),$$

(2.2)

$$\left\{{}^{e}_{b4}\right\} = -\frac{1}{x_4}\delta^e_b, \quad \left\{{}^{4}_{b4}\right\} = 0, \quad \left\{{}^{e}_{44}\right\} = 0,$$
$$\left\{{}^{4}_{44}\right\} = -\frac{1}{2}x_4x_4(1+ax_4x_4)\frac{2(1+2ax_4x_4)}{(x_4)^3(1+ax_4x_4)^2} = -\frac{1}{x_4}\frac{1+2ax_4x_4}{1+ax_4x_4},$$

where we used the Einstein convention for summation. The above Christoffel symbols are the components of the Levi-Civita affine connection made by the pseudo-Riemmanian metric (2.1).

Now we consider the affine connection Γ_a projective to the Levi-Civita connection with the components :

(2.3) $$\Gamma^i_{jh} = \left\{{}^{i}_{jh}\right\} + \delta^i_j p_h + \delta^i_h p_j, \quad p_j = \frac{1}{x_4}\delta^4_j,$$

which is given in the canonical coordinates (x_i) by

(2.4)

$$\Gamma^e_{bc} = -ax_e\left(\delta_{bc} - \frac{ax_bx_c}{1+ar^2}\right), \quad \Gamma^4_{bc} = -\frac{1+ax_4x_4}{x_4}\left(\delta_{bc} - \frac{ax_bx_c}{1+ar^2}\right),$$
$$\Gamma^e_{b4} = 0, \quad \Gamma^4_{b4} = 0, \quad \Gamma^e_{44} = 0, \quad \Gamma^4_{44} = \frac{1}{x_4(1+ax_4x_4)}.$$

Now, looking over the expression of (2.4), we consider a general connection $\overline{\Gamma_a}$ on R^4 with components $\left(P^i_j, \overline{\Gamma^i_{jh}}\right)$ by

(2.5) $$P^i_j = x_4\delta^i_j, \quad \overline{\Gamma^i_{jh}} = x_4\Gamma^i_{jh},$$

which is smooth on the part of R^4 where $1+ar^2 \neq 0$ and $1+ax_4x_4 \neq 0$. The concept of general connection was introduced by the present author in

1960 (see [5] and [7]) and it is now called Otsuki connection. The equations of a geodesic of the space with the general connection $\overline{\Gamma_a}$ is given by

$$\sum_j P_j^i \frac{d^2x^j}{d\tau^2} + \sum_{j,h} \overline{\Gamma_{jh}^i} \frac{dx^j}{d\tau}\frac{dx^h}{d\tau} = 0, \tag{2.6}$$

where τ is the affine parameter with respect to $\overline{\Gamma_a}$. The geodesics of the space are the same of those of the spaces with Γ_a and the Levi-Civita connection of (2.1) on R_+^4 as the loci of moving points.

3. Properties of geodesics of the 4-space with $\overline{\Gamma_a}$

The equations of a geodesic with respect to $\overline{\Gamma_a}$ in the canonical coordinates (x_i) are by (2.4) :

$$\begin{aligned} &\frac{d^2x_e}{d\tau^2} - ax_e\left(\sum_b \frac{dx_b}{d\tau}\frac{dx_b}{d\tau} - \frac{a}{1+ar^2}\left(\sum_b x_b\frac{dx_b}{d\tau}\right)^2\right) = 0, \\ &\frac{d^2x_4}{d\tau^2} - \frac{1+ax_4x_4}{x_4}\left(\sum_b \frac{dx_b}{d\tau}\frac{dx_b}{d\tau} - \frac{a}{1+ar^2}\left(\sum_b x_b\frac{dx_b}{d\tau}\right)^2\right) \\ &\qquad + \frac{1}{x_4(1+ax_4x_4)}\left(\frac{dx_4}{d\tau}\right)^2 = 0, \end{aligned} \tag{3.1}$$

where τ is its affine parameter determined within affine transformation. We shall solve (3.1).

First, setting for simplicity

$$A = \sum_b \frac{dx_b}{d\tau}\frac{dx_b}{d\tau}, \quad B = \sum_b x_b\frac{dx_b}{d\tau}, \quad G = A - \frac{a}{1+ar^2}B^2,$$

we obtain by means of (3.1)

$$\begin{aligned} \frac{dA}{d\tau} &= 2\sum_b \frac{dx_b}{d\tau}\frac{d^2x_b}{d\tau^2} = 2\sum_b \frac{dx_b}{d\tau}(ax_bG) = 2aBG, \\ \frac{dB}{d\tau} &= \sum_b x_b\frac{d^2x_b}{d\tau^2} + \sum_b \frac{dx_b}{d\tau}\frac{dx_b}{d\tau} = \sum_b x_b(ax_bG) + A = ar^2G + A, \\ \frac{dr^2}{d\tau} &= 2\sum_b x_b\frac{dx_b}{d\tau} = 2B \end{aligned}$$

and hence

$$\begin{aligned}\frac{dG}{d\tau} &= \frac{dA}{d\tau} + \frac{a^2}{(1+ar^2)^2}\frac{dr^2}{d\tau}B^2 - \frac{a}{1+ar^2}2B\frac{dB}{d\tau}\\ &= 2aBG + \frac{a^2}{(1+ar^2)^2}2B^3 - \frac{2aB}{1+ar^2}(ar^2G+A)\\ &= 2aBG\left(1-\frac{ar^2}{1+ar^2}\right) + \frac{2a^2B^3}{(1+ar^2)^2} - \frac{2aAB}{1+ar^2}\\ &= \frac{2aBG}{1+ar^2} - \frac{2aB}{1+ar^2}\left(A - \frac{aB^2}{1+ar^2}\right) = \frac{2aBG}{1+ar^2} - \frac{2aBG}{1+ar^2} = 0,\end{aligned}$$

from which we obtain

$$G = \sum_b \frac{dx_b}{d\tau}\frac{dx_b}{d\tau} - \frac{a}{1+ar^2}\left(\sum_b x_b\frac{dx_b}{d\tau}\right)^2 = C, \tag{3.2}$$

where C is an integral constant.

Next substituting (3.2) into (3.1), we obtain

$$\frac{d^2x_b}{d\tau^2} - aCx_b = 0, \tag{3.3}$$

$$\frac{d^2x_4}{d\tau^2} + \frac{1}{x_4(1+ax_4x_4)}\left(\frac{dx_4}{d\tau}\right)^2 - C\frac{1+ax_4x_4}{x_4} = 0. \tag{3.4}$$

The solutions of (3.3) are given as follows.

Case 1: $aC > 0$

$$(x_b) = V_1\cosh(\tau\sqrt{aC}) + V_2\sinh(\tau\sqrt{aC}), \tag{3.5a}$$

Case 2: $aC < 0$

$$(x_b) = V_1\cos(\tau\sqrt{-aC}) + V_2\sin(\tau\sqrt{-aC}), \tag{3.5b}$$

Case 3: $aC = 0$

$$(x_b) = V_1 + \tau V_2, \tag{3.5c}$$

where V_1 and V_2 are two position vectors in R^3. Now, substituting the above results into (3.2), we obtain for each cases the following relations,

Case 1. Since we have

$$\begin{aligned}r^2 &= |V_1|^2\cosh^2(\tau\sqrt{aC}) + V_1\cdot V_2\sinh(2\tau\sqrt{aC}) + |V_2|^2\sinh^2(\tau\sqrt{aC})\\ &= \frac{1}{2}(|V_1|^2+|V_2|^2)\cosh(2\tau\sqrt{aC}) + V_1\cdot V_2\sinh(2\tau\sqrt{aC}) + \frac{1}{2}(|V_1|^2-|V_2|^2),\end{aligned}$$

$$\begin{aligned}&\sum_b \frac{dx_b}{d\tau}\frac{dx_b}{d\tau}\\ &= aC\left\{|V_1|^2\sinh^2(\tau\sqrt{aC}) + V_1\cdot V_2\sinh(2\tau\sqrt{aC}) + |V_2|^2\cosh^2(\tau\sqrt{aC})\right\}\\ &= aC\left\{\frac{1}{2}(|V_1|^2+|V_2|^2)\cosh(2\tau\sqrt{aC}) + V_1\cdot V_2\sinh(2\tau\sqrt{aC}) + \frac{1}{2}(-|V_1|^2+|V_2|^2)\right\},\end{aligned}$$

and

$$\sum_b x_b \frac{dx_b}{d\tau} = \sqrt{aC}\left(V_1 \cosh(\tau\sqrt{aC}) + V_2 \sinh(\tau\sqrt{aC})\right) \cdot \left(V_1 \sinh(\tau\sqrt{aC}) + V_2 \cosh(\tau\sqrt{aC})\right)$$
$$= \sqrt{aC}\left\{\frac{1}{2}(|V_1|^2 + |V_2|^2)\sinh(2\tau\sqrt{aC}) + V_1 \cdot V_2 \cosh(2\tau\sqrt{aC})\right\},$$

where $V_1 \cdot V_2$ denotes the inner product of V_1 and V_2 in R^3, (3.2) is equivalent to

$$\left[aC\left\{\frac{1}{2}(|V_1|^2+|V_2|^2)\cosh(2\tau\sqrt{aC})+V_1\cdot V_2\sinh(2\tau\sqrt{aC})+\frac{1}{2}(-|V_1|^2+|V_2|^2)\right\}-C\right]$$
$$\times\left[1+a\left\{\frac{1}{2}(|V_1|^2+|V_2|^2)\cosh(2\tau\sqrt{aC})+V_1\cdot V_2\sinh(2\tau\sqrt{aC})+\frac{1}{2}(|V_1|^2-|V_2|^2)\right\}\right]$$
$$-a^2C\left\{\frac{1}{2}(|V_1|^2+|V_2|^2)\sinh(2\tau\sqrt{aC})+V_1\cdot V_2\cosh(2\tau\sqrt{aC})\right\}^2 = 0,$$

which is reduced to the relation :

$$a^2(|V_1|^2|V_2|^2 - (V_1 \cdot V_2)^2) - a(|V_1|^2 - |V_2|^2) - 1 = 0 \tag{3.6}$$

by means of $C \neq 0$.

Case 2. Since we have

$$r^2 = |V_1|^2\cos^2(\tau\sqrt{-aC}) + V_1 \cdot V_2 \sin(2\tau\sqrt{-aC}) + |V_2|^2\sin^2(\tau\sqrt{-aC})$$
$$= \frac{1}{2}(|V_1|^2 - |V_2|^2)\cos(2\tau\sqrt{-aC}) + V_1 \cdot V_2 \sin(2\tau\sqrt{-aC}) + \frac{1}{2}(|V_1|^2 + |V_2|^2),$$

$$\sum_b \frac{dx_b}{d\tau}\frac{dx_b}{d\tau}$$
$$= -aC\left\{|V_1|^2\sin^2(\tau\sqrt{-aC}) - V_1 \cdot V_2 \sin(2\tau\sqrt{-aC}) + |V_2|^2\cos^2(\tau\sqrt{-aC})\right\}$$
$$= -aC\left\{\frac{1}{2}(-|V_1|^2 + |V_2|^2)\cos(2\tau\sqrt{-aC}) - V_1 \cdot V_2 \sin(2\tau\sqrt{-aC}) + \frac{1}{2}(|V_1|^2 + |V_2|^2)\right\}$$

and

$$\sum_b x_b \frac{dx_b}{d\tau} = \sqrt{-aC}\left(V_1 \cos(\tau\sqrt{-aC}) + V_2 \sin(\tau\sqrt{-aC})\right) \cdot \left(-V_1 \sin(\tau\sqrt{-aC}) + V_2 \cos(\tau\sqrt{-aC})\right)$$
$$= \sqrt{-aC}\left\{V_1 \cdot V_2 \cos(2\tau\sqrt{-aC}) + \frac{1}{2}(-|V_1|^2 + |V_2|^2)\sin(2\tau\sqrt{-aC})\right\},$$

(3.2) is equivalent to

(3.7)

$$\left[-aC\left\{\frac{1}{2}(-|V_1|^2+|V_2|^2)\cos(2\tau\sqrt{-aC})-V_1\cdot V_2\sin(2\tau\sqrt{-aC})+\frac{1}{2}(|V_1|^2+|V_2|^2)\right\}-C\right]$$
$$\times\left[1+a\left\{\frac{1}{2}(|V_1|^2-|V_2|^2)\cos(2\tau\sqrt{-aC})+V_1\cdot V_2\sin(2\tau\sqrt{-aC})+\frac{1}{2}(|V_1|^2+|V_2|^2)\right\}\right]$$
$$+a^2C\left\{V_1\cdot V_2\cos(2\tau\sqrt{-aC})+\frac{1}{2}(-|V_1|^2+|V_2|^2)\sin(2\tau\sqrt{-aC})\right\}^2=0,$$

which is reduced to the relation :

(3.8) $$a^2(|V_1|^2|V_2|^2-(V_1\cdot V_2)^2)+a(|V_1|^2+|V_2|^2)+1=0.$$

Case 3. Since we have

$$r^2=|V_1|^2+2\tau V_1\cdot V_2+\tau^2|V_2|^2,\quad \sum_b\frac{dx_b}{d\tau}\frac{dx_b}{d\tau}=|V_2|^2,$$

and

$$\sum_b x_b\frac{dx_b}{d\tau}=V_1\cdot V_2+\tau|V_2|^2,$$

(3.2) is equivalent to

$$\begin{aligned}(|V_2|^2-C)&\{1+a(|V_1|^2+2\tau V_1\cdot V_2+\tau^2|V_2|^2)\}-a(V_1\cdot V_2+\tau|V_2|^2)^2\\ =&(|V_2|^2-C)(1+a|V_1|^2)-a(V_1\cdot V_2)^2\\ &+2a\tau\{(|V_2|^2-C)V_1\cdot V_2-(V_1\cdot V_2)|V_2|^2\}+a\tau^2\{(|V_2|^2-C)|V_2|^2-|V_2|^4\}\\ =&|V_2|^2+a(|V_1|^2|V_2|^2-(V_1\cdot V_2)^2)-C\{1+a(|V_1|^2+2\tau V_1\cdot V_2+\tau^2|V_2|^2)\}\\ =&0,\end{aligned}$$

that is

(3.9)

$$|V_2|^2+a(|V_1|^2|V_2|^2-(V_1\cdot V_2)^2)=C\{1+a(|V_1|^2+2\tau V_1\cdot V_2+\tau^2|V_2|^2)\}.$$

Now, we shall solve the differential equation (3.4). First from (3.4) we obtain

$$\frac{d}{d\tau}\sqrt{1+ax_4x_4}=\frac{ax_4}{\sqrt{1+ax_4x_4}}\frac{dx_4}{d\tau},$$

and

$$\begin{aligned}\frac{d^2}{d\tau^2}\sqrt{1+ax_4x_4}=&\frac{ax_4}{\sqrt{1+ax_4x_4}}\frac{d^2x_4}{d\tau^2}+\left(\frac{a}{\sqrt{1+ax_4x_4}}-\frac{a^2x_4x_4}{(1+ax_4x_4)^{3/2}}\right)\left(\frac{dx_4}{d\tau}\right)^2\\ =&\frac{a}{\sqrt{1+ax_4x_4}}\left\{-\frac{1}{1+ax_4x_4}\left(\frac{dx_4}{d\tau}\right)^2+C(1+ax_4x_4)\right\}\\ &+\frac{a}{(1+ax_4x_4)\sqrt{1+ax_4x_4}}\left(\frac{dx_4}{d\tau}\right)^2\\ =&aC\sqrt{1+ax_4x_4},\end{aligned}$$

that is

$$\frac{d^2}{d\tau^2}\sqrt{1+ax_4x_4} = aC\sqrt{1+ax_4x_4}.$$

Therefore, integrating this differential equation we obtain as follows.

Case 1 : $aC > 0$.

$$\sqrt{1+ax_4x_4} = w_1\cosh(\tau\sqrt{aC}) + w_2\sinh(\tau\sqrt{aC}), \tag{3.10}$$

where w_1 and w_2 are integral constants, which implies

$$\begin{aligned} ax_4x_4 =& \frac{1}{2}(w_1^2+w_2^2)\cosh(2\tau\sqrt{aC}) + w_1w_2\sinh(2\tau\sqrt{aC}) \\ &+ \frac{1}{2}(w_1^2-w_2^2) - 1, \end{aligned} \tag{3.11}$$

which determine x_4 as long as its right hand is non-negative. Looking the expression of (3.11), we search for solutions such that

$$x_4 = \lambda_1\cosh(\tau\sqrt{aC}) + \lambda_2\sinh(\tau\sqrt{aC})$$

with constants λ_1, λ_2. Substituting this into (3.11), we obtain the relations

$$\lambda_1^2 = \frac{1}{a}(w_1^2-1), \quad \lambda_2^2 = \frac{1}{a}(w_2^2+1), \quad \lambda_1\lambda_2 = \frac{1}{a}w_1w_2, \tag{3.12}$$

which shows that this setting is admitted only for $a > 0$ and $C > 0$ and $w_1^2 - w_2^2 = 1$. Therefore we can put

$$w_1 = \cosh\beta, \quad w_2 = \sinh\beta$$

and

$$\begin{aligned} x_4 &= \pm\frac{1}{\sqrt{a}}\left\{\sinh\beta\cosh(\tau\sqrt{aC}) + \cosh\beta\sinh(\tau\sqrt{aC})\right\} \\ &= \pm\frac{1}{\sqrt{a}}\sinh(\beta+\tau\sqrt{aC}), \end{aligned} \tag{3.13}$$

where β is a constant.

Case 2 : $aC < 0$.

$$\sqrt{1+ax_4x_4} = w_1\cos(\tau\sqrt{-aC}) + w_2(\tau\sqrt{-aC}), \tag{3.14}$$

where w_1 and w_2 are integral constants, which implies

(3.15)

$$ax_4x_4 = \frac{1}{2}(w_1^2-w_2^2)\cos(2\tau\sqrt{-aC}) + w_1w_2\sin(\tau\sqrt{-aC}) + \frac{1}{2}(w_1^2+w_2^2) - 1,$$

which determines x_4 as long as (its right hand side) $/a$ is non negative. As in the case 1, we shall search for solutions such that

$$x_4 = \lambda_1\cos(\tau\sqrt{-aC}) + \lambda_2\sin(\tau\sqrt{-aC})$$

with constants λ_1, λ_2. Substituting this into (3.15), we obtain the relations

$$\lambda_1^2 = \frac{1}{a}(w_1^2-1), \quad \lambda_2^2 = \frac{1}{a}(w_2^2-1), \quad \lambda_1\lambda_2 = \frac{1}{a}w_1w_2, \tag{3.16}$$

and $w_1^2 + w_2^2 = 1$, which shows that this setting is admitted only for $a < 0$ and $C > 0$. Therefore we can put

$$w_1 = cos\beta, \quad w_2 = \sin\beta$$

and

$$\begin{aligned} x_4 &= \pm\frac{1}{\sqrt{-a}}\left\{\sin\beta\cos(\tau\sqrt{-aC}) - \cos\beta\sin(\tau\sqrt{-aC})\right\} \\ &= \pm\sin(-\beta + \tau\sqrt{-aC}), \end{aligned} \tag{3.17}$$

where β is a constant.

Case 3 : $aC = 0$.

$$\sqrt{1 + ax_4x_4} = w_1 + \tau w_2, \tag{3.18}$$

which determine x_4 as long as $w_1 + \tau w_2$ is non negative.

4. The range of r on geodesics

We shall investigate the range of r on geodesics of the 4-Space treated in Section 3 for each case.

Case 1 : $aC > 0$. In this case, we have

(4.1)

$$r^2 = \frac{1}{2}(|V_1|^2 + |V_2|^2)\cosh(2\tau\sqrt{aC}) + V_1 \cdot V_2 \sinh(2\tau\sqrt{aC}) + \frac{1}{2}(|V_1|^2 - |V_2|^2).$$

Let us define $\Delta_1 > 0$ by

$$\Delta_1^2 = (|V_1|^2 + |V_2|^2)^2 - 4(V_1 \cdot V_2)^2 = (|V_1|^2 - |V_2|^2)^2 + 4S^2,$$

where θ is the angle between V_1 and V_2 and $S = |V_1||V_2|\sin\theta$. We can take a real constant β_1 such that

$$\cosh\beta_1 = \frac{|V_1|^2 + |V_2|^2}{\Delta_1}, \quad \sinh\beta_1 = \frac{2V_1 \cdot V_2}{\Delta_1}.$$

Then we have

$$\begin{aligned} r^2 &= \frac{\Delta_1}{2}\left\{\cosh\beta_1\cosh(2\tau\sqrt{aC}) + \sinh\beta_1\sinh(2\tau\sqrt{aC})\right\} + \frac{1}{2}(|V_1|^2 - |V_2|^2) \\ &= \frac{\Delta_1}{2}\cosh(2\tau\sqrt{aC} + \beta_1) + \frac{1}{2}(|V_1|^2 - |V_2|^2) \\ &\geq \frac{1}{2}(\Delta_1 + |V_1|^2 - |V_2|^2) \end{aligned}$$

and the minimum is attained at $\tau = -\dfrac{\beta_1}{2\sqrt{aC}}$. On the other hand, we have by (3.6)

$$\begin{aligned} &a^2(|V_1|^2|V_2|^2 - (V_1 \cdot V_2)^2) - a(|V_1|^2 - |V_2|^2) - 1 \\ &= a^2S^2 - a(|V_1|^2 - |V_2|^2) - 1 = 0, \end{aligned}$$

from which we obtain

$$\Delta_1^2 = (|V_1|^2 - |V_2|^2)^2 + \frac{4}{a}(|V_1|^2 - |V_2|^2) + \frac{4}{a^2} = \left(|V_1|^2 - |V_2|^2 + \frac{2}{a}\right)^2,$$

and hence

$$\Delta_1 = \left||V_1|^2 - |V_2|^2 + \frac{2}{a}\right|. \tag{4.2}$$

Thus, we see that

$$r^2 \geq \frac{1}{2}\left(\left||V_1|^2 - |V_2|^2 + \frac{2}{a}\right| + |V_1|^2 - |V_2|^2\right). \tag{4.3}$$

Case 2 : $aC < 0$. In this case, we have

(4.4)

$$r^2 = \frac{1}{2}(|V_1|^2 - |V_2|^2)\cos(2\tau\sqrt{-aC}) + V_1 \cdot V_2 \sin(2\tau\sqrt{-aC}) + \frac{1}{2}(|V_1|^2 + |V_2|^2).$$

Let us define $\Delta_2 > 0$ by

$$\Delta_2^2 = (|V_1|^2 - |V_2|^2)^2 + 4(V_1 \cdot V_2)^2 = (|V_1|^2 + |V_2|^2)^2 - 4S^2.$$

Taking a real constant β_2 such that

$$\cos\beta_2 = \frac{|V_1|^2 - |V_2|^2}{\Delta_2}, \quad \sin\beta_2 = \frac{2V_1 \cdot V_2}{\Delta_2},$$

we have

$$\begin{aligned} r^2 &= \frac{\Delta_2}{2}\left\{\cos\beta_2\cos(2\tau\sqrt{-aC}) + \sin\beta_2\sin(2\tau\sqrt{-aC})\right\} + \frac{1}{2}(|V_1|^2 + |V_2|^2) \\ &= \frac{\Delta_2}{2}\cos(2\tau\sqrt{-aC} - \beta_2) + \frac{1}{2}(|V_1|^2 + |V_2|^2), \end{aligned}$$

from which we obtain

$$-\frac{\Delta_2}{2} + \frac{1}{2}(|V_1|^2 + |V_2|^2) \leq r^2 \leq \frac{\Delta_2}{2} + \frac{1}{2}(|V_1|^2 + |V_2|^2).$$

On the other hand, we have by (3.8)

$$\begin{aligned} &a^2(|V_1|^2|V_2|^2 - (V_1 \cdot V_2)^2) + a(|V_1|^2 + |V_2|^2) + 1 \\ &= a^2S^2 + a(|V_1|^2 + |V_2|^2) + 1 = 0, \end{aligned}$$

from which we obtain

$$\Delta_2^2 = (|V_1|^2 + |V_2|^2)^2 + \frac{4}{a}(|V_1|^2 + |V_2|^2) + \frac{4}{a^2} = \left(|V_1|^2 + |V_2|^2 + \frac{2}{a}\right)^2$$

and hence

$$\Delta_2 = \left||V_1|^2 + |V_2|^2 + \frac{2}{a}\right|. \tag{4.5}$$

Thus, we obtain the inequalities

$$(4.6)\quad \begin{aligned} -\frac{1}{2}\left||V_1|^2+|V_2|^2+\frac{2}{a}\right|+\frac{1}{2}(|V_1|^2+|V_2|^2)\le r^2 \le &\frac{1}{2}\left||V_1|^2+|V_2|^2+\frac{2}{a}\right| \\ &+\frac{1}{2}(|V_1|^2+|V_2|^2). \end{aligned}$$

Arranging these results, we obtain the following theorems.

Theorem 1. *For the 4-space with the general connection Γ_a ($a = 1/r_0^2$), the range of r^2 on a geodesics is given by the following inequality :*

$$0 \le |V_1|^2 - |V_2|^2 + r_0^2 \le r^2 < +\infty.$$

Proof. If $C > 0$, then by means of (4.3) we have

$$\frac{1}{2}\left(\left||V_1|^2 - |V_2|^2 + 2r_0^2\right| + |V_1|^2 - |V_2|^2\right) \le r^2 < +\infty.$$

If $|V_1|^2 - |V_2|^2 + 2r_0^2 \ge 0$, this inequality becomes

$$|V_1|^2 - |V_2|^2 + r_0^2 \le r^2 < +\infty$$

and since the left hand side is the minimum of r^2 we have

$$|V_1|^2 - |V_2|^2 + r_0^2 \ge 0.$$

If $|V_1|^2 - |V_2|^2 + 2r_0^2 < 0$, the above inequality becomes

$$-r_0^2 \le r^2 < +\infty,$$

which is impossible.

If $C < 0$, then by means of (4.6) we have

$$-r_0^2 \le r^2 \le |V_1|^2 + |V_2|^2 + r_0^2,$$

which is impossible since the minimum $-r_0^2 < 0$. □

Theorem 2. *For the 4-space with the general connection Γ_a ($a = -1/r_0^2$), the range of r^2 on a geodesics is given by the following inequalities :*

(i) *if $C > 0$,*

$$r_0^2 \le r^2 \le |V_1|^2 + |V_2|^2 - r_0^2$$

or

$$|V_1|^2 + |V_2|^2 - r_0^2 \le r^2 \le r_0^2,$$

(ii) *if $C < 0$,*

$$r_0^2 \le |V_1|^2 - |V_2|^2 - r_0^2 \le r^2 < +\infty$$

or

$$r_0^2 \le r^2 < +\infty.$$

Proof. If $C > 0$, then we have by means of (4.6)

$$-\frac{1}{2}\left||V_1|^2+|V_2|^2-2r_0^2\right|+\frac{1}{2}(|V_1|^2+|V_2|^2)\leq r^2\leq\frac{1}{2}\left||V_1|^2+|V_2|^2-2r_0^2\right| + \frac{1}{2}(|V_1|^2+|V_2|^2).$$

If $|V_1|^2+|V_2|^2-2r_0^2\geq 0$, this inequality becomes

$$r_0^2\leq r^2\leq |V_1|^2+|V_2|^2-r_0^2.$$

If $|V_1|^2+|V_2|^2-2r_0^2< 0$, this inequality becomes

$$|V_1|^2+|V_2|^2-r_0^2\leq r^2\leq r_0^2.$$

If $C < 0$, then we have by means of (4.3)

$$\frac{1}{2}(\left||V_1|^2-|V_2|^2-2r_0^2\right|+|V_1|^2-|V_2|^2)\leq r^2<+\infty.$$

If $|V_1|^2-|V_2|^2-2r_0^2\geq 0$, this inequality becomes

$$|V_1|^2-|V_2|^2-r_0^2\leq r^2<+\infty$$

and $r_0^2\leq |V_1|^2-|V_2|^2-r_0^2$. If $|V_1|^2-|V_2|^2-2r_0^2<0$, this inequality becomes

$$r_0^2\leq r^2<+\infty.$$

Arranging these arguments we obtain the claime of this theorem. □

Note . *For the moving points on geodesics in the 4-space of Theorem 2, the spherical cylinder $r = r_0$ in R^4 is an obstruction to pass through.*

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