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ON MORAVA K-GROUPS OF STUNTED PROJECTIVE SPACES

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1. Introduction

Let $k(n)^*(-)$ be the connective Morava K-theory with coefficient ring $k(n)^* = Z/2[v_n]$ and let $K(n)^*(-)$ be the Morava K-theory with coefficient ring $K(n)^* = Z/2[v_n, v_n{}^{-1}]$, where $|v_n| = -2(2^n - 1)$. In this paper we determine the module structure of $k(n)^*(RP^l_{m+1})$ and the algebra structure of $k(n)^*(RP^l)$ over $k(n)^*$ at the prime 2. Here, the symbol RP^l_{m+1} denotes a stunted real projective space RP^l/RP^m $(0 \leq m < l \leq \infty)$.

Our principal tool for computing $k(n)^*(RP^l_{m+1})$ is the Atiyah-Hirzebruch spectral sequence(AHSS)

$$E_2^{*,*} = H^*(RP^l_{m+1}; k(n)^*) \Longrightarrow k(n)^*(RP^l_{m+1})$$

and we use the result of Yagita([1],Lemma 2.1) for further computation.

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2. Statement of Results And Proofs

In this section, we determine the module structure of the Morava K-theory of the stunted real projective space and the algebra structure of the Morava K-theory of the real projective space over $k(n)^*$ at the prime 2.

The following lemma is needed for computation of the differentials of AHSS for $k(n)^*(RP^\infty_{m+1})$.

Lemma 2.1[Yagita] *Let X be a CW-complex. Let E_r be the Atiyah-Hirzebruch spectral sequence for $k(n)^*(X)$*

$$E_2^{s,t} = H^s(X; k(n)^t) \Longrightarrow k(n)^{s+t}(X)$$

and its differential be $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$. Let $u \in H^*(X)$, then $d_r(u) =$ 0 for $r < 2^{n+1} - 1$ and

$$d_{2^{n+1}-1}(u) = Q_n(u) \otimes v_n.$$

where $Q_0 = Sq^1$, $Q_n = Sq^{2^n}Q_{n-1} + Q_{n-1}Sq^{2^n} (n \geq 1)$. □

First we determine differentials of AHSS for $k(n)^*(RP^\infty_{m+1})$

$$E_2^{p,q}(RP^\infty_{m+1}) = H^p(RP^\infty_{m+1}; k(n)^q) \Longrightarrow k(n)^{p+q}(RP^\infty_{m+1}).$$

Let us consider the following cofibration

$$RP^m \overset{i}{\hookrightarrow} RP^\infty \overset{\pi}{\to} RP^\infty/RP^m = RP^\infty_{m+1}$$

from which we get the following exact sequence

$$0 \longrightarrow \tilde{H}^*(RP^\infty_{m+1}) \xrightarrow{\pi^*} \tilde{H}^*(RP^\infty) \xrightarrow{i^*} \tilde{H}^*(RP^m) \longrightarrow 0.$$

Throughout this paper all cohomology groups have $Z/2$ coeffcients unless otherwise stated. Here we recall that $H^*(RP^\infty) = Z/2[u]$ where $u \in H^1(RP^\infty)$ and, via π^*, we may identify $\tilde{H}^*(RP^\infty_{m+1})$ with the submodule of $\tilde{H}^*(RP^\infty)$ generated by $u^{m+1}, u^{m+2}, \cdots$.

PROPOSITION 2.2 *In the AHSS for* $\tilde{k}(n)^*(RP^\infty_{m+1})$,

1. $d_{2^{n+1}-1}(u^{2i}) = 0$ *for all* $2i \geq m+1$.
2. $d_{2^{n+1}-1}(u^{2j+1}) = v_n u^{2j+2^{n+1}}$ *for all* $2j+1 \geq m+1$.
3. *There are isomorphisms*

$$\tilde{E}^{*,*}_{2^{n+1}} \cong \cdots \cong \tilde{E}^{*,*}_\infty \cong k(n)^*\{u^{2i} | m+1 \leq 2i \leq m+2^{n+1}-1\} \oplus k(n)^*\{u^{2k} | m+2^{n+1} \leq 2k\}/(v_n).$$

PROOF. By induction on n we can easily compute the action of Q_n on $H^*(RP^\infty)$ and hence on $H^*(RP^\infty_{m+1})$. In fact we have $Q_n(u^{2i}) = 0$ and $Q_n(u^{2i+1}) = u^{2i+2^{n+1}}$. Thus by the lemma 2.1 we get

(1) $$d_{2^{n+1}-1}(u^{2i}) = 0 \quad \text{for any } 2i \geq m+1$$

and

(2) $$d_{2^{n+1}-1}(u^{2j+1}) = v_n u^{2j+2^{n+1}} \quad \text{for any } 2j+1 \geq m+1.$$

This proves 1 and 2.
Next we prove 3. From (1), $v_n^a u^{2i}$ is cycle for any $a \geq 0$ and $2i \geq m+1$. If $a = 0$, then u^{2i} can not be boundary. If $a \geq 1$ and $m+1 \leq 2i \leq m+2^{n+1}-1$, then for dimensional reason, $v_n^a u^{2i}$ can not be boundary. If $a \geq 1$ and $2i \geq m+2^{n+1}$, then since $v_n^a u^{2i} = d_{2^{n+1}-1}({v_n}^{a-1} u^{2i-2^{n+1}+1})$, from (2), $v_n^a u^{2i}$ is boundary. Therefore $\tilde{E}_{2^{n+1}} \cong k(n)^*\{u^{2i} | m+1 \leq 2i \leq m+2^{n+1}-1\} \oplus k(n)^*\{u^{2k} | m+2^{n+1} \leq 2k\}/(v_n)$. Since $E_{2^{n+1}}$-term is concentrated in even degree we have $d_r = 0$ for any $r \geq 2^{n+1}$. Thus $\tilde{E}_{2^{n+1}} \cong \cdots \cong \tilde{E}_\infty$. □

Next we consider RP^l_{m+1}. Let $m < l$ and recall that

(3) $$H^*(RP^l_{m+1}) = Z/2\{u^{m+1}, \cdots, u^l\}.$$

Let $\imath : RP^l_{m+1} \hookrightarrow RP^\infty_{m+1}$ be the standard inclusion. Then induced homomorphism

$$\imath^* : H^*(RP^\infty_{m+1}) \to H^*(RP^l_{m+1})$$

is epimorphism.

PROPOSITION 2.3 *$\tilde{E}_\infty$-term of the Atiyah-Hirzebruch spectral sequence for $\tilde{k}(n)^*(RP^l_{m+1})$ is given by:*

$$\tilde{E}_\infty \cong k(n)^*\{u_{2i}, u_{2j+1} | m + 1 \le 2i \le M, L \le 2j + 1 \le l\}$$
$$\oplus k(n)^*\{u_{2k} | m + 2^{n+1} \le 2k \le l\}/(v_n)$$

as $k(n)^*$-module where $M = Min(l, m + 2^{n+1} - 1)$, $L = Max(m + 1, l - 2^{n+1} + 2)$ and u_i denotes the class represented by u^i.

PROOF. For similarity, we only prove the proposition for the case : l is odd and m is even.

(i). If $l < m + 2^{n+1}$, then for dimensional reason, all differentials are zero. Therefore $\tilde{E}_2 = \cdots = \tilde{E}_\infty \cong k(n)^*\{u^{m+1}, \cdots, u^l\}$.

(ii). If $l \ge m + 2^{n+1}$, then since $\imath^*$ is epimorphism, the generators u^{m+2k} are permanent cycle in AHSS for $\tilde{k}(n)^*(RP^l_{m+1})$ for $k = 1, 2, \cdots, l-1$. For dimensional reason, the generators u^{m+2k+1} are cycles for $l - 2^{n+1} + 2 \le m + 2k + 1 \le l$. Also since $v_n^a u^{m+2k+1}$ can not be boundary, from (1) and (2), $v_n^a u^{m+2k+1}$ is cycle and is not boundary for $l - 2^{n+1} + 2 \le m + 2k + 1 \le l$. Therefore

$$\tilde{E}_{2^{n+1}} \cong k(n)^*\{u^{m+2}, \cdots, u^{m+2^{n+1}-2}; u^{l-2^{n+1}+2}, \cdots, u^l\}$$
$$\oplus k(n)^*\{u^{m+2^{n+1}}, \cdots, u^{l-1}\}/(v_n).$$

For dimensional reason and the fact that u^{m+2k} are all permanent cycles, we have $d_r = 0$ for all $r \ge 2^{n+1}$. Thus $\tilde{E}_{2^{n+1}} = \cdots = \tilde{E}_\infty$ and we get the proposition. □

Let us consider the following filtration of $\tilde{k}(n)^*(RP^l_{m+1})$

(4) $$\tilde{k}(n)^*(RP^l_{m+1}) = \tilde{F}^{0,*} \supset \cdots \supset \tilde{F}^{s,*-s} \supset \tilde{F}^{s+1,*-s-1} \supset \cdots$$

where $\tilde{F}^{s,*-s} = ker\{\tilde{k}(n)^*(RP^l_{m+1}) \longrightarrow \tilde{k}(n)^*((RP^l_{m+1})^{s-1})\}$ is a $k(n)^*$-submodule of $\tilde{k}(n)^*(RP^l_{m+1})$. Then since $\tilde{F}^{l+1,*-l-1} = 0$, the filtration (4) is finite. Note that $\tilde{E}^{s,*}_\infty \simeq \tilde{F}^{s,*}/\tilde{F}^{s+1,*-1}$.

LEMMA 2.4 $\tilde{F}^{s,*} \cong \tilde{F}^{s+1,*-1} \oplus \tilde{E}^{s,*}_\infty$ *as $k(n)^*$-module.*

PROOF. We prove that the following exact sequence splits as $k(n)^*$-module;

$$0 \longrightarrow \tilde{F}^{s+1,*-1} \longrightarrow \tilde{F}^{s,*} \xrightarrow{\pi} \tilde{E}_\infty^{s,*} \longrightarrow 0.$$

By Proposition 2.3,

(5)

$$\tilde{E}_\infty^{s,*} \cong \begin{cases} k(n)^* & \text{if } s \text{ is even and } m+1 \le s \le Min(l, m+2^{n+1}-1) \\ k(n)^* & \text{if } s \text{ is odd and } Max(m+1, l-2^{n+1}+2) \le s \le l \\ k(n)^*/(v_n) & \text{if } s \text{ is even and } m+2^{n+1} \le s \le l \\ 0 & \text{otherwise} \end{cases}$$

If s is not in the third case, the exact sequence clearly splits. Suppose that s is even and $m + 2^{n+1} \le s \le l$. Let $u_s \in \tilde{E}_\infty^{s,0}$ be as in Proposition2.3. Then since π is surjective, there exists an element $x \in \tilde{F}^{s,0}$ such that $\pi(x) = u_s$. To prove the lemma it suffices to show that $v_n x = 0$. Since $\pi(v_n x) = v_n \pi(x) = v_n u_s = 0$, $v_n x$ is in $\tilde{F}^{s+1,-2(2^n-1)-1}$. By Proposition 2.3, it is easy to see that $\tilde{E}_\infty^{s+k,-2(2^n-1)-k} = 0$ for all $k \ge 1$. Thus $\tilde{F}^{s+1,-2(2^n-1)-1} = 0$ and hence $v_n x = 0$. □

From Lemma 2.4, we have the following Theorem.

THEOREM 2.5 $\tilde{k}(n)^*(RP^l_{m+1}) \cong \bigoplus_s \tilde{E}_\infty^{s,*-s}$ *as graded* $k(n)^*$*-module.*

COROLLARY 2.6 *As* $k(n)^*$*-module*

$$\tilde{k}(n)^*(RP^l_{m+1}) \cong k(n)^*\{x_{2i}, x_{2j+1} | m+1 \le 2i \le M, L \le 2j+1 \le l\}$$
$$\oplus k(n)^*\{x_{2k} | m+2^{n+1} \le 2k \le l\}/(v_n)$$

where $M = Min(l, m + 2^{n+1} - 1)$, $L = Max(m+1, l - 2^{n+1} + 2)$ *and* x_i *corresponds to* u_i *under the above isomorphism.*

THEOREM 2.7 *There exists a ring isomorphism* $E_\infty \cong k(n)^*(RP^l)$.

PROOF. $E_\infty^{*,*}$ is generated by 1 and u_i as $k(n)^*$-algebra and

$$u_i \cdot u_j = \begin{cases} u_{i+j} & \text{if } i+j \text{ is in the range given in Prop2.3.} \\ 0 & \text{if } i+j \text{ is out of the range.} \end{cases}$$

Let $x_i \in k(n)^*(RP^l)$ be as in Cor2.6. It is clear that both $x_i \cdot x_j$ and x_{i+j} represent u_{i+j} if $i+j$ is in the range given in Prop2.3 and $x_i \cdot x_j$ represents 0 if $i+j$ is out of the range. First we consider first case. In this case, $x_i \cdot x_j - x_{i+j}$ is in the higher filtration. But we can prove that the higher filtration is zero in the degree of $x_i \cdot x_j$ by using Prop2.3 as in the proof of Lem2.4 and hence $x_i \cdot x_j - x_{i+j} = 0$. Therefore $x_i \cdot x_j = x_{i+j}$. Similarly we have $x_i \cdot x_j = 0$ in the second case. This proves the theorem. □

THEOREM 2.8

1. *Let* $l < 2^{n+1}$, *then*
$k(n)^*(RP^l) = k(n)^*[u]/(u^{l+1})$
where the class u *represents* x_1.
2. *Let* l *be an odd integer such that* $l > 2^{n+1}$, *then*
$k(n)^*(RP^l) = k(n)^*[x,y]/(v_n x^{2^n}, y^2 - x^{l-2^{n+1}+2}, x^{\frac{l+1}{2}}, yx^{2^n})$
where $x = x_2$, $y = x_{l-2^{n+1}+2}$.
3. *Let* l *be an even integer such that* $l \geq 2^{n+1}$, *then*
$k(n)^*(RP^l) = k(n)^*[x,z]/(v_n x^{2^n}, z^2 - x^{l-2^{n+1}+3}, x^{\frac{l+2}{2}}, z \cdot x^{2^n-1})$
where $z = x_{l-2^{n+1}+3}$.

PROOF 1. Trivial. 2. Let $\psi : k(n)^*[x,y] \longrightarrow k(n)^*(RP^l)$ be a $k(n)^*$-algebra homomorphism given by $\psi(x) = x_2$ and $\psi(y) = x_{l-2^{n+1}+2}$. Then it is easy to see that ψ is surjective and $(v_n x^{2^n}, y^2 - x^{l-2^{n+1}+2}, x^{\frac{l+1}{2}}, yx^{2^n}) \subset \ker \psi$. By comparing graded dimension over $\mathbb{F}_2$, we see that ψ induces an isomorphism $k(n)^*[x,y]/(v_n x^{2^n}, y^2 - x^{l-2^{n+1}+2}, x^{\frac{l+1}{2}}, yx^{2^n}) \xrightarrow{\cong} k(n)^*(RP^l)$. 3. The proof is similar to 2. □

Using $K(n)^*(X) = k(n)^*(X)[v_n{}^{-1}]$, $K(n)^*(RP^l)$ and $K(n)^*(RP^l_{m+1})$ will immediately be deduced.

THEOREM 2.9 *As* $K(n)^*$*-module* $\tilde{K}(n)^*(RP^l_{m+1})$ *is given by:*

$$\tilde{K}(n)^*(RP^l_{m+1}) \cong K(n)^*\{x_{2i}, x_{2j+1} | m+1 \leq 2i \leq M, L \leq 2j+1 \leq l\}$$

where $M = Min(l, m + 2^{n+1} - 1)$ *and* $L = Max(m+1, l - 2^{n+1} + 2)$.

THEOREM 2.10
1. *Let* $l < 2^{n+1}$, *then*
$K(n)^*(RP^l) = K(n)^*[u]/(u^{l+1})$
where the class u *represents* x_1.
2. *Let* l *be an odd integer such that* $l > 2^{n+1}$, *then*
$K(n)^*(RP^l) = K(n)^*[x,y]/(x^{2^n}, y^2 - x^{l-2^{n+1}+2})$
where $x = x_2$, $y = x_{l-2^{n+1}+2}$.
3. *Let* l *be an even integer such that* $l \geq 2^{n+1}$, *then*
$K(n)^*(RP^l) = K(n)^*[x,z]/(x^{2^n}, z \cdot x^{2^n-1}, z^2 - x^{l-2^{n+1}+3})$
where $z = x_{l-2^{n+1}+3}$.

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