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ON THE WEIGHTED ERGODIC PROPERTIES OF INVERTIBLE LAMPERTI OPERATORS

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ABSTRACT. In this paper we investigate the weighted ergodic properties of invertible Lamperti operators. Some results of Martín-Reyes, de la Torre and others in Málaga (Spain) are unified and generalized.

1. INTRODUCTION

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let $M(\mu)$ denote the space of all complex-valued measurable functions on X. Two functions f and g in $M(\mu)$ are not distinguished provided that $f(x) = g(x)$ for almost all $x \in X$. Hereafter all statements and relations will be assumed to hold modulo sets of measure zero. By a Lamperti operator T on $M(\mu)$ we mean an operator of the form

$$Tf(x) = h(x)\Phi f(x), \tag{1}$$

where $h \in M(\mu)$ is a fixed function and $\Phi : M(\mu) \longrightarrow M(\mu)$ is a linear and multiplicative operator. We recall that Φ is a multiplicative operator if Φ satisfies $\Phi(fg) = (\Phi f)(\Phi g)$ for all $f,\ g \in M(\mu)$.

In this paper we always assume T to be invertible on $M(\mu)$. Hence it follows that $0 < |h| < \infty$ a.e. on X and that Φ is invertible on $M(\mu)$. The following properties of T are known (cf. [11], [13]).

(I) If we put $h_1 = h$, $h_0 = 1$, $h_{-1} = 1/\Phi^{-1}h$, $h_n = h_1 \cdot \Phi h_{n-1}$ and $h_{-n} = h_{-1} \cdot \Phi^{-1}h_{-n+1}$ $(n \geq 2)$, then for each $j, k \in \mathbf{Z}$ we have

$$T^j f = h_j \cdot \Phi^j f \quad \text{and} \quad h_{j+k} = h_j \cdot \Phi^j h_k. \tag{2}$$

(II) By the Radon-Nikodym theorem, for each $j \in \mathbf{Z}$ there exists a positive measurable function J_j in $M(\mu)$ such that if $0 \leq f \in M(\mu)$ then

$$\int J_j \cdot \Phi^j f \, d\mu = \int f \, d\mu \quad \text{and} \quad J_{j+k} = J_j \cdot \Phi^j J_k \quad \text{for} \quad j,k \in \mathbf{Z}. \tag{3}$$

Let $\tau f = |h_1| \cdot \Phi f$ for $f \in M(\mu)$. Then τ is a positive invertible Lamperti operator, and for each $j \in \mathbf{Z}$ we have

$$\tau^j f = |h_j| \cdot \Phi^j f \quad \text{and} \quad |\tau^j f| = |T^j f| \quad \text{for} \quad f \in M(\mu), \tag{4}$$

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so that τ^j becomes the linear modulus of T^j.

We recall that if $T : L^p(\mu) \longrightarrow L^p(\mu)$, where $1 \le p \le \infty$, is a positive linear operator with positive inverse then T has the form (1) for $f \in L^p(\mu)$ (cf. [11]), and thus the operator has a unique extension to an invertible Lamperti operator on $M(\mu)$.

Let w be a nonnegative extended real-valued measurable function on X. Then, since the measure $wd\mu$ on $\mathcal{F}$ is absolutely continuous with respect to μ, $f = g$ a.e. (μ) implies that $f = g$ a.e. $(wd\mu)$. But the converse does not hold. Therefore, as it is easily seen, a Lamperti operator T on $M(\mu)$ is no longer an operator on $M(wd\mu)$ in general. And even though it is the case, the operator T is not necessarily invertible on $M(wd\mu)$. In the case where T is invertible on $M(wd\mu)$ and the measure $wd\mu$ is σ-finite, the study of weighted ergodic properties of T on $M(wd\mu)$ reduces to that of T on $M(\mu)$; and there are many papers investigating successfully invertible Lamperti operators T on $M(\mu)$. See e.g. [1], [2], [3], [5], [15], [17] and [23], etc. However it should seem that the study is not enough for the non-invertible case, although some papers have treated of not necessarily invertible Lamperti operators (see e.g. [11], [12]), and hence the author thinks that it would be interesting to investigate the weighted ergodic properties of T on $M(wd\mu)$, without assuming the invertibility of T on $M(wd\mu)$. This is the starting point of the paper. Here we remark that, by an easy observation, an invertible Lamperti operator T on $M(\mu)$ defined by (1) becomes an operator on $M(wd\mu)$ if and only if $\Phi\chi_A \le \chi_A$, where we let $A = \{x : w(x) = 0\}$ and χ_A denotes the characteristic function of A.

For an invertible Lamperti operator T on $M(\mu)$ we introduce two ergodic maximal operators $M^+(T)$ and $M(T)$ on $M(\mu)$ by the relations

$$M^+(T)f = \sup_{n \ge 0} |T_{0,n}f| \tag{5}$$

and

$$M(T)f = \sup_{m,n \ge 0} |T_{m,n}f|, \tag{6}$$

where we let

$$T_{m,n} = \frac{1}{m+n+1} \sum_{i=-m}^{n} T^i.$$

For simplicity τ will denote a *positive* invertible Lamperti operator on $M(\mu)$, unless the contrary is explained explicitely. In Section 2 we first characterize those τ for which the ergodic maximal operator $M^+(\tau)$ [or $M(\tau)$] is bounded in $L^p(wd\mu), 1 < p < \infty$. Among other things we will observe that $M^+(\tau)$ is bounded in $L^p(wd\mu)$ if and only if τ is an operator on $M(wd\mu)$ and satisfies

$$\sup_{n \ge 0} \|\tau_{0,n}\|_{L^p(wd\mu)} < \infty. \tag{7}$$

This generalizes Martín-Reyes and de la Torre's dominated ergodic theorem [17]; they considered the particular case where τ comes from a positive linear operator in $L^p(\mu), 1 < p < \infty$, with positive inverse and $w = 1$ on X. We then apply the results obtained to prove the a.e. convergence of the ergodic averages $(1/n)\sum_{i=0}^{n-1} T^i f$ and ergodic partial sums $\sum_{k=1}^{n}(T^k f - T^{-k} f)/k$.

In Section 3 we consider an invertible Lamperti operator T on $M(\mu)$ such that

$$K_\infty := \sup_{n\in\mathbf{Z}} \|T^n\|_{L^\infty(\mu)} < \infty. \tag{8}$$

Under the additional hypothesis that Φ has no periodic part (i.e. for any $n \geq 1$ and $E \in \mathcal{F}$ with $\mu E > 0$ there exists a non-null measurable subset A of E such that $\Phi^n \chi_A \neq \chi_A$), we prove that the ergodic maximal operator $M^+(T)$ is of weak type $(p, p), 1 \leq p < \infty$, with respect to the measure $wd\mu$ if and only if the linear modulus τ of T is an operator on $M(wd\mu)$ and satisfies norm condition (7). We also consider the ergodic maximal Hilbert transform $H^*(T)$ on $M(\mu)$ defined by the relation

$$H^*(T)f = \sup_{n\geq 1} \left|\sum_{k=1}^{n} \frac{T^k f - T^{-k} f}{k}\right|. \tag{9}$$

It will be proved that $H^*(T)$ is of weak type $(p, p), 1 \leq p < \infty$, with respect to the measure $wd\mu$ if and only if the linear modulus τ of T is an invertible operator on $M(wd\mu)$ and satisfies

$$\sup_{n\geq 0} \|\tau_{-n,n}\|_{L^p(wd\mu)} < \infty. \tag{10}$$

These generalize results of Atencia, Martín-Reyes and de la Torre (cf. [1], [2], [3]); they considered the case where w and T are such that $0 < w \in L^1(\mu)$ and T is of the form $Tf(x) = (f \circ \phi)(x) = f(\phi x)$, where ϕ is an ergodic invertible measure preserving transformation on a nonatomic probability measure space. Our proof is an adaptation of their arguments.

Lastly we unify the weighted inequalities obtained here and recent results of [4], [5], [15] to prove the a.e. convergence of the ergodic sequence $\{T^n f\}$ and the ergodic partial sums $\{\sum_{k=1}^{n}(T^k f - T^{-k} f)/k\}$ in the sense of Cesàro-α means.

Throughout the paper C will denote a positive constant not necessarily the same at each occurrence.

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2. Weighted strong type inequalities and applications

In this section we first consider a *positive* invertible Lamperti operator τ on $M(\mu)$. Let $\tau f = h_1 \cdot \Phi f$. Then (2) holds with τ instead of T, and we have $0 < h_j < \infty$ on X for each $j \in \mathbf{Z}$.

Theorem 1. *Let $0 \le w \le \infty$ on X and let $1 < p < \infty$. Then the following statements are equivalent for a positive invertible Lamperti operator τ on $M(\mu)$.*
(a) *τ is an operator on $M(wd\mu)$ and there exists a positive constant C such that for any $f \in L^p(wd\mu)$*

$$\int |M^+(\tau)f|^p w \, d\mu \le C \int |f|^p w \, d\mu. \tag{11}$$

(b) *τ is an operator on $M(wd\mu)$ and there exists a positive constant C such that for any $f \in L^p(wd\mu)$*

$$\sup_{n\ge 0} \int |\tau_{0,n} f|^p w \, d\mu \le C \int |f|^p w \, d\mu. \tag{12}$$

(c) *There exists a positive constant C such that for a.e. $x \in X$ and all $k \ge 0$*

$$\left(\sum_{i=0}^{k} h_{-i}(x)^{-p} J_{-i}(x) \Phi^{-i} w(x)\right) \cdot \left(\sum_{i=0}^{k} [h_i(x)^{-p} J_i(x) \Phi^i w(x)]^{\frac{-1}{p-1}}\right)^{p-1} \le C(k+1)^p. \tag{13}$$

Theorem 2. *Let $0 \le w \le \infty$ on X and let $1 < p < \infty$. Then the following statements are equivalent for a positive invertible Lamperti operator τ on $M(\mu)$.*
(a) *τ is an invertible operator on $M(wd\mu)$ and there exists a positive constant C such that for any $f \in L^p(wd\mu)$*

$$\int |M(\tau)f|^p w \, d\mu \le C \int |f|^p w \, d\mu. \tag{14}$$

(b) *τ is an invertible operator on $M(wd\mu)$ and there exists a positive constant C such that for any $f \in L^p(wd\mu)$*

$$\sup_{n\ge 0} \int |\tau_{-n,n} f|^p w \, d\mu \le C \int |f|^p w \, d\mu. \tag{15}$$

(c) *There exists a positive constant C such that for a.e. $x \in X$ and all $k \ge 0$*

$$\left(\sum_{i=0}^{k} h_i(x)^{-p} J_i(x) \Phi^i w(x)\right) \cdot \left(\sum_{i=0}^{k} [h_i(x)^{-p} J_i(x) \Phi^i w(x)]^{\frac{-1}{p-1}}\right)^{p-1} \le C(k+1)^p. \tag{16}$$

As in [16] and [17], to prove these theorems we need the following result about weights on the integers.

Lemma 1 (cf. [14], [18], [21]). *Let $0 \le w \le \infty$ on $\mathbf{Z}$. For a function f on $\mathbf{Z}$, define the functions f^* and f^{**} on $\mathbf{Z}$ by the relations*

$$f^*(i) = \sup_{n \geq 0} \left| \frac{1}{n+1} \sum_{j=0}^{n} f(i+j) \right|$$

and

$$f^{**}(i) = \sup_{m,n \geq 0} \left| \frac{1}{m+n+1} \sum_{j=-m}^{n} f(i+j) \right| .$$

Then we have:

(I) *When* $1 < p < \infty$, *there exists a positive constant* C *such that* $\sum_{i=-\infty}^{\infty} (f^*(i))^p w(i) \leq C \sum_{i=-\infty}^{\infty} |f(i)|^p w(i)$ *for all* f *if and only if there exists a positive constant* C *such that for all* $j \in \mathbf{Z}$ *and* $k \geq 0$

$$\left(\sum_{i=0}^{k} w(j-i) \right) \cdot \left(\sum_{i=0}^{k} w(j+i)^{\frac{-1}{p-1}} \right)^{p-1} \leq C(k+1)^p. \tag{17}$$

(II) *When* $1 < p < \infty$, *there exists a positive constant* C *such that* $\sum_{i=-\infty}^{\infty} (f^{**}(i))^p w(i) \leq C \sum_{i=-\infty}^{\infty} |f(i)|^p w(i)$ *for all* f *if and only if there exists a positive constant* C *such that for all* $j \in \mathbf{Z}$ *and* $k \geq 0$

$$\left(\sum_{i=0}^{k} w(j+i) \right) \cdot \left(\sum_{i=0}^{k} w(j+i)^{\frac{-1}{p-1}} \right)^{p-1} \leq C(k+1)^p. \tag{18}$$

(III) *There exists a positive constant* C *such that for all* f *and* $\lambda > 0$

$$\sum_{\{i: f^*(i) > \lambda\}} w(i) \leq C \frac{1}{\lambda} \sum_{i=-\infty}^{\infty} |f(i)| w(i)$$

if and only if there exists a positive constant C *such that for all* $j \in \mathbf{Z}$

$$\sup_{n \geq 0} \frac{1}{n+1} \sum_{i=0}^{n} w(j-i) \leq C w(j). \tag{19}$$

Proof of Theorem 1. (c) $\Rightarrow$ (a). Let $A = \{x : w(x) = 0\}$. We apply (13) with $k = 1$ to see that $\Phi^{-1}\chi_A \geq \chi_A$. Hence $\Phi\chi_A \leq \chi_A$, and thus τ becomes an operator on $M(w d\mu)$. Let $0 \leq f \in L^p(w d\mu)$. For an $N \geq 1$ we put

$$f_N^* = \max_{0 \leq n \leq N} \tau_{0,n} f .$$

Then for each $L \geq 1$ we have, by (3),

$$\int (f_N^*)^p w \, d\mu = \frac{1}{L+1} \int \sum_{i=0}^{L} (\tau^i f_N^*)^p (h_i^{-p} J_i \, \Phi^i w) \, d\mu,$$

where by (2), (3) and (c),

$$\begin{aligned}\tau^j f_N^* &= h_j \cdot \Phi^j f_N^* = h_j \cdot \max_{0\le n\le N} \frac{1}{n+1}\sum_{i=0}^{n} \Phi^j h_i \cdot \Phi^{j+i} f \\ &= \max_{0\le n\le N} \frac{1}{n+1}\sum_{i=0}^{n} \tau^{j+i} f\end{aligned}$$

and

$$\left(\sum_{i=0}^{k} h_{j-i}^{-p} J_{j-i}\, \Phi^{j-i} w\right) \cdot \left(\sum_{i=0}^{k} [h_{j+i}^{-p} J_{j+i}\, \Phi^{j+i} w]^{\frac{-1}{p-1}}\right)^{p-1} \le C(k+1)^p \quad \text{a.e.}$$

on X for all $j \in \mathbf{Z}$ and $k \ge 0$. Thus we apply Lemma A to obtain that

$$\begin{aligned}\int (f_N^*)^p w\, d\mu &\le \frac{C}{L+1}\int \sum_{i=0}^{L+N} (\tau^i f)^p\, (h_i^{-p} J_i\, \Phi^i w)\, d\mu \\ &= \frac{C}{L+1}\sum_{i=0}^{L+N} \int \Phi^i (f^p w) \cdot J_i\, d\mu \\ &= \frac{C}{L+1}(L+N+1)\int f^p w\, d\mu \qquad \text{(by (3))}.\end{aligned}$$

By letting $L \uparrow \infty$ and then $N \uparrow \infty$, it follows that

$$\int [M^+(\tau) f]^p w\, d\mu \le C \int f^p w\, d\mu.$$

(a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c). Let τ^* denote the invertible Lamperti operator on $M(\mu)$ defined by the relation

$$\tau^* f = \frac{J_{-1}}{h_{-1}}\, \Phi^{-1} f \qquad \text{for} \quad f \in M(\mu).$$

Using (2) and (3), we have

$$\tau^{*i} f = \frac{J_{-i}}{h_{-i}} \cdot \Phi^{-i} f \qquad \text{for} \quad i \in \mathbf{Z}, \tag{20}$$

and

$$\int (\tau^i f) g\, d\mu = \int f(\tau^{*i} g)\, d\mu \qquad \text{for} \quad 0 \le f, g \in M(\mu). \tag{21}$$

Let $1/p + 1/p' = 1$. If $0 \le f \in L^p(\mu)$ and $k \ge 0$ then by (b)

$$\begin{aligned}\int \left[w^{\frac{1}{p}} \cdot \tau_{0,2k}(f w^{\frac{-1}{p}})\right]^p d\mu &= \int w \cdot \left[\tau_{0,2k}(f w^{\frac{-1}{p}})\right]^p d\mu \\ &\le C\int (f^p w^{-1}) w\, d\mu \le C \int f^p\, d\mu,\end{aligned}$$

so that the mapping $f \mapsto w^{\frac{1}{p}} \cdot \tau_{0,2k}(fw^{\frac{-1}{p}})$ is a bounded linear operator from $L^p(\mu)$ into $L^p(\mu)$ with norm less than or equal to $C^{\frac{1}{p}}$; and from (21) it follows that its adjoint operator defined on $L^{p'}(\mu)$ is identical with the mapping $g \mapsto w^{\frac{-1}{p}} \cdot \tau^*_{0,2k}(gw^{\frac{1}{p}})$ for $g \in L^{p'}(\mu)$. Thus if $0 \le f \in L^p(\mu)$ then we have

$$\int \left(w^{\frac{-1}{(p-1)p}} \cdot \left[\tau^*_{0,2k}(f^{p-1}w^{\frac{1}{p}})\right]^{\frac{1}{p-1}}\right)^p d\mu$$

$$= \int \left(w^{\frac{-1}{p}} \cdot \tau^*_{0,2k}(f^{p-1}w^{\frac{1}{p}})\right)^{p'} d\mu \le C \int f^p \, d\mu.$$

Let us assume for the moment that $p \ge 2$. Since $p-1 \ge 1$, the operator $U : L^p(\mu) \longrightarrow L^p(\mu)$ defined by the relation

$$Uf = w^{\frac{1}{p}} \cdot \tau_{0,2k}(|f|w^{\frac{-1}{p}}) + w^{\frac{-1}{(p-1)p}} \cdot \left[\tau^*_{0,2k}(|f|^{p-1}w^{\frac{1}{p}})\right]^{\frac{1}{p-1}}$$

satisfies $U(f_1+f_2) \le Uf_1 + Uf_2$ for $f_1, f_2 \in L^p(\mu)$, and clearly we have

$$\|U\| \le 2C.$$

Then choose a function $g \in L^p(\mu)$ with $g > 0$ on X, and define a function G on X by the relation

$$G = \sum_{i=0}^{\infty} \frac{U^i g}{(3C)^i}.$$

It follows that $0 < G \in L^p(\mu)$ and that

$$UG \le \sum_{i=0}^{\infty} \frac{U^{i+1}g}{(3C)^i} < 3CG < \infty \quad \text{a.e.}$$

on X. Therefore we get

$$\tau_{0,2k}(G \cdot w^{\frac{-1}{p}}) \le (3CG) \cdot w^{\frac{-1}{p}} \quad \text{a.e.} \tag{22}$$

on X, and

$$\tau^*_{0,2k}(G^{p-1} \cdot w^{\frac{1}{p}}) \le (3CG)^{p-1} \cdot w^{\frac{1}{p}} \quad \text{a.e.} \tag{23}$$

on X. Consequently if we put

$$w_1 = G^{p-1} \cdot w^{\frac{1}{p}} \quad \text{and} \quad w_2 = G \cdot w^{\frac{-1}{p}},$$

then

$$w = \left(G^{p-1} \cdot w^{\frac{1}{p}}\right) \cdot \left(G \cdot w^{\frac{-1}{p}}\right)^{1-p} = w_1 \cdot w_2^{1-p},$$

and further by (23) and (22),

$$\tau^*_{0,2k} w_1 \le (3C)^{p-1} w_1 \quad \text{and} \quad \tau_{0,2k} w_2 \le 3C\, w_2 \quad \text{a.e.}$$

on X.

Next, let $1 < p < 2$. Since $p' > 2$ and $w^{-1/p} = \left(w^{\frac{-1}{p-1}}\right)^{1/p'}$, we can apply the above argument to p' and observe that there exist two functions w_1 and w_2 such that

$$w^{\frac{-1}{p-1}} = w_1 \cdot w_2^{1-p'}, \quad \tau_{0,2k} w_1 \le (3C)^{p'-1} w_1 \quad \text{and} \quad \tau^*_{0,2k} w_2 \le 3C\, w_2 .$$

Since $w = \left(w_1 \cdot w_2^{1-p'}\right)^{1-p} = w_2 \cdot w_1^{1-p}$, we conclude that, in any case, w has the representation

$$(24) \qquad w = w_1 \cdot w_2^{1-p} \quad \text{with} \quad \tau^*_{0,2k} w_1 \le C w_1 \quad \text{and} \quad \tau_{0,2k} w_2 \le C w_2,$$

where C is a positive constant independent of $k \ge 0$.

If $0 \le i \le k$ then we have

$$\sum_{s=0}^{k} \tau^s w_2 \le (2k+1)\tau^{-i}(\tau_{0,2k}\, w_2) \le 2C(k+1)\tau^{-i} w_2,$$

whence

$$(25) \qquad \sum_{i=0}^{k} \left[(\tau^{*i} w_1)\cdot(\tau^{-i} w_2)^{1-p}\right] \le \left(\frac{1}{2C}\cdot \tau_{0,k} w_2\right)^{1-p} \sum_{i=0}^{k} \tau^{*i} w_1 .$$

Similarly, since $\sum_{s=0}^{k} \tau^{*s} w_1 \le 2C(k+1)\tau^{*-i} w_1$ for $0 \le i \le k$, we get

$$(26) \qquad \sum_{i=0}^{k} \left[(\tau^{*-i} w)^{1-p'} \cdot \tau^i w_2\right] \le \left(\frac{1}{2C}\cdot \tau^*_{0,k} w_1\right)^{1-p'} \sum_{i=0}^{k} \tau^i w_2 .$$

Now we use the relations

$$\begin{aligned} (\tau^{*i} w_1)\cdot(\tau^{-i} w_2)^{1-p} &= \frac{J_{-i}}{h_{-i}} \cdot \Phi^{-i} w_1 \cdot (h_{-i}\, \Phi^{-i} w_2)^{1-p} \\ &= h_{-i}^{-p}\, J_{-i} \cdot \Phi^{-i}(w_1 w_2^{1-p}) = h_{-i}^{-p}\, J_{-i} \cdot \Phi^{-i} w \end{aligned}$$

and

$$\begin{aligned} (\tau^{*-i} w_1)^{1-p'} \cdot \tau^i w_2 &= h_i^{p'}\, J_i^{1-p'} \cdot \Phi^i(w_1^{1-p'} w_2) \\ &= (h_i^{-p}\, J_i \cdot \Phi^i w)^{1-p'} . \end{aligned}$$

By these together with (25) and (26) we have

$$\begin{aligned} &\left(\sum_{i=0}^{k} h_{-i}^{-p}\, J_{-i}\, \Phi^{-i} w\right) \cdot \left(\sum_{i=0}^{k} [h_i^{-p}\, J_i\, \Phi^i w]^{\frac{-1}{p-1}}\right)^{p-1} \\ &\le \left(\frac{1}{2C}\tau_{0,k}\, w_2\right)^{1-p} \left(\sum_{i=0}^{k} \tau^{*i} w_1\right) \cdot \left(\frac{1}{2C}\tau^*_{0,k}\, w_1\right)^{-1} \left(\sum_{i=0}^{k} \tau^i w_2\right)^{p-1} \\ &\le (2C)^p (k+1)^p \quad \text{a.e.} \end{aligned}$$

on X, which completes the proof. □

Proof of Theorem 2. This is similar to that of Theorem 1, and hence we omit the details. □

Remark 1. (i) Let $A = \{x : w(x) = 0\}$ and $B = \{x : w(x) = \infty\}$. Then each of statements (a), (b) and (c) of Theorem 1 implies that $\Phi\chi_A \le \chi_A$ and $\Phi\chi_B \ge \chi_B$. But in general we have $\Phi\chi_A \ne \chi_A$ and $\Phi\chi_B \ne \chi_B$. On the other hand, each of statements (a), (b) and (c) of Theorem 2 implies that $\Phi\chi_A = \chi_A$ and $\Phi\chi_B = \chi_B$. In this case we may assume without loss of generality that $X = \{x : 0 < w(x) < \infty\}$. Then it follows that $M(wd\mu) = M(\mu)$ and

$$\int \frac{\Phi^i w}{w} J_i \cdot (\Phi^i f)\, wd\mu = \int f\, wd\mu$$

for all $i \in \mathbf{Z}$ and $0 \le f \in M(\mu)$. By using this together with Theorem of [16], we could give another proof of Theorem 2.

(ii) For a function f on $\mathbf{Z}$ if we define the function $f^\natural$ on $\mathbf{Z}$ by

$$f^\natural(i) = \sup_{n\ge 0} \left| \frac{1}{n+1} \sum_{j=0}^{n} f(i-j) \right|,$$

then it follows clearly that

$$f^{**}(i) \le f^*(i) + f^\natural(i) \le 2 f^{**}(i) \qquad (i \in \mathbf{Z}).$$

Using these inequalities together with Lemma A, we could prove that Theorem 1 implies Theorem 2.

Theorem 3. *Let $0 \le w \le \infty$ on X and let $1 < p < \infty$. If τ is the linear modulus of an invertible Lamperti operator T on $M(\mu)$, then the following statements hold.*

(a) *If τ becomes an operator on $M(wd\mu)$ and satisfies $\sup_{n\ge 0} \|\tau_{0,n}\|_{L^p(wd\mu)} < \infty$, then for any $f \in L^p(wd\mu)$ the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} T^i f$$

exists a.e. on the set $\{x : w(x) > 0\}$.

(b) *If τ becomes an invertible operator on $M(wd\mu)$ and satisfies $\sup_{n\ge 0} \|\tau_{-n,n}\|_{L^p(wd\mu)} < \infty$, then for any $f \in L^p(wd\mu)$ the limit*

$$\lim_{n\to\infty} \sum_{k=1}^{n} (T^k f - T^{-k} f)/k$$

exists a.e. on the set $\{x : w(x) > 0\}$.

Proof. (a) By using Theorem 1 it follows from [7] that

$$\lim_{n\to\infty} \frac{1}{n} \tau^n |f| = \lim_{n\to\infty} \frac{1}{n} T^n f = 0 \qquad \text{a.e.}$$

on the set $\{x : w(x) > 0\}$ for any $f \in L^p(wd\mu)$. Since the set $\{g + (f - Tf) : Tg = g, f \in L^p(wd\mu)\}$ is a dense subspace of $L^p(wd\mu)$ by a mean ergodic theorem, we then apply Banach's convergence principle (see e.g. [8]) to infer that (a) holds.

(b) By Remark 1 (i), T and τ can be considered to be invertible Lamperti operators on $M(w d\mu) = M(\mu)$. Thus (b) is a consequence of [19].

The proof is complete. □

3. Weighted weak type inequalities and applications

In this section we assume that an invertible Lamperti operator T on $M(\mu)$ satisfies

$$K_\infty := \sup_{n \in \mathbf{Z}} \|T^n\|_{L^\infty(\mu)} < \infty. \tag{27}$$

Hence from (2) we observe that

$$\frac{1}{K_\infty} \leq |h_n| \leq K_\infty \quad \text{a.e.} \tag{28}$$

on X for each $n \in \mathbf{Z}$. For $f \in M(\mu)$ we let

$$M^+(\Phi)f = \sup_{n \geq 0} |\Phi_{0,n} f| \quad \text{and} \quad M(\Phi)f = \sup_{m,n \geq 0} |\Phi_{m,n} f|,$$

where

$$\Phi_{m,n} f = \frac{1}{m+n+1} \sum_{i=-m}^{n} \Phi^i f.$$

If τ denotes the linear modulus of T, then by (2), (4) and (28) we have

$$\frac{1}{K_\infty} \Phi_{m,n} \leq \tau_{m,n} \leq K_\infty \Phi_{m,n}\,, \tag{29}$$

so that

$$\begin{aligned} &\frac{1}{K_\infty} M^+(\Phi) \leq M^+(\tau) \leq K_\infty M^+(\Phi) \quad \text{and} \\ &\frac{1}{K_\infty} M(\Phi) \leq M(\tau) \leq K_\infty M(\Phi). \end{aligned} \tag{30}$$

Using these relations we first prove the following weighted weak type inequalities.

Theorem 4. *Let $0 \leq w \leq \infty$ on X and let $1 \leq p < \infty$. If T is an invertible Lamperti operator on $M(\mu)$ satisfying (27) and Φ has no periodic part, then the following statements are equivalent.*

(a) *T becomes an operator on $M(w d\mu)$ and there exists a positive constant C such that for any $f \in L^p(w d\mu)$ and $\lambda > 0$*

$$\int_{\{x : M^+(T)f(x) > \lambda\}} w\, d\mu \leq C \frac{1}{\lambda^p} \int |f|^p w\, d\mu. \tag{31}$$

(b) *The linear modulus τ of T becomes an operator on $M(w d\mu)$ and there exists a positive constant C such that for any $f \in L^p(w d\mu)$*

$$\sup_{n \geq 0} \int |\tau_{0,n} f|^p w\, d\mu \leq C \int |f|^p w\, d\mu.$$

Theorem 5. *Let $0 \le w \le \infty$ on X. If T is an invertible Lamperti operator on $M(\mu)$ satisfying (27) and Φ has no periodic part, then the following statements are equivalent when $1 < p < \infty$, and statements (a) and (b) are equivalent when $p = 1$.*

(a) *T becomes an invertible operator on $M(wd\mu)$ and there exists a positive constant C such that for any $f \in L^p(wd\mu)$ and $\lambda > 0$*

$$\int_{\{x : H^*(T)f(x)>\lambda\}} w\, d\mu \le C\frac{1}{\lambda^p}\int |f|^p w\, d\mu. \tag{32}$$

(b) *The linear modulus τ of T becomes an invertible operator on $M(wd\mu)$ and there exists a positive constant C such that for any $f \in L^p(wd\mu)$*

$$\sup_{n\ge 0}\int |\tau_{-n,n} f|^p w\, d\mu \le C\int |f|^p w\, d\mu.$$

(c) *T becomes an invertible operator on $M(wd\mu)$ and there exists a positive constant C such that for any $f \in L^p(wd\mu)$*

$$\int |H^*(T)f|^p w\, d\mu \le C\int |f|^p w\, d\mu.$$

Proof of Theorem 4. Let $1 < p < \infty$.

(b) $\Rightarrow$ (a). Since $|M^+(T)f| \le M^+(\tau)|f|$ for $f \in M(\mu)$, this implication is obvious from Theorem 1.

(a) $\Rightarrow$ (b). By (29) it suffices to prove that

$$\sup_{n\ge 0}\|\Phi_{0,n}\|_{L^p(wd\mu)} < \infty. \tag{33}$$

To do so, we apply Theorem 1. We see that it is enough to prove the existence of a positive constant C such that for a.e. $x \in X$ and all $k \ge 0$

$$\left(\sum_{i=0}^{k} J_{-i}(x)\Phi^{-i}w(x)\right)\cdot\left(\sum_{i=0}^{k}[J_i(x)\Phi^i w(x)]^{\frac{-1}{p-1}}\right)^{p-1} \le C(k+1)^p. \tag{34}$$

As in the proof of Lemma of [20], we may assume without loss of generality that there exists a one-to-one onto mapping S from X to X such that

(i) $A \in \mathcal{F}$ if and only if $SA \in \mathcal{F}$,

(ii) $\mu(SA) > 0$ if and only if $\mu A > 0$,

(iii) $\Phi^i f = f \circ S^i$ for all $i \in \mathbf{Z}$ and $f \in M(\mu)$.

For simplicity, from now on, we will always assume that the one-to-one onto mapping $S : X \longrightarrow X$ satisfies the above conditions (i), (ii) and (iii).

For an integer k with $k \ge 0$ we define a nonnegative extended real-valued function d_k on X by the relation

$$d_k(x) = \sum_{i=0}^{k}\left[J_i(x)w(S^i x)\right]^{\frac{-1}{p-1}}. \tag{35}$$

Write $D_{-\infty} = \{x : d_k(x) = 0\}$, $D_\infty = \{x : d_k(x) = \infty\}$, and

$$D_n = \{x \,:\, 2^n \le \frac{1}{2(k+1)} d_k(x) < 2^{n+1}\} \quad \text{for} \quad n \in \mathbf{Z}. \tag{36}$$

Then we have

$$X = D_{-\infty} \cup D_\infty \cup (\bigcup_{n \in \mathbf{Z}} D_n)\,;$$

and it is clear that (34) holds on $D_{-\infty}$. On the other hand, (a) implies that $\{x : w(Sx) = 0\} \subset \{x : w(x) = 0\}$, and therefore we get

$$\sum_{i=0}^{k} J_{-i}(x) w(S^{-i}x) = 0 \quad \text{on} \quad D_\infty.$$

It follows that (34) holds on D_∞. To prove (34) on each D_n, $n \in \mathbf{Z}$, we apply the hypothesis that Φ has no periodic part. By this hypothesis, D_n has the form

$$D_n = \bigcup_{n=1}^{\infty} B_j\,, \tag{37}$$

where the B_j satisfy

$$B_j \cap S^\ell B_j = \emptyset \quad \text{for} \quad 1 \le \ell \le 2(k+1). \tag{38}$$

Let us fix B_j, and let A denote a measurable subset of B_j with $0 < \mu A < \infty$. Then define a function f on X by the relation

$$f(S^i x) = \begin{cases} h_i(x)^{-1} \cdot [J_i(x)\, w(S^i x)]^{\frac{-1}{p-1}} & \text{if } x \in A \text{ and } 0 \le i \le k \\ 0 & \text{otherwise.} \end{cases}$$

Since $A \subset B_j \subset D_n$ and $h_{i+j}(S^{-j}x) = h_j(S^{-j}x) h_i(x)$ by (2), it follows that for $x \in A$ and $0 \le j \le k$,

$$\begin{aligned} M^+(T) f(S^{-j}x) &\ge \frac{1}{2(k+1)} \left| \sum_{i=0}^{k} h_{i+j}(S^{-j}x) f(S^{i+j}(S^{-j}x)) \right| \\ &= \frac{1}{2(k+1)} \left| \sum_{i=0}^{k} h_j(S^{-j}x) h_i(x) f(S^i x) \right| \\ &\ge \frac{1}{2(k+1)} \cdot \frac{1}{K_\infty} \sum_{i=0}^{k} [J_i(x) w(S^i x)]^{\frac{-1}{p-1}} && \text{(by (28))} \\ &= \frac{1}{K_\infty} \cdot \frac{1}{2(k+1)} d_k(x) \ge \left(\frac{1}{K_\infty}\right) \cdot 2^n && \text{(by (36)).} \end{aligned}$$

Hence if we set

$$E(-1) := \bigcup_{i=0}^{k} S^{-i} A \quad \text{and} \quad E(1) := \bigcup_{i=0}^{k} S^i A\,,$$

then

$$M^+(T)f \geq \left(\frac{1}{K_\infty}\right) 2^n \quad \text{on } E(-1).$$

Thus (a) implies that

$$\int_{E(-1)} w\, d\mu \leq C \left(\frac{K_\infty}{2^n}\right)^p \int |f|^p w\, d\mu, \tag{39}$$

where by the definition of f

$$\int |f|^p w\, d\mu = \int_{E(1)} |f|^p w\, d\mu = \sum_{i=0}^{k} \int_{S^i A} |f|^p w\, d\mu$$

$$= \sum_{i=0}^{k} \int_A |f(S^i x)|^p w(S^i x) J_i(x)\, d\mu \qquad \text{(by (3))}$$

$$\leq K_\infty^p \sum_{i=0}^{k} \int_A [J_i(x) w(S^i x)]^{\frac{-1}{p-1}}\, d\mu \qquad \text{(by (28))},$$

and by (3)

$$\int_{E(-1)} w\, d\mu = \sum_{i=0}^{k} \int_{S^{-i}A} w\, d\mu = \sum_{i=0}^{k} \int_A w(S^{-i}x) J_{-i}(x)\, d\mu.$$

Consequently we get

$$2^{np} \int_A \sum_{i=0}^{k} J_{-i}(x) w(S^{-i}x)\, d\mu \leq C \cdot K_\infty^{2p} \int_A \sum_{i=0}^{k} [J_i(x) w(S^i x)]^{\frac{-1}{p-1}}\, d\mu. \tag{40}$$

On the other hand, since $A \subset B_j \subset D_n$, it follows that

$$\frac{1}{\mu A} \int_A \frac{1}{k+1} \sum_{i=0}^{k} [J_i(x) w(S^i x)]^{\frac{-1}{p-1}}\, d\mu \leq 2^{n+2}.$$

Combining this with (40) yields

$$\left(\frac{1}{\mu A} \int_A \frac{1}{k+1} \sum_{i=0}^{k} J_{-i}(x) w(S^{-i}x)\, d\mu\right) \cdot \left(\frac{1}{\mu A} \int_A \frac{1}{k+1} \sum_{i=0}^{k} [J_i(x) w(S^i x)]^{\frac{-1}{p-1}} d\mu\right)^{p-1} \leq C \cdot 2^{2p} K_\infty^{2p}.$$

Since this holds for every A, arbitrary measurable subset of B_j with positive finite measure, we conclude that for a.e. $x \in B_j$

$$\left(\frac{1}{k+1} \sum_{i=0}^{k} J_{-i}(x) w(S^{-i}x)\right) \cdot \left(\frac{1}{k+1} \sum_{i=0}^{k} [J_i(x) w(S^i x)]^{\frac{-1}{p-1}}\right)^{p-1} \leq C \cdot 2^{2p} K_\infty^{2p},$$

whence (34) holds on D_n, $n \in \mathbf{Z}$, and thus (b) has been established.

Let $p = 1$.

(b) $\Rightarrow$ (a). By (29), (b) is equivalent to

$$C = \sup_{n\geq 0} \|\Phi_{0,n}\|_{L^1(wd\mu)} < \infty. \tag{41}$$

Since (3) implies

$$\int (\Phi_{0,n} f) \cdot w \, d\mu = \int f \cdot \left(\frac{1}{n+1} \sum_{i=0}^{n} J_{-i} \Phi^{-i} w \right) d\mu$$

for $0 \leq f \in M(\mu)$, (41) is equivalent to

$$\sup_{n\geq 0} \frac{1}{n+1} \sum_{i=0}^{n} J_{-i}(x) w(S^{-i}x) \leq C w(x) \quad \text{a.e.} \tag{42}$$

on X. Hence, using (3) again, for a.e. $x \in X$ and all $j \in \mathbf{Z}$ we have

$$\sup_{n\geq 0} \frac{1}{n+1} \sum_{i=0}^{n} J_{j-i}(x) w(S^{j-i}x) \leq C J_j(x) w(S^j x). \tag{43}$$

Let $0 \leq f \in L^1(wd\mu)$. For an $N \geq 0$ we then define

$$f^*_{\Phi,N} = \max_{0\leq n\leq N} \Phi_{0,n} f \quad \left(= \max_{0\leq n\leq N} \frac{1}{n+1} \sum_{i=0}^{n} \Phi^i f \right)$$

It follows that $f^*_{\Phi,N} \uparrow M^+(\Phi) f$ a.e. on X as $N \to \infty$; and for any $L \geq 0$ we have, by (3),

$$\begin{aligned}(L+1) \int_{\{x : f^*_{\Phi,N}(x) > \lambda\}} w \, d\mu &= \sum_{i=0}^{L} \int_{\{x : f^*_{\Phi,N}(S^i x) > \lambda\}} J_i(x) w(S^i x) \, d\mu \\ &= \int \sum_{\{0\leq i\leq L : f^*_{\Phi,N}(S^i x) > \lambda\}} J_i(x) w(S^i x) \, d\mu.\end{aligned}$$

We then apply Lemma A together with (43) to infer that there exists a positive constant C independent of N, $L \geq 0$ such that for a.e. $x \in X$

$$\sum_{\{0\leq i\leq L : f^*_{\Phi,N}(S^i x) > \lambda\}} J_i(x) w(S^i x) \leq \frac{C}{\lambda} \sum_{i=0}^{L+N} f(S^i x) J_i(x) w(S^i x).$$

Hence

$$\begin{aligned}\int_{\{x : f^*_{\Phi,N}(x) > \lambda\}} w \, d\mu &\leq \frac{C}{\lambda} \cdot \frac{1}{L+1} \int \sum_{i=0}^{L+N} f(S^i x) J_i(x) w(S^i x) \, d\mu \\ &= \frac{C}{\lambda} \cdot \frac{L+N+1}{L+1} \int f w \, d\mu.\end{aligned}$$

By letting $L \uparrow \infty$, and then $N \uparrow \infty$, we see that (a) holds.

(a) $\Rightarrow$ (b). Since Φ has no periodic part, if $n \geq 0$ is an integer then X has the form

$$X = \bigcup_{j=1}^{\infty} B_j, \tag{44}$$

where the B_j satisfy

$$B_j \cap S^{\ell} B_j = \emptyset \quad \text{for} \quad 0 \leq \ell \leq n. \tag{45}$$

For the moment let us fix B_j, and let A be a measurable subset of B_j. If we set

$$F(-1) := \bigcup_{i=0}^{n} S^{-i} A$$

and if $x \in F(-1)$ then by (2), (28) and (45) we have

$$\max_{0 \leq k \leq n} \frac{1}{k+1} \left| \sum_{i=0}^{k} T^i \chi_A(x) \right| \geq \frac{1}{n+1} \cdot \frac{1}{K_\infty}.$$

Therefore by (a)

$$\int_{F(-1)} w \, d\mu \ \leq \ C \cdot (n+1) K_\infty \int_A w \, d\mu.$$

Since

$$\int_{F(-1)} w \, d\mu = \sum_{i=0}^{n} \int_{S^{-i}A} w \, d\mu = \sum_{i=0}^{n} \int_A J_{-i}(x) w(S^{-i}x) \, d\mu,$$

we then have

$$\int_A \frac{1}{n+1} \sum_{i=0}^{n} J_{-i}(x) w(S^{-i}x) \, d\mu \leq C \cdot K_\infty \int_A w \, d\mu,$$

which implies, as before, that

$$\frac{1}{n+1} \sum_{i=0}^{n} J_{-i}(x) w(S^{-i}x) \leq C K_\infty \, w(x) \quad \text{a.e.}$$

on B_j and hence on X. Since the constant CK_∞ is independent of $n \geq 0$, this establishes (42) and hence (b).

The proof is complete. □

Proof of Theorem 5. Let $1 < p < \infty$.

(c) $\Rightarrow$ (a) is obvious.

(a) $\Rightarrow$ (b). As in the proof of Theorem 4, it suffices to prove that there exists a positive constant C such that

$$\left(\sum_{i=0}^{k} J_i(x) w(S^i x) \right) \cdot \left(\sum_{i=0}^{k} [J_i(x) w(S^i x)]^{\frac{-1}{p-1}} \right)^{p-1} \leq C(k+1)^p \tag{46}$$

for a.e. $x \in X$ and all $k \geq 0$.

To do so, let d_k, $D_{-\infty}$, D_∞ and D_n ($n \in \mathbf{Z}$) be the same as in the proof of Theorem 4 (cf. (35) , (36)). Since Φ has no periodic part by hypothesis, (a) implies that $\{x : w(Sx) = \infty\} = \{x : w(x) = \infty\}$. Indeed if this is not true, then we can choose an $E \in \mathcal{F}$, with $\mu E > 0$ and $\int_E w d\mu < \infty$, such that

$$SE \subset \{x : w(x) = \infty\} \quad \text{and} \quad S^2(E) \cap (E \cup SE) = \emptyset.$$

Then the function $f = \chi_E$ ($\in L^p(wd\mu)$) satisfies $H^*(T)f(x) \geq 1/K_\infty$ on SE, whence

$$\int_{\{x:H^* f(x)>\lambda\}} wd\mu = \infty \quad \text{for all } \lambda \text{ with } 0 < \lambda < \frac{1}{K_\infty}.$$

This is a contradiction. Similarly (a) implies that $\{x : w(Sx) = 0\} = \{x : w(x) = 0\}$. Therefore we have

$$D_{-\infty} = \{x : w(x) = \infty\},\ D_\infty = \{x : w(x) = 0\},\ SD_{-\infty} = D_{-\infty} \text{ and } SD_\infty = D_\infty.$$

Thus (46) holds clearly on $D_{-\infty} \cup D_\infty$. To prove (46) on each D_n, $n \in \mathbf{Z}$, we represent D_n as

$$D_n = \bigcup_{j=1}^{\infty} B_j,$$

where the B_j satisfy

$$S^\ell B_j \bigcap B_j = \emptyset \quad \text{for} \quad 1 \leq \ell \leq 4(k+1).$$

If A is a measurable subset of B_j with $0 < \mu A < \infty$, then let

$$E(1) := \bigcup_{i=0}^{k} S^i A \quad \text{and} \quad E(2) := \bigcup_{i=k+1}^{2k+1} S^i A.$$

If $0 \leq f \in M(\mu)$ and $\{x : f(x) \neq 0\} \subset E(1)$, then define a function $f^\sim$ on X by the relation

$$\begin{cases} f^\sim(S^{k+1-i}x) = [\operatorname{sgn} h_{-i}(S^{k+1}x)]^{-1} \cdot f(S^{k+1-i}x) & \text{for } x \in A \text{ and } 1 \leq i \leq k+1, \\ f^\sim = 0 & \text{on} \quad X \setminus \bigcup_{i=0}^{k} S^i A, \end{cases}$$

where $\operatorname{sgn} \alpha = \alpha/|\alpha|$ for a complex number $\alpha \neq 0$, and $\operatorname{sgn} 0 = 0$.

Then for $x \in A$ and $k+1 \leq j \leq 2k+1$ we have

$$H^*(T)f^\sim(S^j x) \geq \left| \sum_{i=1}^{k+1} \frac{h_{-i-(j-k-1)}(S^j x) \cdot f^\sim(S^{k+1-i}x)}{i + (j-k-1)} \right|.$$

Since $h_{j-k-1}(S^{k+1}x) \cdot h_{-i-(j-k-1)}(S^{j-k-1}(S^{k+1}x)) = h_{-i}(S^{k+1}x)$ by (2),

$$h_{-i-(j-k-1)}(S^j x) = \frac{h_{-i}(S^{k+1}x)}{h_{j-k-1}(S^{k+1}x)}.$$

Therefore for $x \in A$ and $k+1 \le j \le 2k+1$ we have

$$
\begin{aligned}
(47) \quad H^*(T)f^{\sim}(S^j x) &\ge \frac{1}{h_{j-k-1}(S^{k+1}x)} \cdot \sum_{i=1}^{k+1} \frac{|h_{-i}(S^{k+1}x)| \cdot f(S^{k+1-i}x)}{i+(j-k-1)} \\
&\ge K_\infty^{-2} \cdot \frac{1}{2(k+1)} \sum_{i=0}^{k} f(S^i x) \qquad \text{(by (28))}.
\end{aligned}
$$

In particular, if $0 \le f \in M(\mu)$ is such that

$$
\begin{cases} f(S^i x) = [J_i(x) w(S^i x)]^{\frac{-1}{p-1}} & \text{for } x \in A \text{ and } 0 \le i \le k, \\ f = 0 & \text{on } X \setminus \bigcup_{i=0}^{k} S^i A, \end{cases}
$$

then, by (36) and the fact $A \subset B_j \subset D_n$, we have

$$
\begin{aligned}
H^*(T)f^{\sim}(S^j x) &\ge K_\infty^{-2} \cdot \frac{1}{2(k+1)} \sum_{i=0}^{k} [J_i(x) w(S^i x)]^{\frac{-1}{p-1}} \\
&= K_\infty^{-2} \cdot \frac{1}{2(k+1)} \cdot d_k(x) \ge K_\infty^{-2} \cdot 2^n
\end{aligned}
$$

for $x \in A$ and $k+1 \le j \le 2k+1$.

Thus by (a)

$$
\int_{E(2)} w \, d\mu \le C \cdot K_\infty^{2p} \frac{1}{2^{np}} \int_{E(1)} f^p w \, d\mu.
$$

Since (3) implies

$$
\int_{E(1)} f^p w \, d\mu = \sum_{i=0}^{k} \int_{S^i A} f^p w \, d\mu = \sum_{i=0}^{k} \int_A f^p(S^i x) w(S^i x) J_i(x) \, d\mu,
$$

we can apply the following equations

$$
\sum_{i=0}^{k} f^p(S^i x) w(S^i x) J_i(x) = \sum_{i=0}^{k} [J_i(x) w(S^i x)]^{\frac{-1}{p-1}} = d_k(x),
$$

to obtain that

$$
(48) \qquad \int_{E(2)} w \, d\mu \le C \cdot K_\infty^{2p} \frac{1}{2^{np}} \int_A d_k(x) \, d\mu.
$$

Next, if $0 \le f \in M(\mu)$ and $\{x : f(x) \ne 0\} \subset E(2)$, then define a function $f_\sim$ on X by the relation

$$
\begin{cases} f_\sim(S^{k+i} x) = [\operatorname{sgn} h_i(S^k x)]^{-1} \cdot f(S^{k+i} x) & \text{for } x \in A \text{ and } 1 \le i \le k+1, \\ f_\sim = 0 & \text{on } X \setminus \bigcup_{i=k+1}^{2k+1} S^i A. \end{cases}
$$

Then for $x \in A$ and $0 \le j \le k$ we have

$$
H^*(T) f_\sim(S^j x) \ge \left| \sum_{i=1}^{k+1} \frac{h_{i+(k-j)}(S^j x) \cdot f_\sim(S^{k+i} x)}{i+(k-j)} \right| .
$$

Since $h_{j-k}(S^k x) \cdot h_{i+(k-j)}(S^{j-k}(S^k x)) = h_i(S^k x)$ by (2),

$$h_{i+(k-j)}(S^j x) = \frac{h_i(S^k x)}{h_{j-k}(S^k x)}.$$

Hence it follows that

$$(49) \qquad H^*(T)f_\sim(S^j x) \geq \frac{1}{|h_{j-k}(S^k x)|} \cdot \sum_{i=1}^{k+1} \frac{|h_i(S^k x)| \cdot f(S^{k+i} x)}{i+(k-j)}$$
$$\geq K_\infty^{-2} \cdot \frac{1}{2(k+1)} \sum_{i=k+1}^{2k+1} f(S^i x) \qquad \text{(by (28))}$$

for $x \in A$ and $0 \leq j \leq k$. In particular, if $f = \chi_{E(2)}$ then

$$H^*(T)f_\sim(S^j x) \geq K_\infty^{-2} \cdot \frac{1}{2}$$

for $x \in A$ and $0 \leq j \leq k$. Thus by (a)

$$\int_{E(1)} w \, d\mu \leq C \cdot K_\infty^{2p} \cdot 2^p \int_{E(2)} w \, d\mu .$$

We then use the following equations

$$\int_{E(1)} w \, d\mu = \sum_{i=0}^{k} \int_{S^i A} w \, d\mu = \sum_{i=0}^{k} \int_A w(S^i x) J_i(x) \, d\mu,$$

to obtain that

$$(50) \qquad \int_A \sum_{i=0}^{k} J_i(x) w(S^i x) \, d\mu \leq C \cdot K_\infty^{2p} \cdot 2^p \int_{E(2)} w \, d\mu.$$

Combining this with (48) yields

$$\int_A \sum_{i=0}^{k} J_i(x) w(S^i x) \, d\mu \leq C^2 \cdot K_\infty^{4p} \cdot \frac{1}{2^{(n-1)p}} \int_A d_k(x) \, d\mu.$$

Since $2^n \leq d_k/2(k+1) < 2^{n+1}$ on D_n and $A \subset B_j \subset D_n$, it follows that

$$2^{(n+1)p} \leq \left(\frac{1}{\mu A} \int_A \frac{1}{k+1} d_k(x) \, d\mu \right)^p \leq 2^{(n+2)p}.$$

Thus we obtain

$$\left(\frac{1}{\mu A} \int_A \frac{1}{k+1} \sum_{i=0}^{k} J_i(x) w(S^i x) \, d\mu \right) \cdot \left(\frac{1}{\mu A} \int_A \frac{1}{k+1} d_k(x) \, d\mu \right)^{p-1}$$
$$\leq C^2 \cdot K_\infty^{4p} \cdot 2^{3p},$$

and therefore

$$\left(\frac{1}{k+1}\sum_{i=0}^{k} J_i(x)w(S^ix)\right)\cdot\left(\frac{1}{k+1}\sum_{i=0}^{k}[J_i(x)w(S^ix)]^{\frac{-1}{p-1}}\right)^{p-1} \le C^2\cdot K_\infty^{4p}\cdot 2^{3p} \quad \text{a.e.}$$

on B_j (and hence on D_n). Since the constant $C^2\cdot K_\infty^{4p}\cdot 2^{3p}$ is independent of $k\ge 0$, we have proved (46) and hence (b).

(b) $\Rightarrow$ (c). By Remark 1 (i), we may assume without loss of generality that $X = \{x : 0 < w(x) < \infty\}$. Then T and τ can be considered to be invertible Lamperti operators on $M(w d\mu) = M(\mu)$, whence (b) $\Rightarrow$ (c) follows from Lemma of [19].

Let $p = 1$.

(a) $\Rightarrow$ (b). As in the proof of Theorem 4 (cf. (41), (42)), (b) is equivalent to the existence of a positive constant C such that

$$\sup_{n\ge 0}\frac{1}{2n+1}\sum_{i=-n}^{n} J_i(x)w(S^ix) \le Cw(x) \quad \text{a.e.} \tag{51}$$

on X. To prove (51), let $N\ge 1$ be fixed arbitralily. Since Φ has no periodic part by hypothesis, X has the form

$$X = \bigcup_{j=0}^{\infty} B_j\,,$$

where the B_j satisfy

$$B_j\cap S^\ell B_j = \emptyset \quad \text{for} \quad 1\le \ell\le 2N.$$

If A is a measurable subset of B_j such that $0<\mu A<\infty$, and if $x\in S^iA$ for some i with $1<|i|\le N$, then by (28) we have

$$H^*(T)\chi_A(x) \ge \frac{1}{K_\infty}\cdot\frac{1}{N}.$$

Hence (a) implies

$$\sum_{|i|=1}^{N}\int_{S^iA} w\,d\mu \le C\cdot K_\infty N\int_A w\,d\mu\,.$$

We now apply (3) to infer that

$$\int_A \frac{1}{2N+1}\sum_{i=-N}^{N} J_i(x)w(S^ix)\,d\mu \le (CK_\infty+1)\int_A w\,d\mu\,;$$

therefore

$$\frac{1}{2N+1}\sum_{i=-N}^{N} J_i(x)w(S^ix) \le (CK_\infty+1)w \quad \text{a.e.}$$

on B_j and hence on X, completing the proof of (51).

(b) $\Rightarrow$ (a). By (51) and (3), we have

$$\frac{1}{2n+1}\sum_{i=-n}^{n} J_{j+i}(x)w(S^{j+i}x) \le C\, J_j(x)w(S^j x) \tag{52}$$

for a.e. $x \in X$ and all $j \in \mathbf{Z}$ and $n \ge 0$. For an $N \ge 1$ we then define the truncated maximal operator $H_N^*(T)$ on $M(\mu)$ by the relation

$$H_N^*(T)f = \max_{1\le n\le N}\left|\sum_{k=-n}^{n}{}' \frac{T^k f}{k}\right|,$$

where the prime means that the term with zero denominator is omitted.

Clearly we have

$$H_N^*(T)f(x) \uparrow H^*(T)f(x) \qquad \text{a.e.} \tag{53}$$

on X as $N \to \infty$. If $j \in \mathbf{Z}$, then

$$\begin{aligned} |h_j(x)|\, H_N^*(T)f(S^j x) &= \max_{1\le n\le N}\left|\sum_{k=-n}^{n}{}' \frac{h_j(x)h_k(S^j x)f(S^{j+k}x)}{k}\right| \\ &= \max_{1\le n\le N}\left|\sum_{k=-n}^{n}{}' \frac{h_{j+k}(x)f(S^{j+k}x)}{k}\right| \qquad \text{(by (2))}, \end{aligned}$$

so that

$$\begin{aligned} H_N^*(T)f(S^j x) &= \frac{1}{|h_j(x)|}\cdot \max_{1\le n\le N}\left|\sum_{k=-n}^{n}{}' \frac{h_{j+k}(x)f(S^{j+k}x)}{k}\right| \\ &\le K_\infty \cdot \max_{1\le n\le N}\left|\sum_{k=-n}^{n}{}' \frac{h_{j+k}(x)f(S^{j+k}x)}{k}\right|. \end{aligned} \tag{54}$$

By this together with (3) we observe that for $L \ge 1$ and $\lambda > 0$

$$\begin{aligned} (2L+1)\int_{\{x\,:\,H_N^*(T)f(x)>\lambda\}} w\, d\mu &= \sum_{j=-L}^{L}\int_{\{x\,:\,H_N^*(T)f(S^j x)>\lambda\}} J_j(x)w(S^j x)\, d\mu \\ &= \int \sum_{\{-L\le j\le L\,:\,H_N^*(T)f(S^j x)>\lambda\}} J_j(x)w(S^j x)\, d\mu \\ &= \int \sum_{\left\{-L\le j\le L\,:\,\max_{1\le n\le N}\left|\sum'^{n}_{k=-n} \frac{h_{j+k}(x)f(S^{j+k}x)}{k}\right|>\lambda/K_\infty\right\}} J_j(x)w(S^j x)\, d\mu. \end{aligned}$$

Next we apply (52) together with a known result about the classical discrete Hilbert transform (see e.g. Theorem 10 of [10]) to infer that there exists a positive constant C such that

$$\sum_{\{-L\leq j\leq L\,:\,\max_{1\leq n\leq N}|\sum'^{n}_{k=-n}\frac{h_{j+k}(x)f(S^{j+k}x)}{k}|>\lambda/K_\infty\}} J_j(x)w(S^jx)$$

$$\leq C\,\frac{K_\infty}{\lambda}\cdot\sum_{j=-N-L}^{N+L}|h_j(x)f(S^jx)|\cdot J_j(x)w(S^jx)$$

for a.e. $x\in X$ and all $\lambda>0$ and $N,\,L\geq 1$. Thus by (28) and (3)

$$\begin{aligned}
(2L+1)&\int_{\{x\,:\,H_N^*(T)f(x)>\lambda\}} w\,d\mu\\
&\leq \int_X C\cdot\frac{K_\infty}{\lambda}\cdot\left(\sum_{j=-N-L}^{N+L}|h_j(x)f(S^jx)|\cdot J_j(x)w(S^jx)\right)\,d\mu\\
&\leq C\cdot\frac{K_\infty^2}{\lambda}\int_X\sum_{j=-N-L}^{N+L}|f(S^jx)|\cdot J_j(x)w(S^jx)\,d\mu\\
&= C\cdot\frac{K_\infty^2}{\lambda}\cdot(2N+2L+1)\int_X|f|w\,d\mu\,.
\end{aligned}$$

Letting $L\uparrow\infty$ yields

$$\int_{\{x\,:\,H_N^*(T)f(x)>\lambda\}} w\,d\mu\ \leq\ C\cdot\frac{K_\infty^2}{\lambda}\int_X|f|w\,d\mu\,.$$

Hence (a) follows from (53), and this completes the proof of Theorem 5. □

Remark 2. The hypothesis that Φ has no periodic part was used only in the proof of implication (a) ⇒ (b) of Theorems 4 and 5. Thus, without this hypothesis, implication (b) ⇒ (a) of Theorem 4 and implications (b) ⇒ (c) ⇒ (a) of Theorem 5 are true.

In the remainder of the paper we investigate the a.e. convergence of the ergodic sequence $\{T^nf\}$ and the ergodic partial sums $\{\sum_{k=1}^n(T^kf-T^{-k}f)/k\}$ in the sense of Cesàro-α means. For the basic properties of Cesàro-α means we refer the reader to Zygmund [24].

Following [4], for a real number α with $-1<\alpha\leq 0$ we write

$$R_{n,1+\alpha}(T)f\ =\ \frac{1}{A_n^{1+\alpha}}\sum_{k=0}^{n}A_{n-k}^{\alpha}T^kf$$

and

$$H_{n,\alpha}(T)\ =\ \frac{1}{A_n^{\alpha}}\sum_{k=1}^{n}A_{n+1-k}^{\alpha}\left(\frac{T^kf-T^{-k}f}{k}\right),$$

where the Cesàro numbers A_n^β are given as

$$A_n^\beta = \frac{(\beta+1)\dots(\beta+n)}{n!} \quad \text{and} \quad A_0^\beta = 1\,.$$

Two maximal operators $M_{1+\alpha}^+(T)$ and $H_\alpha^*(T)$ on $M(\mu)$ are defined by the relations

$$M_{1+\alpha}^+(T)f = \sup_{n\geq 0} |R_{n,1+\alpha}(T)f|$$

and

$$H_\alpha^*(T)f = \sup_{n\geq 0} |H_{n,\alpha}(T)f|\,.$$

Note that $M_1^+(T)f = M^+(T)f$ and $H_0^*(T)f = H^*(T)f$. In the theorems below we use the Lorentz spaces $L_{r,1}(wd\mu)$ with $1 \leq r < \infty$. Recall that $f \in L_{r,1}(wd\mu)$ if and only if

$$\|f\|_{r,1;\,wd\mu} := \int_0^\infty \left(\int_{\{x:|f(x)|>t\}} wd\mu\right)^{1/r} dt < \infty,$$

that $\|\chi_E\|_{r,1;\,wd\mu} = \left(\int_E w\,d\mu\right)^{1/r}$ for $E \in \mathcal{F}$ with $\int_E wd\mu < \infty$, and that $L_{r,1}(wd\mu) \subset L_{r,r}(wd\mu) = L^r(wd\mu)$. These properties of Lorentz spaces are explained in Hunt [9].

Theorem 6. *Let $0 \leq w \leq \infty$ on X and let $1 \leq p < \infty$. If T is an invertible Lamperti operator on $M(\mu)$ satisfying (27) and if the linear modulus τ of T becomes an operator on $M(wd\mu)$ and satisfies*

$$\sup_{n\geq 0} \|\tau_{0,n}\|_{L^p(wd\mu)} < \infty\,, \tag{55}$$

then the following statements hold.

(a) *When $1 < p \leq r < \infty$, the limit*

$$\lim_{n\to\infty} R_{n,\,p/r}(T)f$$

exists a.e. on the set $\{x : w(x) > 0\}$ for all $f \in L^r(wd\mu)$; further there exists a positive constant C such that

$$\|M_{p/r}^+(T)f\|_{L^r(wd\mu)} \leq C\,\|f\|_{L^r(wd\mu)} \tag{56}$$

for all $f \in L^r(wd\mu)$.

(b) *When $1 = p \leq r < \infty$, the limit*

$$\lim_{n\to\infty} R_{n,\,1/r}(T)f$$

exists a.e. on the set $\{x : w(x) > 0\}$ for all $f \in L_{r,1}(wd\mu)$.

Theorem 7. *Let $0 \leq w \leq \infty$ on X and let $1 \leq p < \infty$. If T is an invertible Lamperti operator on $M(\mu)$ satisfying (27) and if the linear modulus τ of T becomes an invertible operator on $M(wd\mu)$ and satisfies*

$$\sup_{n\geq 0} \|\tau_{n,n}\|_{L^p(wd\mu)} < \infty\,, \tag{57}$$

then the following statements hold.

(a) *When* $1 < p \le r < \infty$, *the limit*

$$\lim_{n\to\infty} H_{n,(p/r)-1}(T)f$$

exists a.e. on the set $\{x : w(x) > 0\}$ *for all* $f \in L^r(wd\mu)$; *further there exists a positive constant* C *such that*

$$\|H^*_{(p/r)-1}(T)f\|_{L^r(wd\mu)} \le C\,\|f\|_{L^r(wd\mu)} \tag{58}$$

for all $f \in L^r(wd\mu)$.

(b) *When* $1 = p \le r < \infty$, *the limit*

$$\lim_{n\to\infty} H_{n,(1/r)-1}(T)f$$

exists a.e. on the set $\{x : w(x) > 0\}$ *for all* $f \in L_{r,1}(wd\mu)$.

Proof of Theorem 6. (a) By (29), Φ becomes an operator on $M(wd\mu)$ and satisfies

$$\sup_{n\ge 0} \|\Phi_{0,n}\|_{L^p(wd\mu)} < \infty,$$

whence we can apply Theorem 1 together with (28) to infer that there exists a positive constant C such that

$$\left(\sum_{i=0}^{k} |h_{-i}(x)|^{-r} J_{-i}(x) w(S^{-i}x)\right)\cdot\left(\sum_{i=0}^{k} [\,|h_i(x)|^{-r} J_i(x) w(S^i x)]^{\frac{-1}{p-1}}\right)^{p-1} \le C\,(k+1)^p$$

for a.e. $x \in X$ and all $k \ge 0$. Since $0 < p/r \le 1$ and $1 < p = (p/r)r$, it follows from [15] (cf. especially the proofs of Corollary 3.4 and Theorem 3.1 of [15]) that

(i) the limit $\lim_{n\to\infty} R_{n,p/r}(\tau)f$ exists a.e. on the set $\{x : w(x) > 0\}$ for all $f \in L^r(wd\mu)$, and

(ii) the maximal operator $M^+_{p/r}(\tau)$ is bounded in $L^r(wd\mu)$.

Since $0 \le M^+_{p/r}(T)f \le M^+_{p/r}(\tau)|f|$ for $f \in L^r(wd\mu)$, (56) holds. And the a.e. convergence of $R_{n,p/r}(T)f$ on the set $\{x : w(x) > 0\}$ follows from Banach's convergence principle, because $\{g + (f - Tf) : Tg = g, f \in L^r(wd\mu)\}$ is a dense subspace of $L^r(wd\mu)$ by a mean ergodic theorem, and for $f \in L^r(wd\mu)$ we have

$$\lim_{n\to\infty} R_{n,p/r}(T)[f - Tf] = 0 \quad \text{a.e.} \tag{59}$$

on the set $\{x : w(x) > 0\}$. Indeed (59) holds for f of the form $f = \chi_E$ by the proof of Proposition 3.2 of [15], and thus an approximation argument together with (56) can be used to see that (59) holds for any $f \in L^r(wd\mu)$.

(b) Let $r < s < \infty$, where $1 = p \le r < \infty$. Then $p = 1 < s/r \le s$, and the Marcinkiewicz interpolation theorem implies that

$$\sup_{n\ge 0} \|\tau_{0,n}\|_{L^{s/r}(wd\mu)} < \infty.$$

Since $1 < s/r \le s$, we then apply (a) to infer that the limit $\lim_{n\to\infty} R_{n,1/r}(T)f$ exists a.e. on the set $\{x : w(x) > 0\}$ for all $f \in L^s(wd\mu)$.

Since the Lorentz space $L_{r,1}(wd\mu)$ is a Banach space and $L^s(wd\mu) \cap L_{r,1}(wd\mu)$ is a dense subspace of $L_{r,1}(wd\mu)$, it is enough to prove by the Banach convergence principle that

$$M^+_{1/r}(T)f < \infty \quad \text{a.e.}$$

on the set $\{x : w(x) > 0\}$ for all $f \in L_{r,1}(wd\mu)$. By (29) and (4) it suffices to prove the following weak type inequality:

(W) *There exists a positive constant C such that*

$$\int_{\{x : M^+_{1/r}(\Phi)f(x) > \lambda\}} w\, d\mu \le C\, \frac{1}{\lambda^r}\, \|f\|^r_{r,1;\, wd\mu} \tag{60}$$

for all $f \in L_{r,1}(wd\mu)$ and $\lambda > 0$.

If $r = 1$ then, since Φ satisfies (41), (W) follows from Theorem 4 (cf. also Remark 2).

If $1 < r < \infty$ then, by the proof of Theorem 3.13 of Chapter V of [22], it suffices to prove the existence of a positive constant C such that

$$\int_{\{x : M^+_{1/r}(\Phi)\chi_E(x) > \lambda\}} w\, d\mu \le \frac{C}{\lambda^r} \int_E w\, d\mu \tag{61}$$

for all $E \in \mathcal{F}$ and $\lambda > 0$. To do so , we adapt the argument of Bernardis and Martín-Reyes [4] as follows.

Let $f = \chi_E$, where $E \in \mathcal{F}$. If we define, for an $N \ge 1$,

$$M^+_{1/r}(\Phi)_N\, \chi_E(x) = \sup_{0 \le n \le N} \left| \frac{1}{A_n^{1/r}} \sum_{k=0}^{n} A_{n-k}^{(1/r)-1} \chi_E(S^k x) \right|$$

then $M^+_{1/r}(\Phi)_N\, \chi_E \uparrow M^+_{1/r}(\Phi)\chi_E$ a.e. on X as $N \to \infty$. For the moment let us fix an $N \ge 1$. If we set

$$A := \{x : M^+_{1/r}(\Phi)_N\, \chi_E(x) > \lambda\},$$

then by (3)

$$\begin{aligned} (L+1) \int_A w\, d\mu &= \int \sum_{i=0}^{L} \chi_A(S^i x) w(S^i x) J_i(x)\, d\mu \\ &= \int \sum_{\{0 \le i \le L :\, M^+_{1/r}(\Phi)_N \chi_E(S^i x) > \lambda\}} J_i(x) w(S^i x)\, d\mu. \end{aligned}$$

On the other hand, we know (cf. (41), (42), (43)) that there exists a positive constant C such that

$$\sup_{n \ge 0} \frac{1}{n+1} \sum_{i=0}^{n} J_{j-i}(x) w(S^{j-i} x) \le C \cdot J_j(x) w(S^j x)$$

for a.e. $x \in X$ and all $j \in \mathbf{Z}$. Thus by Lemma 2.6 and Theorem E of [4] there exists a positive constant C such that

$$\sum_{\{0\le i\le L:\, M^+_{1/r}(\Phi)_N\chi_E(S^ix)>\lambda\}} J_i(x)w(S^ix)$$

$$\le \frac{C}{\lambda^r}\cdot\left(\int_0^\infty\left[\sum_{\{0\le i\le N+L:\,\chi_E(S^ix)>t\}} J_i(x)w(S^ix)\right]^{1/r} dt\right)^r .$$

Therefore we have

$$(L+1)\int_A w\,d\mu \le \frac{C}{\lambda^r}\cdot\int_X\left(\int_0^1\left[\sum_{\{0\le i\le N+L:\,\chi_E(S^ix)>t\}} J_i(x)w(S^ix)\right]^{1/r} dt\right)^r d\mu$$

$$\le \frac{C}{\lambda^r}\int_X\left(\sum_{i=0}^{N+L} J_i(X)w(S^ix)\chi_E(S^ix)\right) d\mu \qquad \text{(by Hölder's inequality)}$$

$$= \frac{C}{\lambda^r}\cdot(N+L+1)\int_E w\,d\mu \qquad \text{(by (3))}.$$

Letting $L\uparrow\infty$ and then $N\uparrow\infty$, we see that (61) holds, and this completes the proof of Theorem 6. □

Proof of Theorem 7. By (57) we may assume without loss of generality that $X=\{x: 0<w(x)<\infty\}$. Then T and τ can be regarded as invertible Lamperti operators on $M(wd\mu)=M(\mu)$.

Let $p\le r<\infty$. Then by the Marcinkiewicz interpolation theorem

$$\sup_{n\ge 0}\|\tau_{n,n}\|_{L^r(wd\mu)}<\infty . \tag{62}$$

Hence T becomes a bounded and invertible operator on $L^r(wd\mu)$. Let $\tau_{p/r}$ denote the invertible (positive) Lamperti operator on $M(wd\mu)=M(\mu)$ defined by the relation

$$\tau_{p/r}f = |h_1|^{r/p}\cdot\Phi f .$$

Then we have

$$\tau^i_{p/r}f = |h_i|^{r/p}\cdot\Phi^i f = |h_i|^{(r-p)/p}\cdot\tau^i f \qquad (i\in\mathbf{Z})$$

and by (28)

$$\tau^i_{p/r} \le K_\infty^{(r-p)/p}\cdot\tau^i \qquad (i\in\mathbf{Z}).$$

Thus

$$\sup_{n\ge 0}\left\|\frac{1}{2n+1}\sum_{i=-n}^{n}\tau^i_{p/r}\right\|_{L^p(wd\mu)}<\infty. \tag{63}$$

Since $0 < p/r \leq 1$ and $p = (p/r)r$, (a) now follows from [5] when $1 < p < r < \infty$, and from [19] when $1 < p = r < \infty$. (b) is a consequence of Theorem 1.4 of [4]. □

Remark 3. (i) In statement (b) of Theorems 6 and 7, the function f in $L_{r,1}(wd\mu)$ cannot be replaced by a function in $L^r(wd\mu)$ when $1 = p < r < \infty$. In fact, if we consider an ergodic invertible measure preserving transformation ϕ on a nonatomic probability measure space $(X, \mathcal{F}, \mu)$ and an operator T on $M(\mu)$ of the form $Tf = f \circ \phi$, then clearly $\|T^n\|_{L^p(\mu)} = 1$ for all $n \in \mathbf{Z}$ and $1 \leq p \leq \infty$. Déniel proved in [6] that if $1 < r < \infty$ then there exists a function $f \in L^r(\mu)$ for which the a.e. convergence of the sequence $\{R_{n,1/r}(T)f(x)\}_{n=0}^{\infty}$ fails to hold. Later, modifying the idea of Déniel [6], Bernardis, Martín-Reyes and Sarrión Gavilán proved in [5] that if $1 < r < \infty$ then there exists an $f \in L^r(\mu)$ for which the a.e. convergence of the sequence $\{H_{n,(1/r)-1}(T)f(x)\}_{n=1}^{\infty}$ fails to hold.

(ii) Statement (b) of Theorem 6 is not true if the hypothesis (27) is omitted. A counterexample can be found in [4].

(iii) Statement (b) of Theorem 7 is not true at least for the case $1 = p = r$ if the hypothesis (27) is omitted. This can be seen from [19].

4. Concluding remarks

The purpose of this section is to prove the following weighted ergodic theorem, without assuming that T satisfies (27).

Theorem 8. *Let $0 \leq w \leq \infty$ on X and let $1 < p < \infty$. Then the following statements hold for an invertible Lamperti operator T on $M(\mu)$.*

(a) *If T is an operator on $M(wd\mu)$ and satisfies*

$$K^+(p) := \sup_{n \geq 0} \|T^n\|_{L^p(wd\mu)} < \infty,$$

then for any r with $1/p < r \leq 1$ the limit

$$\lim_{n \to \infty} R_{n,r}(T)f$$

exists a.e. on the set $\{x : w(x) > 0\}$ for every $f \in L^p(wd\mu)$; and the maximal operator $M_r^+(T)$ is bounded in $L^p(wd\mu)$.

(b) *If T is an invertible operator on $M(wd\mu)$ and satisfies*

$$K(p) := \sup_{n \in \mathbf{Z}} \|T^n\|_{L^p(wd\mu)} < \infty,$$

then for any r with $1/p < r \leq 1$ the limit

$$\lim_{n \to \infty} H_{n,r-1}(T)f$$

exists a.e. on the set $\{x : w(x) > 0\}$ for every $f \in L^p(wd\mu)$; and the maximal operator $H_{r-1}^*(T)$ is bounded in $L^p(wd\mu)$.

Remark 4. In the above theorem we cannot take $r = 1/p$. See Remark 3 (i).

Proof of Theorem 8. (a) If τ_r denotes the invertible Lamperti operator on $M(\mu)$ defined by

$$\tau_r f = |h_1|^{1/r} \cdot \Phi f,$$

then we have

$$\tau_r^i f(x) = |h_i(x)|^{1/r} \cdot \Phi^i f(x) \qquad (i \in \mathbf{Z}).$$

If $0 \le f \in M(\mu)$ then, since $rp > 1$, it follows from Hölder's inequality that

$$\begin{aligned}\left(\frac{1}{n+1}\sum_{i=0}^{n} \tau_r^i f\right)^{rp} &\le \frac{1}{n+1}\sum_{i=0}^{n} (\tau_r^i f)^{rp} \\ &= \frac{1}{n+1}\sum_{i=0}^{n} \left[|h_i| \cdot \Phi^i(f^r)\right]^p = \frac{1}{n+1}\sum_{i=0}^{n} [\tau^i(f^r)]^p,\end{aligned}$$

whence

$$\begin{aligned}\int_X \left(\frac{1}{n+1}\sum_{i=0}^{n} \tau_r^i f\right)^{rp} \cdot w\, d\mu &\le \frac{1}{n+1}\sum_{i=0}^{n} \int_X [\tau^i(f^r)]^p \cdot w\, d\mu \\ &\le (K^+(p))^p \cdot \int f^{rp} \cdot w\, d\mu.\end{aligned}$$

Therefore τ_r becomes an operator on $M(wd\mu)$ and satisfies

$$\sup_{n\ge 0} \left\| \frac{1}{n+1}\sum_{i=0}^{n} \tau_r^i \right\|_{L^{rp}(wd\mu)} < \infty .$$

Thus by Theorem 1 there exists a positive constant C such that

$$\begin{aligned}\left(\sum_{i=0}^{k} |h_{-i}(x)|^{-p} J_{-i}(x)\, w(S^{-i}x)\right) \cdot \left(\sum_{i=0}^{k} [\,|h_i(x)|^{-p} J_i(x)\, w(S^i x)\,]^{\frac{-1}{(rp-1)}}\right)^{rp-1} \\ \le C\,(k+1)^{rp},\end{aligned}$$

for a.e. $x \in X$ and all $k \ge 0$. Since $0 < r \le 1$ and $1 < rp$, it follows from [15], as in the above proof of (a) of Theorem 6, that

(i) the limit $\lim_{n\to\infty} R_{n,r}(\tau) f$ exists a.e. on the set $\{x : w(x) > 0\}$ for every f in $L^p(wd\mu)$, where τ is the linear modulus of T, and

(ii) the maximal operator $M_r^+(\tau)$ is bounded in $L^p(wd\mu)$.

Thus (a) follows similarly, as in (a) of Theorem 6.

(b) We may assume as before that $X = \{x : 0 < w(x) < \infty\}$, and hence T can be considered to be an invertible Lamperti operator on $M(wd\mu) = M(\mu)$. As in (a), we observe that

$$\sup_{n\ge 0} \left\| \frac{1}{2n+1}\sum_{i=-n}^{n} \tau_r^i \right\|_{L^{rp}(wd\mu)} \le K(p)^{1/r}.$$

Thus (b) follows from [5] when $1/p < r < 1$, and from [19] when $1/p < r = 1$.

This completes the proof of Theorem 8. □

The next proposition may be considered to be a supplementary result to Theorem 1.

Proposition. *Let $0 \le w \le \infty$ on X and let $1 \le p < \infty$. Then the following statements hold for an invertible Lamperti operator T on $M(\mu)$.*

(a) *T becomes an operator on $M(wd\mu)$ and satisfies the norm condition*

$$K^+(p) := \sup_{n \ge 0} \|T^n\|_{L^p(wd\mu)} < \infty$$

if and only if there exists a positive constant C such that for a.e. $x \in X$ and all $n \ge 0$

$$|h_{-n}(x)|^{-p} J_{-n}(x)\, \Phi^{-n} w(x) \le C\, w(x)\,. \tag{64}$$

(b) *The linear modulus τ of T becomes an operator on $M(wd\mu)$ and satisfies the norm condition*

$$\sup_{n \ge 0} \|\tau_{0,n}\|_{L^1(wd\mu)} < \infty$$

if and only if there exists a positive constant C such that for a.e. $x \in X$ and all $n \ge 0$

$$\frac{1}{n+1} \sum_{i=0}^{n} |h_{-i}(x)|^{-1} J_{-i}(x)\, \Phi^{-i} w(x) \le C\, w(x)\,. \tag{65}$$

Proof. (a) By (4) we may assume without loss of generality that T is positive. Then for $0 \le f \in M(\mu)$ and $n \ge 0$ we have , by (2) and (3),

$$\|T^n f\|^p_{L^p(wd\mu)} = \int (T^n f)^p \cdot w\, d\mu = \int f^p \cdot (|h_{-n}|^{-p} J_{-n}\, \Phi^{-n} w)\, d\mu\,. \tag{66}$$

Thus (64) implies that T becomes an operator on $M(wd\mu)$ and satisfies the norm condition: $K^+(p) < \infty$. Conversely if T is an operator on $M(wd\mu)$ and satisfies the norm condition: $K^+(p) < \infty$, then for $f = \chi_A$ with $A \in \mathcal{F}$ we have by (66)

$$\int_A (|h_{-n}|^{-p} J_{-n}\, \Phi^{-n} w)\, d\mu = \int (T^n \chi_A)^p\, w\, d\mu \tag{67}$$

$$\le \|T^n\|^p_{L^p(wd\mu)} \cdot \int_A w\, d\mu \le (K^+(p))^p \cdot \int_A w d\mu\,.$$

This completes the proof of (a).

(b) We may assume, as above, that $\tau = T$. Then for $0 \le f \in M(\mu)$ and

$n \geq 0$ we have, using (66) with $p = 1$, that

$$(68) \qquad \|\tau_{0,n}f\|_{L^1(wd\mu)} = \int (\tau_{0,n}f) \cdot w \; d\mu = \int f \cdot \left(\frac{1}{n+1} \sum_{i=0}^{n} |h_{-i}|^{-1} J_{-i} \, \Phi^{-i} w \right) \; d\mu .$$

Thus (65) implies that τ becomes an operator on $M(wd\mu)$ and satisfies the norm condition.

Conversely if τ is an operator on $M(wd\mu)$ and satisfies

$$C := \sup_{n \geq 0} \|\tau_{0,n}\|_{L^1(wd\mu)} < \infty,$$

then for $f = \chi_A$ with $A \in \mathcal{F}$ we have

$$(69) \quad \int_A \left(\frac{1}{n+1} \sum_{i=0}^{n} |h_{-i}|^{-1} J_{-i} \, \Phi^{-i} w \right) \; d\mu = \int (\tau_{0,n}\chi_A) \cdot w \; d\mu \leq \|\tau_{0,n}\|_{L^1(wd\mu)} \cdot \int_A w \; d\mu \leq C \int_A w d\mu.$$

Hence (65) follows, and the proof is complete. □

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