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# ACTIONS ON SPACES OF POLYNOMIALS

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## §1. Introduction

For $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, we denote by $Q^d_{(n)}(\mathbb{K})$ the space consisting of all $n$-tuples $(p_1(z),\cdots,p_n(z))\in\mathbb{K}[z]^n$ of $\mathbb{K}$-coefficients monic polynomials of degree $d$ such that $p_1(z),\cdots,p_n(z)$ have no common *real* roots (but may have common *complex* roots). It is important and valuable to study its topology from the point of view of singularity theory and algebraic topology ([3], [7]). For example, R. Cohen, J.D.S Jones and G. Segal considered the topology of $Q^d_{(n)}(\mathbb{C})$ for their study of Floer homotopy types in section 4 of [2]. Recently A. Kozlowski and the author investigated the homotopy types of $Q^d_{(n)}(\mathbb{K})$ in [5].

In this paper, we shall only consider the case $\mathbb{K}=\mathbb{R}$. Let $Q^d_{(n)}$ be the space consisting of all $n$-tuples $(p_1(z),\cdots,p_n(z))\in\mathbb{R}[z]^n$ of real coefficients polynomials which satisfy the following 3 conditions

(i) $p_n(z)$ is a monic polynomial of degree $d$.
(ii) $p_1(z),\cdots,p_n(z)$ has no common *real* roots.
(iii) $\max\{\deg(p_j):1\leq j\leq n-1\}<d$.

Since the map

$$\begin{array}{ccc} Q^d_{(n)}(\mathbb{R}) & \longrightarrow & Q^d_{(n)} \\ (p_1(z),\cdots,p_{n-1}(z),p_n(z)) & \longrightarrow & (p_n(z)-p_1(z),\cdots,p_n(z)-p_{n-1}(z),p_n(z)) \end{array}$$

gives a homeomorphism $Q^d_{(n)}(\mathbb{R})\cong Q^d_{(n)}$ and it is convenient to consider $Q^d_{(n)}$ instead from the point of view of group actions, we shall only consider the space $Q^d_{(n)}$. For an integer $d$, let $[d]_2$ be 0 or 1 according as $d$ is even or odd. Let us consider the map

$$j^d_{(n)}:Q^d_{(n)}\to\Omega_{[d]_2}\mathbb{R}\mathrm{P}^{n-1}\simeq\Omega S^{n-1}$$

given by

$$j^d_{(n)}(p_1,\cdots,p_n)(t)=\begin{cases}[p_1(t):p_2(t):\cdots:p_n(t)] & \text{if} \quad t\in\mathbb{R}\\ [0:0:\cdots:0:0:1] & \text{if} \quad t=\infty\end{cases}$$

for $t\in S^1=\mathbb{R}\cup\infty$, $(p_1,\cdots,p_n)\in Q^d_{(n)}(\mathbb{K})$.

Then note the following result.

**Theorem 1.1** ([5],[8]). *If $n\geq 3$, the map $j^d_{(n)}:Q^d_{(n)}\to\Omega S^{n-1}$ is a homotopy equivalence up to dimension $D_n(d)=(n-2)(d+1)-1$, where a*

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*map $f : X \to Y$ is called a homotopy equivalence up to dimension $N$ if the induced homomorphism $f_* : \pi_k(X) \to \pi_k(Y)$ is bijective when $k < N$ and surjective when $k = N$.* □

Since $D_n(d) \to \infty$ (when $d \to \infty$), $Q^d_{(n)}$ can be regarded as a finite dimensional model of infinite dimensional space $\Omega S^{n-1}$. In fact, more precisely we have proved:

**Theorem 1.2** ([5],[8]). (i). *If $n \geq 4$, there is a homotopy equivalence $Q^d_{(n)} \simeq J_d(\Omega S^{n-1})$.*

(ii). *If $n = 3$ and $d = 2m+1$, there is a homotopy equivalence $Q^{2m+1}_{(3)} \simeq S^1 \times J_m(\Omega S^3)$.*

(iii). *If $n = 3$ and $d = 2m$, there is a homotopy equivalence $\Sigma Q^{2m}_{(3)} \simeq \Sigma J_{2m}(\Omega S^2)$, where $\Sigma$ denotes the reduced suspension.*

*Here we denote by $J_m(\Omega S^{k+1})$ the $m$-th stage James filtration of the loop space $\Omega S^{k+1}$,*

$J_m(\Omega S^{k+1}) = S^k \cup e^{2k} \cup \cdots \cup e^{(m-1)k} \cup e^{mk} \subset S^k \cup e^{2k} \cup e^{3k} \cup \cdots = \Omega S^{k+1}$ □

The author would like to study group actions on the space $Q^d_{(n)}$ and the homotopy type of its orbit space for $n = 1 + 2^l$ ($l = 1, 2, 3$), for then there is a homotopy equivalence $\Omega S^{n-1} \simeq S^{n-2} \times \Omega S^{2n-3}$ ([1]). For the case $l = 1$ (i.e. the case $n = 3$) we already obtained the following result:

**Theorem 1.3** ([8]). *The multiplication of $\mathbb{C}$ induces a $SO_2$ action on $Q^d_{(3)}$ such that there exists a homotopy equivalence $Q^{2m+1}_{(3)}/SO_2 \simeq J_m(\Omega S^3)$.* □

So the author hopes to study the remaining case $n = 1 + 2^l$ (with $l = 2, 3$), since this problem would be related to the multiplication of quaternion field $\mathbb{H}$ if $l = 2$ and that of Cayley division ring $C$ if $l = 3$. However, because the distributive law does not hold on $C$, the case $l = 3$ seems difficult. So in this paper, we shall only study the only case $l = 2$, i.e. the case $n = 5$.

Let us consider the $SU_2$ action on $Q^d_{(5)}$ given by

$$\text{(1.4)} \qquad \begin{array}{ccc} Q^d_{(5)} \times SU_2 & \longrightarrow & Q^d_{(5)} \\ ((p_1, p_2, p_3, p_4, p_5), A) & \longrightarrow & (q_1, q_2, q_3, q_4, p_5) \end{array}$$

where for $A = \begin{pmatrix} \alpha & -\beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU_2$ (with $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$), the real coefficients polynomials $q_k(z) \in \mathbb{R}[z]$ ($k = 1, 2, 3, 4$) are given by the equation

$$\begin{pmatrix} q_1(z) + i \cdot q_2(z) \\ q_3(z) + i \cdot q_4(z) \end{pmatrix} = A \begin{pmatrix} p_1(z) + i \cdot p_2(z) \\ p_3(z) + i \cdot p_4(z) \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} p_1(z) + i \cdot p_2(z) \\ p_3(z) + i \cdot p_4(z) \end{pmatrix}$$

If $d$ is odd, then there is a fibration sequence

$$\text{(1.5)} \qquad S^3 = SU_2 \xrightarrow{\gamma} Q^d_{(5)} \xrightarrow{q} Q^d_{(5)}/SU_2$$

The main results of this paper are as follows:

**Theorem 1.4.** *If $d = 2m+1$, then the fibration (1.5) is trivial. Hence there is a homeomorphism $Q_{(5)}^{2m+1} \cong SU_2 \times Q_{(5)}^{2m+1}/SU_2$.*

**Theorem 1.5.** *There is a homotopy equivalence $Q_{(5)}^{2m+1}/SU_2 \simeq J_m(\Omega S^7)$.*

The idea of the proof is to define the splitting of the fibraion sequence (1.5) using the multiplication of $\mathbb{H}$.

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## §2. Proofs of theorems A and B

Let $\mathbb{H} = \mathbb{R} \oplus i \cdot \mathbb{R} \oplus j \cdot \mathbb{R} \oplus k \cdot \mathbb{R}$ be the quaternion filed. From now on we identify

$$SU_2 \cong S^3 = \{x_1 + i \cdot x_2 + j \cdot x_3 + k \cdot x_4 \in \mathbb{H} : x_l \in \mathbb{R}, \sum_{m=1}^{4} x_m^2 = 1\}$$

First, we prove theorem A.

*Proof of theorem A.*
Remark that any element $w \in S^3$ can be written as

$$\begin{aligned} w &= \cos\theta_1 \cdot e^{i\theta_2} + j \cdot \sin\theta_1 \cdot e^{i\theta_3} \\ &= \cos\theta_1 \cos\theta_2 + i \cdot \cos\theta_1 \sin\theta_2 + j \cdot \sin\theta_1 \cos\theta_3 - k \cdot \sin\theta_1 \sin\theta_3 \end{aligned}$$

Hence we can choose the inclusion map $\gamma : S^3 \to Q_{(5)}^{2m+1}$ as

$$\gamma(w) = (r_1(z), r_2(z), r_3(z), r_4(z), z(z^2+1)^m)$$

for $w = \cos\theta_1 \cdot e^{i\theta_2} + j \cdot \sin\theta_1 \cdot e^{i\theta_3} \in S^3$, where polynomials $r_s(z) \in \mathbb{R}[z]$ $(s = 1,2,3,4)$ are given by

$$r_s(z) = \begin{cases} z + \cos\theta_1 \cos\theta_2 & (s=1) \\ z + \cos\theta_1 \sin\theta_2 & (s=2) \\ z + \sin\theta_1 \cos\theta_3 & (s=3) \\ z - \sin\theta_1 \sin\theta_3 & (s=4) \end{cases}$$

Let $(p_1(z), p_2(z), p_3(z), p_4(z), p_5(z)) \in Q_{(5)}^{2m+1}$ be any element. The monic polynomial $p_5(z) \in R[z]$ can be written as

$$p_5(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_s)g(z)$$

where $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s$ and $g(z) \in \mathbb{R}[z]$ is a monic polynomila of degree $d-s$ such that $g(z)=0$ has no real roots.

We put

$$R_1(p_1,p_2,p_3,p_4,p_5) = Q(\alpha_1)^{\epsilon(1)}Q(\alpha_2)^{\epsilon(2)}\cdots Q(\alpha_s)^{\epsilon(s)}$$

where $\epsilon(t) = (-1)^{t-1}$ and $Q(\alpha) = p_1(\alpha)+i\cdot p_2(\alpha)+j\cdot p_3(\alpha)+k\cdot p_4(\alpha) \in \mathbb{H}$.

Since $p_1(z) = \cdots = p_5(z) = 0$ has no common real roots, $Q(\alpha_t) \neq 0$ for any $1 \leq t \leq s$. Moreover, if $\alpha_t = \alpha_{t+1}$, $Q(\alpha_t)^{\epsilon(t)}Q(\alpha_{t+1})^{\epsilon(t+1)} = 1$. Hence the map

$$R_1 : Q_{(5)}^{2m+1} \longrightarrow \mathbb{H}^* = \mathbb{H} - \{0\}$$

is continuous. Define the map $R : Q^{2m+1} \to S^3$ by

$$R(p_1,p_2,p_3,p_4,p_5) = \frac{R_1(p_1,p_2,p_3,p_4,p_5)}{|R_1(p_1,p_2,p_3,p_4,p_5)|}$$

An easy computation shows that $R \circ \gamma = \mathrm{id} : S^3 \to S^3$. Hence the fibration $S^3 = SU_2 \xrightarrow{\gamma} Q_{(5)}^{2m+1} \xrightarrow{q} Q_{(5)}^{2m+1}/SU_2$ is a trivial fibration. This complete the proof. □

***Remark.*** *The above proof does not work if $d$ is an even integer. In fact, if $d$ is an even integer, a monic polynomial $p_5(z)$ of degree $d$ does not necessarily have real roots. So $R_1$ is not well-defined in general.*

Next, we probe theorem B.

*Proof of theorem B.*
It follows from theorem 1.2 that there is a homotopy equivalence $Q_{(5)}^{2m+1} \simeq J_{2m+1}(\Omega S^4)$. Since there is a homotopy equivalence $\Omega S^4 \simeq S^3 \times \Omega S^7$, there is a homotopy equivalence $J_{2m+1}(\Omega S^4) \simeq S^3 \times J_m(\Omega S^7)$. Hence $Q_{(5)}^{2m+1}$ and $S^3 \times J_m(\Omega S^7)$ are homotopy equivalent. Then from theorem A, there is a homotopy equivalence

$$f : Q_{(5)}^{2m+1}/S^3 \times S^3 \xrightarrow{\simeq} J_m(\Omega S^7) \times S^3$$

Let $\phi : Q_{(5)}^{2m+1}/S^3 \to J_m(\Omega S^7)$ be the composite of maps

$$Q_{(5)}^{2m+1}/S^3 \xrightarrow{i_1} Q_{(5)}^{2m+1}/S^3 \times S^3 \xrightarrow{\simeq} J_m(\Omega S^7) \times S^3 \xrightarrow{\pi_1} J_m(\Omega S^7)$$

where $i_1$ and $\pi_1$ denote the injection and projection to the first facter respectively. It is easy to see that $\phi_* : H_*(Q_{(5)}^{2m+1}/S^3, \mathbb{Z}) \xrightarrow{\cong} H_*(J_m(\Omega S^7), \mathbb{Z})$ is an isomorphism. Because both spaces are simply connected, $\phi$ is a homotopy equivalence. □

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