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THE QUASI KO_*-TYPES OF STUNTED MOD 8 LENS SPACES

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0. Introduction

Let E be a ring spectrum with unit. Given CW-spectra X and Y we say that X is quasi E_*-equivalent to Y if $E \wedge X$ is equivalent to $E \wedge Y$ as an E-module spectrum (see [14]). In this case there exists a map $f : Y \to E \wedge X$ inducing an isomorphism $f_* : E_*Y \to E_*X$, which is called a quasi E_*-equivalence. For the real K-spectrum KO we are interested in the quasi KO_*-types of the mod q lens space L^n and its stunted space $L^n_{l+1} = L^n/L^l$ when q is a power of 2. Here we simply denote by L^{2m+1} the $(2m+1)$-dimensional standard mod q lens space $L^m(q)$ and by L^{2m} its $2m$-skeleton $L^m_0(q)$. In [16] and [18] we have completely determined the quasi KO_*-types of the stunted mod 2 and mod 4 lens spaces. In this paper we shall determine the quasi KO_*-types of the stunted mod 8 lens spaces.

The complex K-spectrum KU possesses the conjugation ψ^{-1}_C. Given CW-spectra X and Y we say that X has the same $\mathcal{C}$-type as Y if KU_*X is isomorphic to KU_*Y as an abelian group with involution ψ^{-1}_C. In the previous papers [16] and [18] we investigated all the $\mathcal{C}$-types of the stunted mod 2 and mod 4 lens spaces L^n_{l+1} in order to determine their quasi KO_*-types. However it is very bothersome to investigate all the $\mathcal{C}$-types of the stunted mod 8 lens spaces L^n_{l+1} because the behaviour of the conjugations ψ^{-1}_C on $KU^0L^n_{l+1}$ is complicated. To avoid a wasted effort in this paper we shall discuss several basic properties about the quasi KO_*-types of the stunted mod q lens spaces L^n_{l+1} (Propositions 1 and 2) by using Thom complexes and weighted projective spaces as stated below. These results assert that it is sufficient for us to determine only the quasi KO_*-types of $L^{2m} = L^m_0(8)$ and $L^{4s}_3 = L^{2s}_0(8)/L^1_0(8)$ in order to establish our purpose (Theorems 2.8, 2.10 and 2.11) completely.

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Let ξ_{2m+1} be the canonical complex line bundle over $L^m(q)$ and ξ_{2m} its restriction onto $L_0^m(q)$. The stunted mod q lens space L_{2k}^{2k+n} is cellular homeomorphic to the Thom complex $T(k\epsilon_O(\xi_n))$, and L_{2k+1}^{2k+n} is so to $T(k\epsilon_O(\xi_n))/\Sigma^{2k}$, where ϵ_O stands for the realification. On the other hand, the S-dual $DL_{2k}^{2k+2m+1}$ is (stably) homotopy equivalent to the Thom complex $T(-k\epsilon_O(\xi_{2m+1}) - \tau)$ where τ denotes the tangent vector bundle over $L^m(q)$. Using these facts we can show

Proposition 1. i) *The stunted mod q lens space L_{l+4}^{n+4} is quasi KO_*-equivalent to $\Sigma^4 L_l^n$. In particular, L_{4m+1}^{4m+n} and L_{4m}^{4m+n} are quasi KO_*-equivalent to $\Sigma^{4m}L^n$ and $\Sigma^{4m} \vee \Sigma^{4m}L^n$, respectively. (Proposition 2.3).*

ii) *The S-dual DL_l^{4s+r} is quasi KO_*-equivalent to $\Sigma^{-4s-3}L_{3-r}^{4s+3-l}$ when $0 \leq r \leq 3$. (Proposition 2.4).*

The mod q lens spaces $L_0^m(q)$ and $L^m(q)$ are exhibited as the fibers of certain maps $i_0 : P^m \to P^{1,m-1}$ and $i : P^m \to P^{1,m}$, respectively. Here $P^m = P^m(1,1,\cdots,1)$ is the usual complex projective space of dimension m, and $P^{1,m-1} = P^m(q,1,\cdots,1)$ is the weighted projective space of type $(q,1,\cdots,1)$. Since the quasi KO_*-types of weighted projective spaces have been determined in [12], we can show

Proposition 2. *When q is even, the stunted mod q lens space L_l^{4s-1} is quasi KO_*-equivalent to the wedge sum $L_l^{4s-2} \vee \Sigma^{4s-1}$, but L_l^{4s+1} is not so to $L_l^{4s} \vee \Sigma^{4s+1}$. (Proposition 2.6).*

In §1 we construct new small spectra $PP'_{r,s,t,p,q}$, $vPP'_{r,s,t,p,q}$, $MPP'_{r,s,t,p,q}$ and so on appearing in Theorems 2.8, 2.10 and 2.11, and then study the behaviour of the conjugations ψ_C^{-1} on their KU-homology groups. Moreover we characterize the quasi KO_*-types of CW-spectra having the same C-type as $PP'_{r,s,t,p,q}$ ($vPP'_{r,s,t,p,q}$) or $MPP'_{r,s,t,p,q}$ by developing the same method as adopted in [15] or [18] (Theorem 1.7). In the first part of §2 we discuss such several properties of the stunted mod q lens spaces L_{l+1}^n as Propositions 1 and 2 by using Thom complexes and weighted projective spaces. In the latter part of §2 we investigate the behaviour of the conjugations ψ_C^{-1} on $KU^0(L^m(8))$ and $KU^0(L^{2s}(8)/L^1(8))$, and then dualize their results to determine the $\mathcal{C}$-types of $L^{2m} = L_0^m(8)$ and $L_3^{4s} = L_0^{2s}(8)/L_0^1(8)$ (Propositions 2.7 and 2.9). Applying the characterization given in Theorem 1.7 to L^m and L_3^{4s} we can easily determine their quasi KO_*-types. Consequently we can prove our main results (Theorems 2.8, 2.10 and 2.11) by virtue of Propositions 1 and 2, as is stated above.

1. Small spectra $PP'_{r,s,t,p,q}$ and $MPP'_{r,s,t,p,q}$

1.1. Let $SZ/2^m$ ($m \geq 1$) be the Moore spectrum of type $Z/2^m$, and $i : \Sigma^0 \to SZ/2^m$ and $j : SZ/2^m \to \Sigma^1$ be the bottom cell inclusion and the top cell projection, respectively. The stable Hopf map $\eta : \Sigma^1 \to \Sigma^0$ of order 2 admits an extension $\bar{\eta} : \Sigma^1 SZ/2^m \to \Sigma^0$ and a coextension $\tilde{\eta} : \Sigma^2 \to SZ/2^m$ satisfying $\bar{\eta} i = \eta$ and $j\tilde{\eta} = \eta$. Let us denote by $P_{m,n}$, $P'_{m,n}$, $P''_{m,n}$, ${}_VP_{m,n}$ and ${}_VP'_{m,n}$ the small spectra constructed as the cofibers of the following maps $\tilde{\eta} j$, $i\bar{\eta}$, $i\bar{\eta} + \tilde{\eta} j : \Sigma^1 SZ/2^n \to SZ/2^m$, $i_V\tilde{\eta} j : \Sigma^1 SZ/2^n \to V_m$ and $i\bar{\eta} j'_V : \Sigma^{-1} V'_n \to SZ/2^m$, respectively. Here we adopt the notations V_m and V'_n in place of $P'_{m-1,1}$ and $P_{1,n-1}$, respectively, and $i_V : SZ/2^{m-1} \to V_m$ is the canonical inclusion and $j'_V : V'_n \to \Sigma^2 SZ/2^{n-1}$ is the canonical projection. According to [11, Proposition 3.2] and its dual, the spectra ${}_VP_{m,n}$ and ${}_VP'_{m,n}$ are quasi KO_*-equivalent to $\Sigma^2 P_{n+1,m-1}$ and $\Sigma^6 P'_{n-1,m+1}$, respectively.

As in [13] (or [18]) we denote by $PP'_{r,s,p}$, $P'P_{p,r,s}$, ${}_VPP'_{r,s,p}$ and ${}_VP'P_{p,r,s}$ the small spectra constructed as the cofibers of the following maps

$$(\tilde{\eta} j, i\bar{\eta}) : \Sigma^1 SZ/2^p \to SZ/2^r \vee SZ/2^s, \quad i\bar{\eta} \vee \tilde{\eta} j : \Sigma^1 SZ/2^r \vee \Sigma^1 SZ/2^s \to SZ/2^p,$$
$$(i_V\tilde{\eta} j, i\bar{\eta}) : \Sigma^1 SZ/2^p \to V_r \vee SZ/2^s, \quad i\bar{\eta} j'_V \vee \tilde{\eta} j : \Sigma^{-1} V'_r \vee \Sigma^1 SZ/2^s \to SZ/2^p,$$

respectively. Evidently there hold the S-dualities $\Sigma^3 DPP'_{r,s,p} = P'P_{p,r,s}$ and $\Sigma^3 D_VPP'_{r,s,p} = {}_VP'P_{p,r,s}$. Note that $PP'_{r,0,p} = P_{r,p}$, ${}_VPP'_{r,0,p} = {}_VP_{r,p}$, ${}_VPP'_{1,s,p} = P'_{s,p+1}$ and their duals hold. In [18, Propositions 2.1 and 2.3] these small spectra were written as $U_{s,r,p}$, $U'_{p,r,s}$, $V_{s,r,p}$ and $V'_{p,r,s}$. According to [18, Corollary 3.4] the spectra $P'P_{p,r,s}$ and ${}_VP'P_{p,r,s}$ are quasi KO_*-equivalent to $\Sigma^2 PP'_{s+1,r-1,p}$ and $\Sigma^6 {}_VPP'_{s+1,r-1,p}$, respectively.

We introduce new small spectra $P_{r,s,p,q}$, $P'_{p,q,r,s}$, ${}_VP_{r,s,p,q}$, ${}_VP'_{p,q,r,s}$, $PP'_{r,s,t,p,q}$, $P'P_{p,q,r,s,t}$, ${}_VPP'_{r,s,t,p,q}$ and ${}_VP'P_{p,q,r,s,t}$ constructed as the cofibers of the following maps

$$(1.1)\quad \begin{array}{lcl} i\tilde{\eta}j_P \vee \tilde{\eta}j & : & \Sigma^{-1}P_{r,p} \vee \Sigma^1 SZ/2^q \to SZ/2^s, \\ (i'_P\tilde{\eta}j, i\tilde{\eta}) & : & \Sigma^1 SZ/2^s \to P'_{p,r} \vee SZ/2^q, \\ i\tilde{\eta}_V j_P \vee \tilde{\eta}j & : & \Sigma^{-1}{}_V P_{r,p} \vee \Sigma^1 SZ/2^q \to SZ/2^s, \\ ({}_V i'_P\tilde{\eta}j, i\tilde{\eta}) & : & \Sigma^1 SZ/2^s \to {}_V P'_{p,r} \vee SZ/2^q, \\ i\tilde{\eta}j_P \vee \tilde{\eta}jj'_P & : & \Sigma^{-1}P_{r,p} \vee \Sigma^{-1}P'_{t,q} \to SZ/2^s, \\ (i'_P\tilde{\eta}j, i_P i\tilde{\eta}) & : & \Sigma^1 SZ/2^s \to P'_{p,r} \vee P_{q,t}, \\ i\tilde{\eta}_V j_P \vee \tilde{\eta}jj'_P & : & \Sigma^{-1}{}_V P_{r,p} \vee \Sigma^{-1}P'_{t,q} \to SZ/2^s, \\ ({}_V i'_P\tilde{\eta}j, i_P i\tilde{\eta}) & : & \Sigma^1 SZ/2^s \to {}_V P'_{p,r} \vee P_{q,t}, \end{array}$$

respectively. Here $i_P : SZ/2^q \to P_{q,t}$, $i'_P : SZ/2^p \to P'_{p,r}$, ${}_V i'_P : SZ/2^p \to {}_V P'_{p,r}$ are the canonical inclusions, and $j_P : P_{r,p} \to \Sigma^2 SZ/2^p$, ${}_V j_P : {}_V P_{r,p} \to \Sigma^2 SZ/2^p$, $j'_P : P'_{t,q} \to \Sigma^2 SZ/2^q$ are the canonical projections. Evidently there hold the S-dualities $\Sigma^3 DP_{r,s,p,q} = P'_{p,q,r,s}$, $\Sigma^3 D_V P_{r,s,p,q} = {}_V P'_{p,q,r,s}$, $\Sigma^3 DPP'_{r,s,t,p,q} = P'P_{p,q,r,s,t}$ and $\Sigma^3 D_V PP'_{r,s,t,p,q} = {}_V P'P_{p,q,r,s,t}$. Note that $P_{r,s,p,0} = PP'_{r,s,p}$, ${}_V P_{r,s,p,0} = {}_V PP'_{r,s,p}$, ${}_V P_{1,s,p,q} = P'P_{s,p+1,q}$, $PP'_{r,s,0,p,q} = P_{r,s,p,q}$, ${}_V PP'_{r,s,0,p,q} = {}_V P_{r,s,p,q}$, ${}_V PP'_{1,s,t,p,q} = P'_{s,t,p+1,q}$ and their duals hold. The spectra $P_{r,s,p,q}$, $P'_{p,q,r,s}$, ${}_V P_{r,s,p,q}$, ${}_V P'_{p,q,r,s}$, $PP'_{r,s,t,p,q}$, $P'P_{p,q,r,s,t}$, ${}_V PP'_{r,s,t,p,q}$ and ${}_V P'P_{p,q,r,s,t}$ are exhibited as the cofibers of the following maps

$$(1.2)\quad \begin{array}{ll} \tilde{\eta}jj_{P'P} : \Sigma^{-1}P'P_{s,p,q} \to SZ/2^r, & i_{PP'}i\tilde{\eta} : \Sigma^1 SZ/2^r \to PP'_{p,q,s}, \\ i_V\tilde{\eta}jj_{P'P} : \Sigma^{-1}P'P_{s,p,q} \to V_r, & i_{PP'}i\tilde{\eta}j'_V : \Sigma^{-1}V'_r \to PP'_{p,q,s}, \\ \tilde{\eta}jj_{P'P}\pi'_P : \Sigma^{-1}P'_{s,t,p,q} \to SZ/2^r, & l_P i_{PP'}i\tilde{\eta} : \Sigma^1 SZ/2^r \to P_{p,q,s,t}, \\ i_V\tilde{\eta}jj_{P'P}\pi'_P : \Sigma^{-1}P'_{s,t,p,q} \to V_r, & l_P i_{PP'}i\tilde{\eta}j'_V : \Sigma^{-1}V'_r \to P_{p,q,s,t}, \end{array}$$

respectively. Here $i_{PP'} : SZ/2^p \to PP'_{p,q,s}$, $l_P : PP'_{p,q,s} \to P_{p,q,s,t}$ are the canonical inclusions, and $j_{P'P} : P'P_{s,p,q} \to \Sigma^2 SZ/2^p$, $\pi'_P : P'_{s,t,p,q} \to PP'_{s,p,q}$ are the canonical projections.

For $Y = P_{r,s,p,q}$ or ${}_V P_{r,s,p,q}$ we observe that $KU_1 Y = 0$ and $KU_0 Y$ is isomorphic to the direct sum $KU_0 PP'_{r,s,p} \oplus KU_0 \Sigma^2 SZ/2^q$ if $s > q$, to $KU_0 SZ/2^r \oplus KU_0 P'P_{s,p,q}$ if $r > p$, and to $KU_0 P_{r,p} \oplus KU_0 P_{s,q}$ if $r > p > s$, $r \le p \ge s$, $p > s > q$ or $p \ge s \le q$. For $X = PP'_{r,s,t,p,q}$ or ${}_V PP'_{r,s,t,p,q}$ similarly we observe that $KU_1 X = 0$ and $KU_0 X$ is isomorphic to the direct sum $KU_0 P_{r,s,p,q} \oplus KU_0 SZ/2^t$ if $q > t$, to $KU_0 SZ/2^r \oplus KU_0 P'_{s,t,p,q}$ if $r > p$, to $KU_0 PP'_{r,s,p} \oplus KU_0 P'_{t,q}$ if $s > q$, $s \ge q \le t$, $r > p \le s \ge q$ or $r \le p < s \ge q$, and to $KU_0 P_{r,p} \oplus KU_0 PP'_{s,t,q}$ if $p > s$, $r \le p \ge s$, $p \ge s \le q > t$ or $p \ge s < q \le t$. In order to investigate the behavior of the conjugations ψ_C^{-1}

on KU_0Y and KU_0X we recall the following result shown in [18, Proposition 2.1 i)] (or [13, Proposition 1.1]).

Proposition 1.1. *When $X = PP'_{r,s,p}$ or $_VPP'_{r,s,p}$ ($r, p \geq 1$ and $s \geq 0$), the conjugation ψ_C^{-1} on KU_0X is represented by the following matrix $A_{r,s,p}$:*

$$
\begin{array}{ccccc}
 & (1) & r > p > s & (2) & r \geq p \leq s \\
KU_0X & \cong & Z/2^r \oplus Z/2^p \oplus Z/2^s & & Z/2^r \oplus Z/2^{p-1} \oplus Z/2^{s+1} \\
\psi_C^{-1} & = & \begin{pmatrix} 1 & 2^{r-p} & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} & & \begin{pmatrix} 1 & 2^{r-p+1} & -2^{r-p} \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
 & (3) & r \leq p \geq s & (4) & r \leq p \leq s \\
KU_0X & \cong & Z/2^{r-1} \oplus Z/2^{p+1} \oplus Z/2^s & & Z/2^{r-1} \oplus Z/2^p \oplus Z/2^{s+1} \\
\psi_C^{-1} & = & \begin{pmatrix} 1 & 0 & 0 \\ -2^{p-r+2} & -1 & 0 \\ -2^{p-r+1} & -1 & 1 \end{pmatrix} & & \begin{pmatrix} 1 & 0 & 0 \\ -2^{p-r+1} & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

When $X = P'P_{p,r,s}$ or $_VP'P_{p,r,s}$ ($r, p \geq 1$ and $s \geq 0$), the conjugation ψ_C^{-1} on KU_0X is immediately given as a dual of Proposition 1.1 (see [18, Proposition 2.1 ii)]), by making use of the universal coefficient sequence

$$0 \to \mathrm{Ext}(KU_{-1}DX, Z) \to KU_0X \to \mathrm{Hom}(KU_0DX, Z) \to 0$$

where DX stands for the S-dual of X. By a routine argument using Proposition 1.1 and its dual we can easily show

Proposition 1.2. *When $X = PP'_{r,s,t,p,q}$ or $_VPP'_{r,s,t,p,q}$ ($r, s, p, q \geq 1$ and $t \geq 0$), the conjugation ψ_C^{-1} on KU_0X is represented by the following matrix $A_{r,s,t,p,q}$:*

$$
\begin{array}{cc}
(1)\ r > p > s > q > t & (2)\ r > p > s \geq q \leq t \\
Z/2^r \oplus Z/2^p \oplus Z/2^s \oplus Z/2^q \oplus Z/2^t & Z/2^r \oplus Z/2^p \oplus Z/2^s \oplus Z/2^{q-1} \oplus Z/2^{t+1} \\
\begin{pmatrix} 1 & 2^{r-p} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 2^{s-q} & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2^{r-p} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 2^{s-q+1} & -2^{s-q} \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

(3) $r > p \geq s \leq q > t,\ r > p > s \leq q \geq t$

$Z/2^r \oplus Z/2^p \oplus Z/2^{s-1} \oplus Z/2^{q+1} \oplus Z/2^t$

$$\begin{pmatrix} 1 & 2^{r-p} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 2^{q-s+1} & -2^{q-s+2} & -1 & 0 \\ 0 & 0 & -2^{q-s+1} & -1 & 1 \end{pmatrix}$$

(4) $r > p > s \leq q \leq t,\ r > p \geq s < q \leq t$

$Z/2^r \oplus Z/2^p \oplus Z/2^{s-1} \oplus Z/2^q \oplus Z/2^{t+1}$

$$\begin{pmatrix} 1 & 2^{r-p} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 2^{q-s} & -2^{q-s+1} & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(5) $r > p \leq s \geq q > t,\ r \geq p \leq s > q > t$

$Z/2^r \oplus Z/2^{p-1} \oplus Z/2^{s+1} \oplus Z/2^q \oplus Z/2^t$

$$\begin{pmatrix} 1 & 2^{r-p+1} & -2^{r-p} & 0 & 0 \\ 0 & -1 & 1 & 2^{s-q} & 0 \\ 0 & 0 & 1 & 2^{s-q+1} & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

(6) $r \geq p \leq s \geq q \leq t$

$Z/2^r \oplus Z/2^{p-1} \oplus Z/2^{s+1} \oplus Z/2^{q-1} \oplus Z/2^{t+1}$

$$\begin{pmatrix} 1 & 2^{r-p+1} & -2^{r-p} & 0 & 0 \\ 0 & -1 & 1 & 2^{s-q+1} & -2^{s-q} \\ 0 & 0 & 1 & 2^{s-q+2} & -2^{s-q+1} \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(7) $r > p \leq s \leq q > t,\quad r > p \leq s < q \geq t$
$r \geq p < s \leq q > t,\quad r \geq p < s < q \geq t$

$Z/2^r \oplus Z/2^{p-1} \oplus Z/2^s \oplus Z/2^{q+1} \oplus Z/2^t$

$$\begin{pmatrix} 1 & 2^{r-p+1} & -2^{r-p} & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2^{q-s+1} & -1 & 0 \\ 0 & 0 & -2^{q-s} & -1 & 1 \end{pmatrix}$$

(8) $r > p \leq s < q \leq t,\ r \geq p < s < q \leq t$

$Z/2^r \oplus Z/2^{p-1} \oplus Z/2^s \oplus Z/2^q \oplus Z/2^{t+1}$

$$\begin{pmatrix} 1 & 2^{r-p+1} & -2^{r-p} & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2^{q-s} & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(9) $r \leq p \geq s > q > t$

$Z/2^{r-1} \oplus Z/2^{p+1} \oplus Z/2^s \oplus Z/2^q \oplus Z/2^t$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2^{p-r+2} & -1 & 0 & 0 & 0 \\ -2^{p-r+1} & -1 & 1 & 2^{s-q} & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

(10) $r \leq p \geq s \geq q \leq t$

$Z/2^{r-1} \oplus Z/2^{p+1} \oplus Z/2^s \oplus Z/2^{q-1} \oplus Z/2^{t+1}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2^{p-r+2} & -1 & 0 & 0 & 0 \\ -2^{p-r+1} & -1 & 1 & 2^{s-q+1} & -2^{s-q} \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(11) $r \leq p \geq s \leq q \geq t$

$Z/2^{r-1} \oplus Z/2^{p+1} \oplus Z/2^{s-1} \oplus Z/2^{q+1} \oplus Z/2^t$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2^{p-r+2} & -1 & 0 & 0 & 0 \\ -2^{p-r+1} & -1 & 1 & 0 & 0 \\ 0 & 2^{q-s+1} & -2^{q-s+2} & -1 & 0 \\ 0 & 2^{q-s} & -2^{q-s+1} & -1 & 1 \end{pmatrix}$$

(12) $r \leq p \geq s \leq q \leq t$

$Z/2^{r-1} \oplus Z/2^{p+1} \oplus Z/2^{s-1} \oplus Z/2^q \oplus Z/2^{t+1}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2^{p-r+2} & -1 & 0 & 0 & 0 \\ -2^{p-r+1} & -1 & 1 & 0 & 0 \\ 0 & 2^{q-s} & -2^{q-s+1} & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(13) $r \leq p \leq s > q > t,\ r \leq p < s \geq q > t$

$Z/2^{r-1} \oplus Z/2^p \oplus Z/2^{s+1} \oplus Z/2^q \oplus Z/2^t$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2^{p-r+1} & -1 & 1 & 2^{s-q} & 0 \\ 0 & 0 & 1 & 2^{s-q+1} & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

(14) $r \leq p \leq s \geq q \leq t$

$Z/2^{r-1} \oplus Z/2^p \oplus Z/2^{s+1} \oplus Z/2^{q-1} \oplus Z/2^{t+1}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2^{p-r+1} & -1 & 1 & 2^{s-q+1} & 0 \\ 0 & 0 & 1 & 2^{s-q+2} & -2^{s-q+1} \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(15) $r \leq p < s \leq q > t,\ r \leq p < s < q \geq t$

$Z/2^{r-1} \oplus Z/2^p \oplus Z/2^s \oplus Z/2^{q+1} \oplus Z/2^t$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2^{p-r+1} & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2^{q-s+1} & -1 & 0 \\ 0 & 0 & -2^{q-s} & -1 & 1 \end{pmatrix}$$

(16) $r \leq p < s < q \leq t$

$Z/2^{r-1} \oplus Z/2^p \oplus Z/2^s \oplus Z/2^q \oplus Z/2^{t+1}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2^{p-r+1} & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2^{q-s} & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

When $X = P'P_{p,q,r,s,t}$ or $_VP'P_{p,q,r,s,t}$ $(r,s,p,q \geq 1$ and $t \geq 0)$, the conjugation ψ_C^{-1} on KU_0X is immediately given as a dual of Proposition 1.2. As an immediate result of Proposition 1.2 and its dual we obtain

Corollary 1.3. i) *The spectra $P_{r,s,p,q}$ and $_VP_{r,s,p,q}$ $(r,s,p \geq 1$ and $q \geq 0)$ have the same C-type as $\Sigma^2 P_{q+1,p,s,r-1}$.*

ii) *The spectra $P'P_{p,q,r,s,t}$ and $_VP'P_{p,q,r,s,t}$ $(r,s,p,q \geq 1$ and $t \geq 0)$ have the same C-type as $\Sigma^2 PP'_{t+1,s,r-1,q,p}$.*

1.2. Let us denote by M_m, M'_m, $MP_{m,n}$, $P'M'_{n,m}$, $MPP'_{r,s,p}$ and $P'PM'_{p,r,s}$ the small spectra constructed as the cofibers of the following maps

$$\begin{array}{ll} i\eta : \Sigma^1 \to SZ/2^m, & \eta j : SZ/2^m \to \Sigma^0, \\ i\eta \vee \bar{\eta} j : \Sigma^1 \vee \Sigma^1 SZ/2^n \to SZ/2^m, & (\eta j, i\bar{\eta}) : \Sigma^1 SZ/2^m \to \Sigma^1 \vee SZ/2^n, \\ i\eta \vee \bar{\eta} j j'_P : \Sigma^1 \vee \Sigma^{-1} P'_{s,p} \to SZ/2^r, & (\eta j, i_P i\bar{\eta}) : \Sigma^1 SZ/2^r \to \Sigma^1 \vee P_{p,s}, \end{array}$$

respectively. In [18, Proposition 2.4] the spectra $MP_{m,n}$ and $MPP'_{r,s,p}$ were written as $MV'_{m,n}$ and $MU_{s,r,p}$, respectively.

Recall that the spectrum V_m is exhibited as the cofiber of the map $2^{m-1}\bar{i} : \Sigma^0 \to C(\bar{\eta})$ where $C(\bar{\eta})$ denotes the cofiber of the map $\bar{\eta} : \Sigma^1 SZ/2 \to \Sigma^0$ and $\bar{i} : \Sigma^0 \to C(\bar{\eta})$ is the bottom cell inclusion. Denote by $_VM_m$, $_VMP_{m,n}$ and $_VMPP'_{r,s,p}$ the small spectra constructed as the cofibers of the following maps

$$\begin{array}{l} \bar{i}_V(\eta \wedge 1) : \Sigma^1 C(\bar{\eta}) \to V_m, \\ \bar{i}_V(\eta \wedge 1) \vee i_V \bar{\eta} j : \Sigma^1 C(\bar{\eta}) \vee \Sigma^1 SZ/2^n \to V_m, \\ \bar{i}_V(\eta \wedge 1) \vee i_V \bar{\eta} j j'_P : \Sigma^1 C(\bar{\eta}) \vee \Sigma^{-1} P'_{s,p} \to V_r, \end{array}$$

respectively, where $\bar{i}_V : C(\bar{\eta}) \to V_m$ is the canonical inclusion. According to [17, Lemma 1.5] the spectrum $_VM_m$ is quasi KO_*-equivalent to M_m. By virtue of [18, Theorem 3.3] (or [13, Theorem 1.2]) we observe that the spectra $_VMP_{m,n}$ and $_VMPP'_{r,s,p}$ are quasi KO_*-equivalent to $MP_{m,n}$ and $MPP'_{r,s,p}$, respectively.

Using the maps given in (1.2) we introduce new small spectra $MP_{r,s,p,q}$, $_VMP_{r,s,p,q}$, $MPP'_{r,s,t,p,q}$, $_VMPP'_{r,s,t,p,q}$, $P'M'_{p,q,r,s}$ and $P'PM'_{p,q,r,s,t}$ constructed as the cofibers of the following maps

$$(1.3)\quad \begin{array}{rcl} i\eta \vee \tilde{\eta} j j_{P'P} & : & \Sigma^1 \vee \Sigma^{-1} P'P_{s,p,q} \to SZ/2^r, \\ \bar{i}_V(\eta\wedge 1) \vee i_V \tilde{\eta} j j_{P'P} & : & \Sigma^1 C(\bar{\eta}) \vee \Sigma^{-1} P'P_{s,p,q} \to V_r, \\ i\eta \vee \tilde{\eta} j j_{P'P} \pi'_P & : & \Sigma^1 \vee \Sigma^{-1} P'_{s,t,p,q} \to SZ/2^r, \\ \bar{i}_V(\eta\wedge 1) \vee i_V \tilde{\eta} j j_{P'P} \pi'_P & : & \Sigma^1 C(\bar{\eta}) \vee \Sigma^{-1} P'_{s,t,p,q} \to V_r, \\ (\eta j, i_{PP'} i\bar{\eta}) & : & \Sigma^1 SZ/2^r \to \Sigma^1 \vee PP'_{p,q,s}, \\ (\eta j, l_P i_{PP'} i\bar{\eta}) & : & \Sigma^1 SZ/2^r \to \Sigma^1 \vee P_{p,q,s,t}, \end{array}$$

respectively. Evidently there hold the S-dualities $\Sigma^3 DMP_{r,s,p,q} = P'M'_{p,q,r,s}$ and $\Sigma^3 DMPP'_{r,s,t,p,q} = P'PM'_{p,q,r,s,t}$. Note that $MP_{r,s,p,0} = MPP'_{r,s,p}$, $_VMP_{r,s,p,0} = {}_VMPP'_{r,s,p}$, $MPP'_{r,s,0,p,q} = MP_{r,s,p,q}$, $_VMPP'_{r,s,0,p,q} = {}_VMP_{r,s,p,q}$ and their duals hold. The spectra $MP_{r,s,p,q}$, $_VMP_{r,s,p,q}$, $MPP'_{r,s,t,p,q}$, $_VMPP'_{r,s,t,p,q}$, $P'M'_{p,q,r,s}$ and $P'PM'_{p,q,r,s,t}$ are exhibited as the cofibers of the following maps

$$(1.4)\quad \begin{array}{ll} i_P i\eta : \Sigma^1 \to P_{r,s,p,q}, & {}_Vi_P \bar{i}_V(\eta\wedge 1) : \Sigma^1 C(\bar{\eta}) \to {}_VP_{r,s,p,q}, \\ i_{PP'} i\eta : \Sigma^1 \to PP'_{r,s,t,p,q}, & {}_Vi_{PP'} \bar{i}_V(\eta\wedge 1) : \Sigma^1 C(\bar{\eta}) \to {}_VPP'_{r,s,t,p,q}, \\ \eta j j'_P : \Sigma^{-1} P'_{p,q,r,s} \to \Sigma^1, & \eta j j_{P'P} : \Sigma^{-1} P'P_{p,q,r,s,t} \to \Sigma^1, \end{array}$$

respectively. Here $i_P : SZ/2^r \to P_{r,s,p,q}$, $_Vi_P : V_r \to {}_VP_{r,s,p,q}$, $i_{PP'} : SZ/2^r \to PP'_{r,s,t,p,q}$, $_Vi_{PP'} : V_r \to {}_VPP'_{r,s,t,p,q}$ are the canonical inclusions, and $j'_P : P'_{p,q,r,s} \to \Sigma^2 SZ/2^r$ and $j_{P'P} : P'P_{p,q,r,s,t} \to \Sigma^2 SZ/2^r$ are the canonical projections.

For $X = MPP'_{r,s,t,p,q}$ or $_VMPP'_{r,s,t,p,q}$ it is obvious that $KU_0X \cong Z \oplus KU_0PP'_{r,s,t,p,q}$ and $KU_1X = 0$. In order to investigate the behavior of the conjugation ψ_C^{-1} on KU_0X we recall the following result shown in [18, Proposition 2.3] (or [13, Proposition 1.1]).

Proposition 1.4. *When $X = MPP'_{r,s,p}$ or $_VMPP'_{r,s,p}$ ($r,p \ge 1$ and $s \ge 0$), the conjugation ψ_C^{-1} on $KU_0X \cong Z \oplus KU_0PP'_{r,s,p}$ is represented by the matrix $\begin{pmatrix} -1 & 0 \\ \mathbf{b} & A_{r,s,p} \end{pmatrix}$ for a certain column vector $\mathbf{b}$ transposed $(-1,x,0)$. Here the matrix $A_{r,s,p}$ is expressed separately into four cases (1) $\sim$ (4) in Proposition 1.1, and the integer x is defined to be 0, 0, 2^{p-r+1} or 2^{p-r} according as (1) $r > p > s$, (2) $r \ge p \le s$, (3) $r \le p \ge s$ or (4) $r \le p \le s$.*

By a routine argument using Proposition 1.4 we can show

Proposition 1.5. *When $X = MPP'_{r,s,t,p,q}$ or $vMPP'_{r,s,t,p,q}$ ($r,s,p,q \geq 1$ and $t \geq 0$), the conjugation ψ_C^{-1} on $KU_0X \cong Z \oplus KU_0PP'_{r,s,t,p,q}$ is represented by the matrix $\begin{pmatrix} -1 & 0 \\ \mathbf{b} & A_{r,s,t,p,q} \end{pmatrix}$ for a certain column vector* $\mathbf{b}$ *transposed $(-1,x,y,0,0)$. Here the matrix $A_{r,s,t,p,q}$ is expressed separately into sixteen cases (1) $\sim$ (16) in Proposition 1.2, and the integers x, y are given in each case as follows :*

$$(x,\ y) = \begin{cases} (0,\ 0) & \textit{in cases of (1)} \sim \textit{(8)} \\ (2^{p-r+1},\ 0) & \textit{in cases of (9)} \sim \textit{(10)} \\ (2^{p-r+1},\ 2^{p-r}) & \textit{in cases of (11)} \sim \textit{(12)} \\ (2^{p-r},\ 0) & \textit{in cases of (13)} \sim \textit{(16)}. \end{cases}$$

For $X = P_{m,n}$, $vP_{m,n}$, $MP_{m,n}$, $PP'_{r,s,p}$, $vPP'_{r,s,p}$ or $MPP'_{r,s,p}$ ($m,r,s,p \geq 1$ and $n \geq 0$) we recall the KO-homology groups KO_iX ($0 \leq i \leq 7$) tabled below (see [17, Proposition 2.2], [18, Propositions 2.2 and 2.4] or [13, Proposition 1.1]:

$i \backslash X$	$P_{m,n}$	$vP_{m,n}$	$MP_{m,n}$	$PP'_{r,s,p}$	$vPP'_{r,s,p}$	$MPP'_{r,s,p}$
0	$Z/2^m$	$Z/2^{m-1}$	$Z/2^m$	$Z/2^r \oplus Z/2^s$	$Z/2^{r-1} \oplus Z/2^s$	$Z/2^r \oplus Z/2^s$
1	$Z/2$	0	0	$Z/2$	0	0
2	$(*)_{n,m}$	$Z/2^{n+1}$	$Z \oplus Z/2^{n+1}$	$(*)_{p-1,r} \oplus Z/2$	$Z/2^p \oplus Z/2$	$Z \oplus Z/2^p \oplus Z/2$
3	$Z/2$	$Z/2$	$Z/2$	$Z/2$	$Z/2$	$Z/2$
4	$(*)_{m-2,n+1}$	$(*)_{m-1,n+1}$	$(*)_{m-1,n+1}$	$Z/2^{r-1} \oplus Z/2^{s+1}$	$Z/2^r \oplus Z/2^{s+1}$	$Z/2^r \oplus Z/2^{s+1}$
5	0	$Z/2$	0	0	$Z/2$	0
6	$Z/2^n$	$(*)_{n-1,m}$	$Z \oplus Z/2^n$	$Z/2^p$	$(*)_{p-1,r}$	$Z \oplus Z/2^p$
7	0	0	0	0	0	0

where $(*)_{k,1} \cong Z/2^{k+2}$ and $(*)_{k,l} \cong Z/2^{k+1} \oplus Z/2$ if $l \geq 2$.

For $X = P'_{n,m}$, $vP'_{n,m}$, $P'M'_{n,m}$, $P'P_{p,r,s}$, $vP'P_{p,r,s}$, $P'PM'_{p,r,s}$ ($m,r,s,p \geq 1$ and $n \geq 0$) the KO-homology groups KO_iX ($0 \leq i \leq 7$) are immediately given by making use of the universal coefficient sequence

$$0 \to \mathrm{Ext}(KO_{3-i}DX, Z) \to KO_iX \to \mathrm{Hom}(KO_{4-i}DX, Z) \to 0$$

where DX stands for the S-dual of X. Using these results we can easily compute

Proposition 1.6. *When $X = PP'_{r,s,t,p,q}$, $vPP'_{r,s,t,p,q}$ or $MPP'_{r,s,t,p,q}$ ($r,s,p,q \geq 1$ and $t \geq 0$), the KO-homology groups KO_iX ($0 \leq i \leq 7$) are tabled as* *follows:*

$i\backslash X$	$PP'_{r,s,t,p,q}$	$vPP'_{r,s,t,p,q}$	$MPP'_{r,s,t,p,q}$
0	$Z/2^r \oplus Z/2^s \oplus Z/2^t$	$Z/2^{r-1} \oplus Z/2^s \oplus Z/2^t$	$Z/2^r \oplus Z/2^s \oplus Z/2^t$
1	$Z/2$	0	0
2	$(*)_{p-1,r} \oplus (*)_{q-1,t+1}$	$Z/2^p \oplus (*)_{q-1,t+1}$	$Z \oplus Z/2^p \oplus (*)_{q-1,t+1}$
3	$Z/2$	$Z/2$	$Z/2$
4	$Z/2^{r-1} \oplus Z/2^s \oplus Z/2^{t+1}$	$Z/2^r \oplus Z/2^s \oplus Z/2^{t+1}$	$Z/2^r \oplus Z/2^s \oplus Z/2^{t+1}$
5	0	$Z/2$	0
6	$Z/2^p \oplus Z/2^q$	$(*)_{t-1,r} \oplus Z/2^q$	$Z \oplus Z/2^p \oplus Z/2^q$
7	0	0	0

where $(*)_{k,1} \cong Z/2^{k+2}$ *and* $(*)_{k,l} \cong Z/2^{k+1} \oplus Z/2$ *if* $l \geq 2$.

When $X = P'P_{p,q,r,s,t}$, $vP'P_{p,q,r,s,t}$ or $P'PM_{p,q,r,s,t}$ ($r, s, p, q \geq 1$ and $t \geq 0$) the KO-homology groups KO_iX ($0 \leq i \leq 7$) are immediately given as a dual of Proposition 1.6.

Recall that the conjugation ψ_C^{-1} on $KU_0P''_{m,n}$ is represented by the following matrix:

$$m > n \qquad Z/2^{m+1} \oplus Z/2^{n-1} \qquad \begin{pmatrix} 1-2^{m-n+1} & 2^{m-n+2}(1-2^{m-n}) \\ 1 & -1+2^{m-n+1} \end{pmatrix}$$

$$m = n \qquad Z/2^m \oplus Z/2^n \qquad \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

$$m < n \qquad Z/2^{m-1} \oplus Z/2^{n+1} \qquad \begin{pmatrix} 1-2^{n-m+1} & 1 \\ 2^{n-m+2}(1-2^{n-m}) & -1+2^{n-m+1} \end{pmatrix}$$

and $KO_iP''_{m,n} \cong Z/2^m,\ 0,\ Z/2^n,\ 0$ according as $i \equiv 0, 1, 2, 3 \bmod 4$ (see [17, Propositions 2.1 and 2.2]). Note that $\Sigma^2 P''_{n,m}$ and $\Sigma^4 P''_{m,n}$ have the same quasi KO_*-type as $P''_{m,n}$.

Let X be a CW-spectrum having the same C-type as the wedge sum $Y \vee (\vee_i P''_{m_i,n_i})$, where $Y = SZ/2^r$, M_r ($r \geq 1$), $PP'_{r,s,p}$, $MPP'_{r,s,p}$ ($r, p \geq 1$ and $s \geq 0$), $PP'_{r,s,t,p,q}$ or $MPP'_{r,s,t,p,q}$ ($r, s, p, q \geq 1$ and $t \geq 0$). Then we note that $KO_1X \oplus KO_5X \cong Z/2$ or 0, and $KO_3X \oplus KO_7X \cong Z/2$. Applying the same method as adopted in [15, Theorems 3.3 and 4.2] or [18, Theorem 3.3] we can show

Theorem 1.7. i) *Let Y be the small spectrum $SZ/2^r$ $(r \geq 1)$, $PP'_{r,s,p}$ $(r,p \geq 1$ and $s \geq 0)$ or $PP'_{r,s,t,p,q}$ $(r,s,p,q \geq 1$ and $t \geq 0)$. If a CW-spectrum X has the same $\mathcal{C}$-type as the wedge sum $Y \vee (\vee_i P''_{m_i,n_i})$, then it is quasi KO_*-equivalent to the wedge sum $W \vee (\vee_i P''_{m_i,n_i})$. Here W is one of the following four spectra:*

(1) $SZ/2^r$, $\Sigma^4 SZ/2^r$, V_r and $\Sigma^4 V_r$ when $Y = SZ/2^r$;

(2) $PP'_{r,s,p}$, $\Sigma^4 PP'_{r,s,p}$, $_VPP'_{r,s,p}$ and $\Sigma^4 {}_VPP'_{r,s,p}$ when $Y = PP'_{r,s,p}$;

(3) $PP'_{r,s,t,p,q}$, $\Sigma^4 PP'_{r,s,t,p,q}$, $_VPP'_{r,s,t,p,q}$ and $\Sigma^4 {}_VPP'_{r,s,t,p,q}$ when $Y = PP'_{r,s,t,p,q}$.

ii) *Let Y be the small spectrum M_r $(r \geq 1)$, $MPP'_{r,s,p}$ $(r,p \geq 1$ and $s \geq 0)$ or $MPP'_{r,s,t,p,q}$ $(r,s,p,q \geq 1$ and $t \geq 0)$. If a CW-spectrum X has the same $\mathcal{C}$-type as the wedge sum $Y \vee (\vee_i P''_{m_i,n_i})$, then it is quasi KO_*-equivalent to the wedge sum $W \vee (\vee_i P''_{m_i,n_i})$. Here W is either of the following two spectra:*

(1) M_r and $\Sigma^4 M_r$ when $Y = M_r$;

(2) $MPP'_{r,s,p}$ and $\Sigma^4 MPP'_{r,s,p}$ when $Y = MPP'_{r,s,p}$;

(3) $MPP'_{r,s,t,p,q}$ and $\Sigma^4 MPP'_{r,s,t,p,q}$ when $Y = MPP'_{r,s,t,p,q}$.

Combining Theorem 1.7 with Corollarly 1.3 and Proposition 1.5 we can immediately obtain

Corollary 1.8. i) *The spectrum $_VP_{r,s,p,q}$ $(r,s,p \geq 1$ and $q \geq 0)$ is quasi KO_*-equivalent to $\Sigma^2 P_{q+1,p,s,r-1}$.*

ii) *The spectra $P'P_{p,q,r,s,t}$ and $_VP'P_{p,q,r,s,t}$ $(r,s,p,q \geq 1$ and $t \geq 0)$ are quasi KO_*-equivalent to $\Sigma^2 PP'_{t+1,s,r-1,q,p}$ and $\Sigma^6 {}_VPP'_{t+1,s,r-1,q,p}$, respectively.*

iii) *The spectrum $_VMPP'_{r,s,t,p,q}$ $(r,s,p,q \geq 1$ and $t \geq 0)$ is quasi KO_*-equivalent to $MPP'_{r,s,t,p,q}$.*

2. THE STUNTED MOD 8 LENS SPACES

2.1. Let E be an (associative) ring spectrum with unit and ξ be an n-dimensional real vector bundle over a CW-complex X. Let us denote by $T(\xi)$ the Thom complex of ξ, thus $T(\xi) = D(\xi)/S(\xi)$ where $D(\xi)$ and $S(\xi)$ are the associated disc and sphere bundle, respectively. We say ξ to be E-orientable if there exists a Thom class $u_\xi \in E^nT(\xi)$ such that the composite map $(u_\xi \wedge p_+)\Delta : T(\xi) \to \Sigma^n E \wedge X_+$ is a quasi E_*-equivalence. Here $\Delta : T(\xi) \to T(\xi) \wedge D(\xi)_+$ is induced by the diagonal map and $p : D(\xi) \to X$ denotes the projection, and Y_+ stands for the based CW-complex with the additional base point $+$ for any CW-complex Y.

Proposition 2.1. *Let ξ be an n-dimensional real vector bundle over X. If ξ is E-orientable, then the Thom complex $T(\xi \oplus \alpha)$ is quasi E_*-equivalent to $\Sigma^n T(\alpha)$ for any real vector bundle α over X.*

Proof. Consider the composite map $(u_\xi \wedge 1)\varphi : T(\xi \oplus \alpha) \to \Sigma^n E \wedge T(\alpha)$ where $\varphi : T(\xi \oplus \alpha) \to T(\xi \times \alpha) \cong T(\xi) \wedge T(\alpha)$ is the canonical map. If α is the trivial bundle of dimension m, then $(u_\xi \wedge 1)\varphi = (u_\xi \wedge p_+)\Delta : \Sigma^m T(\xi) \to \Sigma^{n+m} E \wedge X_+$, thus $(u_\xi \wedge 1)\varphi$ is a quasi E_*-equivalence. For a general α we apply the Mayer-Vietoris exact sequence to observe that the map $(u_\xi \wedge 1)\varphi$ is a quasi E_*-equivalence. □

Let $L^m(q)$ be the $(2m+1)$-dimensional standard mod q lens space and $L_0^m(q)$ its $2m$-skeleton. For simplicity we set $L^{2m+1} = L^m(q)$ and $L^{2m} = L_0^m(q)$. Let ξ_{2m+1} be the canonical complex line bundle over $L^m(q)$ and ξ_{2m} denote the restriction of ξ_{2m+1} onto $L_0^m(q)$. As is well known, the 8-dimensional real vector bundle $2\epsilon_0(\xi_n) \oplus 4\theta$ over L^n is KO-orientable where ϵ_0 stands for the realification and θ denotes the trivial real line bundle over L^n. Hence we see

Corollary 2.2. *The Thom complex $T(2\epsilon_0(\xi_n) \oplus \alpha)$ is quasi KO_*-equivalent to $\Sigma^4 T(\alpha)$ for any real vector bundle α over L^n.*

The stunted mod q lens space L^n/L^l $(n > l \geq 0)$ is simply written to be L_{l+1}^n as usual. Recall that the stunted mod q lens spaces L_{2k}^{2k+n} is cellular homeomorphic to the Thom complex $T(k\epsilon_0(\xi_n))$, and L_{2k+1}^{2k+n} is so to $T(k\epsilon_0(\xi_n))/\Sigma^{2k}$ (see [3, Theorem 1] or [6, Theorem 4.7 and Corollary 4.8]). From Corollary 2.2 we can immediately show

Proposition 2.3. *The stunted mod q lens space L_{l+4}^{n+4} is quasi KO_*-equivalent to $\Sigma^4 L_l^n$. In particular, L_{4m+1}^{4m+n} and L_{4m}^{4m+n} are quasi KO_*-equivalent to $\Sigma^{4m}L^n$ and $\Sigma^{4m} \vee \Sigma^{4m}L^n$, respectively.*

According to the duality theorem [2, Theorem 3.3] for Thom complexes, the S-dual $DL_{2k}^{2k+2m+1} = DT(k\epsilon_0(\xi_{2m+1}))$ is (stably) homotopy equivalent to $T(-k\epsilon_0(\xi_{2m+1}) - \tau)$ where τ denotes the tangent vector bundle over $L^m(q)$. Choose a positive integer N such that it is divisible by the J-order of $\epsilon_0(\xi_{2m+1}) - 2\theta$. Then we can observe that the S-dual $DL_{2k}^{2k+2m+1}$ is (stably) homotopy equivalent to $\Sigma^{1-2N}L_{2N-2k-2m-2}^{2N-2k-1}$ because $\tau \oplus \theta \cong (m+1)\epsilon_0(\xi_{2m+1})$. This implies directly that the S-dual DL_l^n is (stably) homotopy equivalent to $\Sigma^{1-2N}L_{2N-n-1}^{2N-l-1}$ even if $(n, l) = (2k+2m, 2k)$, $(2k+2m+1, 2k+1)$ or $(2k+2m, 2k+1)$. (See [10, Proposition 5] or [8, Lemma 2.9]). Applying Proposition 2.3 we can immediately obtain

Proposition 2.4. *The S-dual DL_l^{4s+r} is quasi KO_*-equivalent to $\Sigma^{-4s-3}L_{3-r}^{4s+3-l}$ when $0 \leq r \leq 3$.*

Let $S^{2n+1}(q_0, \cdots, q_n)$ denote the unit sphere $S^{2n+1} \subset C^{n+1}$ with S^1-action defined by $\lambda \cdot (x_0, \cdots, x_n) = (\lambda^{q_0}x_0, \cdots, \lambda^{q_n}x_n) \in C^{n+1}$ for any $\lambda \in S^1 \subset C$. The orbit space $P^n(q_0, \cdots, q_n) = S^{2n+1}(q_0, \cdots, q_n)/S^1$ is called a weighted projective space. For simplicity we set $P^n = P^n(1, 1, \cdots, 1)$ and $P^{1,n-1} = P^n(q, 1, \cdots, 1)$ for a fixed positive integer q. Hereafter we shall assume that q is even. Of course, P^n is the usual complex projective space CP^n of dimension n. In [12, Theorem 2.4] we have determined the quasi KO_*-types of weighted projective spaces $P^n(q_0, \cdots, q_n)$. In fact, the spaces P^{2m}, P^{2m+1}, $P^{1,2m-1}$ and $P^{1,2m}$ are quasi KO_*-equivalent to the wedge sum $\vee_m C(\eta)$, $(\vee_m C(\eta)) \vee \Sigma^{4m+2}$, $\Sigma^2 \vee (\vee_{m-1} C(\eta)) \vee \Sigma^{4m}$ and $\Sigma^2 \vee (\vee_m C(\eta))$, respectively, where $C(\eta)$ denotes the cofiber of the stable Hopf map $\eta : \Sigma^1 \to \Sigma^0$.

The mod q lens spaces $L^{2n} = L_0^n(q)$ and $L^{2n+1} = L^n(q)$ are related to the weighted projective spaces P^n, $P^{1,n-1}$ and $P^{1,n}$ by the following (homotopy) commutative diagram

$$\begin{array}{ccccc} L^{2n} & \longrightarrow & P^n & \xrightarrow{i_0} & P^{1,n-1} \\ \downarrow i_L & & \| & & \downarrow \tilde{i} \\ L^{2n+1} & \xrightarrow{\pi} & P^n & \xrightarrow{i} & P^{1,n} \end{array}$$

with two cofiber sequences (see [4]). Here i, $\bar{i}$ and i_L are the canonical inclusions, π is the natural surjection and the map i_0 is defined by $i_0[x_0, \cdots, x_{n-1}, x_n] = [x_n^q, x_0, \cdots, x_{n-1}]$. Notice that the stunted weighted projective spaces P^{2m+n}/P^{2m}, P^{2m+n+1}/P^{2m+1}, $P^{1,2m+n-1}/P^{1,2m-1}$ and $P^{1,2m+n}/P^{1,2m}$ are quasi KO_*-equivalent to $\Sigma^{4m}P^n$, $\Sigma^{4m+2}P^{1,n-1}$, $\Sigma^{4m}P^{1,n-1}$ and $\Sigma^{4m+2}P^n$, respectively. Then we have the following cofiber sequences of KO-module spectra with $\epsilon = 0$ or 1:

(2.1)

$$\begin{array}{ccccc}
KO \wedge L_{4m+1}^{4m+2n+\epsilon} & \to & \Sigma^{4m}KO \wedge P^n & \to & \Sigma^{4m}KO \wedge P^{1,n-1+\epsilon} \\
KO \wedge L_{4m}^{4m+2n+\epsilon} & \to & \Sigma^{4m-2}KO \wedge P^{1,n} & \to & \Sigma^{4m}KO \wedge P^{1,n-1+\epsilon} \\
KO \wedge L_{4m-1}^{4m+2n-2+\epsilon} & \to & \Sigma^{4m-2}KO \wedge P^{1,n-1} & \to & \Sigma^{4m-2}KO \wedge P^{n+\epsilon} \\
KO \wedge L_{4m-2}^{4m+2n-2+\epsilon} & \to & \Sigma^{4m-2}KO \wedge P^{n+1} & \to & \Sigma^{4m-2}KO \wedge P^{n+\epsilon}.
\end{array}$$

Note that $KO_{2i}(L_{4m\pm1}^{4m+n})$ and $\mathrm{Tor}KO_{2i}(L_{4m-2}^{4m+n})$ are $Z/2$-modules because $KU_0(L_{4m\pm1}^{4m+n}) = 0$ and $KU_0(L_{4m-2}^{4m+n}) \cong Z$, where $\mathrm{Tor}G$ stands for the torsion subgroup of G. By means of (2.1) we can immediately compute

Lemma 2.5. *When q is even, the stunted mod q lens spaces L_{4m+k}^{4m+n} satisfies*

i) $KO_{4m}(L_{4m\pm1}^{4m+n}) = 0 = \mathrm{Tor}KO_{4m}(L_{4m-2}^{4m+n})$ *if* $n \equiv 1, 2, 3, 4, 5 \bmod 8$;

ii) $KO_{4m+4}(L_{4m\pm1}^{4m+n}) = 0 = \mathrm{Tor}KO_{4m+4}(L_{4m-2}^{4m+n})$ *if* $n \equiv 0, 1, 5, 6, 7 \bmod 8$;

iii) $KO_{4m+6}(L_{4m\pm1}^{4m+n}) = 0 = \mathrm{Tor}KO_{4m+6}(L_{4m-2}^{4m+n})$; *and*

iv) $\mathrm{Tor}KO_{4m+2}(L_{4m-2}^{4m+n}) = 0$.

Proposition 2.6. *Assume that q is even.*

i) *The stunted mod q lens space L_l^{4s-1} is quasi KO_*-equivalent to the wedge sum $L_l^{4s-2} \vee \Sigma^{4s-1}$.* (*Cf.* [13, Proposition 3.4]).

ii) *The stunted mod q lens space L_l^{4s+1} is never quasi KO_*-equivalent to the wedge sum $L_l^{4s} \vee \Sigma^{4s+1}$.*

Proof. i) When $l = 4m + 1$ or $4m$ we consider the following commutative diagram

$$\begin{array}{ccccc}
\Sigma^{4m+2n}KO & \xrightarrow{1\wedge\tilde{\alpha}} & \Sigma^{4m-1}KO \wedge P^{1,n-1} & \xrightarrow{1\wedge\bar{i}} & \Sigma^{4m-1}KO \wedge P^{1,n} \\
\| & & \downarrow & & \downarrow \\
\Sigma^{4m+2n}KO & \xrightarrow{1\wedge\alpha_L} & KO \wedge L_l^{4m+2n} & \xrightarrow{1\wedge i_L} & KO \wedge L_l^{4m+2n+1}
\end{array}$$

with two cofiber sequences. Recall that $P^{1,n-1}$ and $P^{1,n}$ are quasi KO_*-equivalent to the wedge sum $\Sigma^2 \vee (\vee_{s-m-1}C(\eta))$ and $\Sigma^2 \vee (\vee_{s-m-1}C(\eta)) \vee \Sigma^{4(s-m)}$, respectively, when $n = 2(s-m) - 1$. Then we can immediately observe that the upper left map $1 \wedge \tilde{\alpha}$ is trivial. When $l = 4m-1$ or $4m-2$ we use the following commutative diagram

$$\begin{array}{ccccc}
\Sigma^{4m+2n-2}KO & \xrightarrow{1\wedge\alpha} & \Sigma^{4m-3}KO\wedge P^n & \xrightarrow{1\wedge i} & \Sigma^{4m-3}KO\wedge P^{n+1} \\
\| & & \downarrow & & \downarrow \\
\Sigma^{4m+2n-2}KO & \xrightarrow{1\wedge\alpha_L} & KO\wedge L_l^{4m+2n-2} & \xrightarrow{1\wedge i_L} & KO\wedge L_l^{4m+2n-1}
\end{array}$$

with two cofiber sequences. Since P^n is quasi KO_*-equivalent to the wedge sum $\vee_{s-m}C(\eta)$ when $n = 2(s-m)$, it is obvious that the upper left map $1\wedge\alpha$ is trivial.

ii) By virtue of Proposition 2.3 we may assume that $l \not\equiv 0 \bmod 4$. Using (2.1) we can easily compute that $KO_{4m}L_{4m+1}^{4m+8t} \cong KO_{4m+4}L_{4m\pm1}^{4m+8t+4} \cong \mathrm{Tor}KO_{4m}L_{4m-2}^{4m+8t} \cong \mathrm{Tor}KO_{4m+4}L_{4m-2}^{4m+8t+4} \cong Z/2$, but $KO_{4m}L_{4m-1}^{4m+8t} \cong Z/2$ or 0. Moreover we can see that $KO_{4m}L_{4m-1}^{4m+8t-2} \cong Z/2$. From the above i) it follows that $KO_{4m}L_{4m-1}^{4m+8t-1} \cong Z/2 \oplus Z/2$. This implies that $KO_{4m}L_{4m-1}^{4m+8t}$ must be $Z/2$. By means of Lemma 2.5 we can conclude that $\mathrm{Tor}\{KO_0L_l^{4s} \oplus KO_4L_l^{4s}\} \cong Z/2$, but $\mathrm{Tor}\{KO_0L_l^{4s+1} \oplus KO_4L_l^{4s+1}\} = 0$. Now our result is immediate. □

2.2. For the canonical complex line bundle ξ_{2n+1} over the mod 8 lens space $L^n(8)$ we set $\sigma = \xi_{2n+1} - \theta_C$ and $\sigma(k) = (\xi_{2n+1})^{\otimes 2^k} - \theta_C$ where θ_C denotes the trivial complex line bundle over $L^n(8)$. Recall that the (reduced) KU-cohomology groups $KU^0L^n(8) \cong Z[\sigma]/(\sigma^{n+1}, \sigma(3))$ are given as follows [7, Proposition 3.7]:

(2.2)i) $KU^0L^1(8) \cong Z/8$, generated by σ;

ii) $KU^0L^2(8) \cong Z/16 \oplus Z/4$, generated by σ and $\sigma(1)$;

iii) $KU^0L^{4t-1}(8) \cong Z/2^{4t+1} \oplus Z/2^{2t} \oplus Z/2^{2t} \oplus Z/2^{t-1} \oplus Z/2^{t-1} \oplus Z/2^{t-1} \oplus Z/2^{t-1}$, generated by σ, $\sigma(1) + 2^t\sigma$, $\sigma(1)\sigma$, $\sigma(2) + 2^t\sigma(1) + 2^{3t}\sigma$, $\sigma(2)\sigma - 2^t\sigma(1)\sigma + 2^{3t+1}\sigma$, $\sigma(2)\sigma(1) - 2^{t+1}\sigma(1)$, $\sigma(2)\sigma(1)\sigma$;

iv) $KU^0L^{4t}(8) \cong Z/2^{4t+2} \oplus Z/2^{2t+1} \oplus Z/2^{2t} \oplus Z/2^t \oplus Z/2^{t-1} \oplus Z/2^{t-1} \oplus Z/2^{t-1}$, generated by σ, $\sigma(1)$, $\sigma(1)\sigma - 2^{2t+1}\sigma$, $\sigma(2)$, $\sigma(2)\sigma - 2^t\sigma(1)\sigma + 2^{3t+1}\sigma$, $\sigma(2)\sigma(1) - 2^{t+1}\sigma(1)$, $\sigma(2)\sigma(1)\sigma - 2^{3t+2}\sigma$;

v) $KU^0L^{4t+1}(8) \cong Z/2^{4t+3} \oplus Z/2^{2t+1} \oplus Z/2^{2t+1} \oplus Z/2^t \oplus Z/2^t \oplus Z/2^{t-1} \oplus Z/2^{t-1}$, generated by σ, $\sigma(1) + 2^{2t+1}\sigma$, $\sigma(1)\sigma$, $\sigma(2) - 2^{t+1}\sigma(1)$, $\sigma(2)\sigma$, $\sigma(2)\sigma(1) - 2^{t+1}\sigma(1)\sigma - 2^{t+1}\sigma(1) + 2^{3t+2}\sigma$, $\sigma(2)\sigma(1)\sigma + 2^{t+1}\sigma(1)\sigma + 2^{t+2}\sigma(1) - 2^{3t+3}\sigma$;

vi) $KU^0L^{4t+2}(8) \cong Z/2^{4t+4} \oplus Z/2^{2t+2} \oplus Z/2^{2t+1} \oplus Z/2^t \oplus Z/2^t \oplus Z/2^t \oplus Z/2^{t-1}$, generated by σ, $\sigma(1)$, $\sigma(1)\sigma+2^{2t+2}\sigma$, $\sigma(2)+2^{t+1}\sigma(1)+2^{3t+3}\sigma$, $\sigma(2)\sigma+2^{3t+3}\sigma$, $\sigma(2)\sigma(1)$, $\sigma(2)\sigma(1)\sigma + 2^{t+1}\sigma(1)\sigma - 2^{t+2}\sigma(1) - 2^{3t+3}\sigma$.

Set $\Sigma = (\sigma,\sigma(1),\sigma(1)\sigma,\sigma(2),\sigma(2)\sigma,\sigma(2)\sigma(1),\sigma(2)\sigma(1)\sigma)$. Then the conjugation ψ_C^{-1} on $KU^0L^n(8)$ behaves as $\psi_C^{-1}\Sigma = \Sigma P$ when the matrix P is given as follows:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & -2 & -2 & 2 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 & -1 & -1 & 2 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

Therefore the conjugation ψ_C^{-1} on $KU^0L^n(8)$ is represented by the matrix $B^{-1}PB$ if $\Sigma B = (g_1, \cdots, g_7)$ forms a basis of $KU^0L^n(8)$.

In order to express concisely the matrix representing the conjugation ψ_C^{-1} on $KU^0L^n(8)$ we here modify the direct sum decompositions given in (2.2) slightly as follows:

(2.3) i) $KU^0L^1(8) \cong Z/8$, generated by σ;

ii) $KU^0L^2(8) \cong Z/16 \oplus Z/4$, generated by σ and $\sigma(1) - 4\sigma$;

iii) $KU^0L^{4t-1}(8) \cong Z/2^{4t+1} \oplus Z/2^{2t} \oplus Z/2^{2t} \oplus Z/2^{t-1} \oplus Z/2^{t-1} \oplus Z/2^{t-1} \oplus Z/2^{t-1}$, generated by $(1+5\cdot 2^{3t-1})\sigma + 5\cdot 2^{t-1}\sigma(1) + 3\cdot 2^{t-1}\sigma(1)\sigma + \sigma(2)\sigma - \sigma(2)\sigma(1)$, $2^{2t}(1-2^{2t})\sigma + (1-3\cdot 2^t)\{\sigma(1)+\sigma(1)\sigma\} + \sigma(2) - \sigma(2)\sigma + 2\sigma(2)\sigma(1)$, $2^{2t}(3+2^t)\sigma + \sigma(1)$, $3\cdot 2^{3t}\sigma + 2^t\sigma(1) + \sigma(2) + \sigma(2)\sigma(1)$, $3\cdot 2^{3t}\sigma + 2^t\{\sigma(1) - \sigma(1)\sigma\} + \sigma(2) + \sigma(2)\sigma$, $-3\cdot 2^{3t+1}\sigma - 2^{t+2}\sigma(1) + \sigma(2)\sigma(1) + \sigma(2)\sigma(1)\sigma$, $2^{3t}\sigma + 2^t\sigma(1) + \sigma(2)$;

iv) $KU^0L^{4t}(8) \cong Z/2^{4t+2}\oplus Z/2^{2t+1}\oplus Z/2^{2t}\oplus Z/2^t\oplus Z/2^{t-1}\oplus Z/2^{t-1}\oplus Z/2^{t-1}$, generated by $(1+2^{3t})\sigma + 2^{2t}\sigma(1) - 2^{t-1}(1+3\cdot 2^t)\sigma(1)\sigma + \sigma(2)\sigma + 2^{t-2}\sigma(2)\sigma(1)$, $(1+2^t)\{\sigma(1)+\sigma(1)\sigma\} - \sigma(2) - \sigma(2)\sigma$, $-2^{2t+1}\sigma + 5\cdot 2^t\sigma(1) + (1+2^t)\sigma(1)\sigma - \sigma(2)\sigma - \sigma(2)\sigma(1)$, $\sigma(2)$, $-2^{t+1}\sigma(1) + \sigma(2)\sigma(1)$, $2^{3t+1}\sigma - 2^t\sigma(1)\sigma + \sigma(2)\sigma$, $-2^{3t+2}\sigma - 2^{t+2}\sigma(1) + 2\sigma(2) + \sigma(2)\sigma(1) + \sigma(2)\sigma(1)\sigma$;

v) $KU^0L^{4t+1}(8) \cong Z/2^{4t+3} \oplus Z/2^{2t+1} \oplus Z/2^t \oplus Z/2^{t-1} \oplus Z/2^{2t+1} \oplus Z/2^t \oplus Z/2^{t-1}$, generated by $(1+2^{3t}-2^{4t+2})\sigma - 2^{t-1}(1-3\cdot 2^t)\sigma(1) + (1-2^{t-1}+3\cdot 2^{2t-1})\sigma(1)\sigma + 2^{t-1}\{\sigma(2)+\sigma(2)\sigma-\sigma(2)\sigma(1)\}$, $2^{2t+1}\sigma - (1-2^t-2^{2t})\{\sigma(1) + \sigma(1)\sigma\} + \sigma(2)\sigma - \sigma(2)\sigma(1)$, $2^{3t+2}\sigma - \sigma(2) - \sigma(2)\sigma - \sigma(2)\sigma(1) - \sigma(2)\sigma(1)\sigma$, $-2^{3t+2}\sigma - 2^{t+1}\sigma(1) + 2\sigma(2) + 2\sigma(2)\sigma + \sigma(2)\sigma(1) + \sigma(2)\sigma(1)\sigma$, $-2^{2t+1}(1+$

$2^{t+1}-2^{2t})\sigma-(1-2^{t+1}-2^{2t})\sigma(1)+2^t\sigma(1)\sigma-\sigma(2)\sigma(1)$, $-2^{t+1}\{\sigma(1)+\sigma(1)\sigma\}+\sigma(2)+\sigma(2)\sigma(1)$, $2^{3t+2}\sigma-2^{t+1}\{\sigma(1)+\sigma(1)\sigma\}+\sigma(2)\sigma$;

vi) $KU^0L^{4t+2}(8)\cong Z/2^{4t+4}\oplus Z/2^{2t+2}\oplus Z/2^{2t+1}\oplus Z/2^t\oplus Z/2^{t-1}\oplus Z/2^t\oplus Z/2^t$, generated by $(1-2^{3t+4}+2^{4t+3})\sigma+3\cdot 2^t\sigma(1)+2^t\sigma(1)\sigma+\sigma(2)+\sigma(2)\sigma$, $(1-2^{t+1})\{\sigma(1)+\sigma(1)\sigma\}-\sigma(2)-\sigma(2)\sigma$, $2^{2t+2}(1-7\cdot 2^t)\sigma-2^t\sigma(1)+(1-2^t)\sigma(1)\sigma-2\sigma(2)-\sigma(2)\sigma-\sigma(2)\sigma(1)$, $-2^{t+1}\{\sigma(1)+\sigma(1)\sigma\}+\sigma(2)+\sigma(2)\sigma(1)$, $-2^{3t+3}\sigma-2^{t+3}\sigma(1)-2^{t+1}\sigma(1)\sigma+2\sigma(2)+2\sigma(2)\sigma(1)+\sigma(2)\sigma(1)\sigma$, $-2^{t+1}\sigma(1)+\sigma(2)+\sigma(2)\sigma+\sigma(2)\sigma(1)$, $2^{t+2}\sigma(1)-\sigma(2)\sigma(1)$.

By means of (2.3) we can easily see that the conjugation ψ_C^{-1} on $KU^0L^n(8)$ is represented by the following matrix:

(2.4) i) $\psi_C^{-1}=1$ on $Z/8$ when $n=1$;

ii) $\psi_C^{-1}=\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$ on $Z/16\oplus Z/4$ when $n=2$;

iii) $\psi_C^{-1}=\begin{pmatrix}1-2^{2t} & 2^{2t+1}(1-2^{2t-1})\\ 1 & -1\end{pmatrix}\oplus\begin{pmatrix}1-2^t & 2^{t+1}\\ 1 & -1\end{pmatrix}\oplus\begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix}\oplus 1$

on $Z/2^{4t+1}\oplus Z/2^{2t}\oplus Z/2^{2t}\oplus Z/2^{t-1}\oplus Z/2^{t-1}\oplus Z/2^{t-1}\oplus Z/2^{t-1}$ when $n=4t-1$;

iv) $\psi_C^{-1}=\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 1 & -1 & -2 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1 & -2\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}\oplus\begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix}$,

on $Z/2^{4t+2}\oplus Z/2^{2t+1}\oplus Z/2^{2t}\oplus Z/2^t\oplus Z/2^{t-1}\oplus Z/2^{t-1}\oplus Z/2^{t-1}$ when $n=4t$;

v) $\psi_C^{-1}=\begin{pmatrix}1-2^{2t+1} & 2^{2t+2}(1-2^{2t})\\ 1 & -1\end{pmatrix}\oplus\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 1 & -1 & 0 & 0 & 0\\ -2^{t+1} & 2^{t+2} & 1 & 0 & 0\\ 0 & 0 & 1 & -1 & -2\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}$

on $Z/2^{4t+3}\oplus Z/2^{2t+1}\oplus Z/2^t\oplus Z/2^{t-1}\oplus Z/2^{2t+1}\oplus Z/2^t\oplus Z/2^{t-1}$ when $n=4t+1$;

vi) $\psi_C^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$

on $Z/2^{4t+4} \oplus Z/2^{2t+2} \oplus Z/2^{2t+1} \oplus Z/2^t \oplus Z/2^{t-1} \oplus Z/2^t \oplus Z/2^t$ when $n = 4t+2$.

Dualizing (2.4) to use Proposition 1.2 (1) and (5) we obtain

Proposition 2.7. *The suspended mod 8 lens space* $\Sigma^1 L_0^n(8) = \Sigma^1 L^{2n}$ *has the same* C*-type as the following small spectrum:*
$\Sigma^2 SZ/8$, $P_{4,2}$, $\Sigma^2 SZ/4 \vee P''_{4,3}$, $PP'_{4t+2,2t,t-1,2t+1,t} \vee P''_{t-1,t-1}$, $PP'_{t,2t,t-1,t,t} \vee P''_{4t+2,2t+2}$, $PP'_{4t+4,2t+1,t-1,2t+2,t} \vee P''_{t,t}$, $SZ/2^t \vee P''_{4t+4,2t+3} \vee P''_{2t+1,t+1} \vee P''_{t,t}$
according as $n = 1, 2, 3, 4t, 4t+1, 4t+2, 4t+3$ $(t \geq 1)$.

Using Theorem 1.7, Propositions 2.3, 2.6 and 2.7 and Lemma 2.5 we can show the following result.

Theorem 2.8. *The suspended stunted mod 8 lens spaces* $\Sigma^{1-4m} L_{4m+1}^{4m+n}$ *is quasi* KO_**-equivalent to the small spectrum:*
Σ^2, $\Sigma^2 SZ/8$, $\Sigma^4 \vee \Sigma^2 SZ/8$, $VP_{4,2}$, $MP_{4,2}$, $\Sigma^6 V_2 \vee P''_{4,3}$, $\Sigma^0 \vee \Sigma^6 V_2 \vee P''_{4,3}$, $PP'_{4t+2,2t,t-1,2t+1,t} \vee P''_{t-1,t-1}$, $MPP'_{4t+2,2t,t-1,2t+1,t} \vee P''_{t-1,t-1}$, $VPP'_{t,2t,t-1,t,t} \vee P''_{4t+2,2t+2}$, $\Sigma^4 \vee VPP'_{t,2t,t-1,t,t} \vee P''_{4t+2,2t+2}$, $VPP'_{4t+4,2t+1,t-1,2t+2,t} \vee P''_{t,t}$, $MPP'_{4t+4,2t+1,t-1,2t+2,t} \vee P''_{t,t}$, $SZ/2^t \vee P''_{4t+4,2t+3} \vee P''_{2t+1,t+1} \vee P''_{t,t}$, $\Sigma^0 \vee SZ/2^t \vee P''_{4t+4,2t+3} \vee P''_{2t+1,t+1} \vee P''_{t,t}$
according as $n = 1, 2, \cdots, 7, 8t, 8t+1, \cdots, 8t+7$ $(t \geq 1)$.

Proof. By virtue of Proposition 2.3 we may assume that $m = 0$. When n is even, we use Proposition 2.7 combined with Lemma 2.5 and then apply Theorem 1.7 to obtain our result. When $n \equiv 3 \bmod 4$ our result is immediate from Proposition 2.6 i). When $n \equiv 1 \bmod 4$ the attaching map $\alpha_L : \Sigma^{n-1} \to L^{n-1}$ is not KO_*-trivial because of Proposition 2.6 ii). Since $KO_{n-1} L^{n-1} \cong Z/2$, our result is easily observed by use of (1.4) and Corollary 1.8 iii). □

2.3. Consider the homomorphism $i^* : KU^0 L^{2m}(8) \to KU^0 L^1(8)$ induced by the inclusion $i : L^1(8) \to L^{2m}(8)$. The induced homomorphism i^* carries $\Sigma = (\sigma, \sigma(1), \sigma(1)\sigma, \sigma(2), \sigma(2)\sigma, \sigma(2)\sigma(1), \sigma(2)\sigma(1)\sigma)$ to $(\sigma, 2\sigma, 0, 4\sigma, 0, 0, 0)$. Hence the (reduced) KU-cohomology groups $KU^0(L^{2m}(8)/L^1(8))$ are given as follows:

(2.5) i) $KU^0(L^2(8)/L^1(8)) \cong Z/8$, generated by $2\sigma - \sigma(1)$;
ii) $KU^0(L^{4t}(8)/L^1(8)) \cong Z/2^{4t+1} \oplus Z/2^{2t} \oplus Z/2^{2t} \oplus Z/2^{t-1} \oplus Z/2^{t-1} \oplus Z/2^{t-1} \oplus Z/2^{t-1}$, generated by $2\sigma-\sigma(1)$, $2\sigma(1)-\sigma(2)$, $-2^{2t+1}\sigma+\sigma(1)\sigma, 2\sigma(2)$, $2^{3t+1}\sigma - 2^t\sigma(1)\sigma + \sigma(2)\sigma$, $-2^{t+1}\sigma(1) + \sigma(2)\sigma(1)$, $2^{3t+2}\sigma + \sigma(2)\sigma(1)\sigma$;
iii) $KU^0(L^{4t+2}(8)/L^1(8)) \cong Z/2^{4t+3} \oplus Z/2^{2t+1} \oplus Z/2^{2t+1} \oplus Z/2^{t-1} \oplus Z/2^t \oplus Z/2^t \oplus Z/2^{t-1}$, generated by $2\sigma - \sigma(1)$, $-2^{3t+3}\sigma + 2(1-2^t)\sigma(1) - \sigma(2)$, $2^{2t+2}\sigma+\sigma(1)\sigma$, $2^{3t+4}\sigma+2^{t+2}\sigma(1)+2\sigma(2)$, $2^{3t+3}\sigma+\sigma(2)\sigma$, $\sigma(2)\sigma(1)$, $-2^{3t+3}\sigma-2^{t+2}\sigma(1) + 2^{t+1}\sigma(1)\sigma + \sigma(2)\sigma(1)\sigma$.

In order to express concisely the matrix representing the conjugation ψ_C^{-1} on $KU^0(L^{2m}(8)/L^1(8))$ we here modify the direct sum decompositions given in (2.5) slightly as follows:

(2.6) i) $KU^0(L^2(8)/L^1(8)) \cong Z/8$, generated by $2\sigma - \sigma(1)$;
ii) $KU^0(L^{4t}(8)/L^1(8)) \cong Z/2^{4t+1} \oplus Z/2^{2t} \oplus Z/2^{2t} \oplus Z/2^{t-1} \oplus Z/2^{t-1} \oplus Z/2^{t-1} \oplus Z/2^{t-1}$, generated by $2(1-2^{2t})\sigma - (1-2^{t+2})\sigma(1) + \sigma(1)\sigma + \sigma(2)\sigma - \sigma(2)\sigma(1)$, $-2^{2t+1}\sigma + 2^{t+1}\sigma(1) + \sigma(1)\sigma$, $2^{2t+2}\sigma + 2(1-2^{t+1})\sigma(1) + \sigma(2) + 2\sigma(2)\sigma(1) + \sigma(2)\sigma(1)\sigma$, $-2^{t+1}\sigma(1) + \sigma(2)\sigma(1)$, $2^{t+2}\sigma(1) + 2\sigma(2)$, $2^{3t+1}\sigma - 2^t\sigma(1)\sigma + \sigma(2)\sigma$, $-2^{3t+2}\sigma - 2^{t+1}\sigma(1) + 2\sigma(2) + \sigma(2)\sigma(1) + \sigma(2)\sigma(1)\sigma$;
iii) $KU^0(L^{4t+2}(8)/L^1(8)) \cong Z/2^{4t+3} \oplus Z/2^{2t+1} \oplus Z/2^{2t+1} \oplus Z/2^t \oplus Z/2^t \oplus Z/2^{t-1} \oplus Z/2^{t-1}$, generated by $2(1-2^{2t+1}-9\cdot 2^{4t+1})\sigma-(1+9\cdot 2^t+9\cdot 2^{2t})\sigma(1)-(1-2^{t+1})\sigma(1)\sigma - 9\cdot 2^{t-1}\sigma(2)-\sigma(2)\sigma$, $-2^{2t+2}(1-15\cdot 2^{2t})\sigma+2^t(3+5\cdot 2^{t+1})\sigma(1)+\sigma(1)\sigma+5\cdot 2^{t-1}\sigma(2)+\sigma(2)\sigma$, $2^{2t+3}(1+3\cdot 2^t)\sigma+2(1+3\cdot 2^t)\sigma(1)-2^{t+1}\sigma(1)\sigma+\sigma(2) - \sigma(2)\sigma(1) - \sigma(2)\sigma(1)\sigma$, $-2^{t+2}\sigma(1) + 2^{t+1}\sigma(1)\sigma + \sigma(2)\sigma - \sigma(2)\sigma(1)$, $2^{3t+3}\sigma - 2^{t+1}\sigma(1)\sigma + \sigma(2)\sigma(1)$, $-2^{3t+3}\sigma - 2^{t+3}\sigma(1) + 3\cdot 2^{t+1}\sigma(1)\sigma - 2\sigma(2) + \sigma(2)\sigma(1)\sigma$, $-2^{t+2}\sigma(1) - 2^{t+2}\sigma(1)\sigma + 2\sigma(2)$.

By a routine computation we can easily see that the conjugation ψ_C^{-1} on $KU^0(L^{2m}(8)/L^1(8))$ is represented by the following matrix:

(2.7) i) $\psi_C^{-1} = 1$ on $Z/8$ when $m = 1$;

ii) $\psi_C^{-1} = 1 \oplus \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \oplus \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \oplus \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$

on $Z/2^{4t+1} \oplus Z/2^{2t} \oplus Z/2^{2t} \oplus Z/2^{t-1} \oplus Z/2^{t-1} \oplus Z/2^{t-1} \oplus Z/2^{t-1}$ when $m = 2t$;

iii) $\psi_C^{-1} = 1 \oplus \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \oplus \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \oplus \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$
on $Z/2^{4t+3} \oplus Z/2^{2t+1} \oplus Z/2^{2t+1} \oplus Z/2^t \oplus Z/2^t \oplus Z/2^{t-1} \oplus Z/2^{t-1}$ when $m = 2t+1$.

Dualizing (2.7) we can immediately obtain

Proposition 2.9. *The suspended stunted mod 8 lens space $\Sigma^1 L_3^{4m}$ has the same C-type as the following small spectrum:*
$SZ/8, SZ/2^{4t+1} \vee P''_{2t,2t} \vee P''_{t-1,t-1} \vee P''_{t-1,t-1},\ SZ/2^{4t+3} \vee P''_{2t+1,2t+1} \vee P''_{t,t} \vee P''_{t-1,t-1}$
according as $m = 1, 2t, 2t+1$ $(t \geq 1)$.

Using Theorems 1.7 and 2.8, Propositions 2.3, 2.4, 2.6 and 2.9 and Lemma 2.5 we can show the following result.

Theorem 2.10. *The suspended stunted mod 8 lens space $\Sigma^{1-4m} L_{4m-1}^{4m+n-4}$ is quasi KO_*-equivalent to the following small spectrum:*
$\Sigma^0,\ SZ/8,\ M_3,\ P'_{2,4},\ \Sigma^4 \vee P'_{2,4},\ V_{4t+1} \vee P''_{2t,2t} \vee P''_{t-1,t-1} \vee P''_{t-1,t-1},\ M_{4t+1} \vee P''_{2t,2t} \vee P''_{t-1,t-1} \vee P''_{t-1,t-1},\ PP'_{t,2t,4t+1,t,2t+1} \vee P''_{t-1,t-1},\ \Sigma^0 \vee PP'_{t,2t,4t+1,t,2t+1} \vee P''_{t-1,t-1},\ SZ/2^{4t+3} \vee P''_{2t+1,2t+1} \vee P''_{t,t} \vee P''_{t-1,t-1},\ M_{4t+3} \vee P''_{2t+1,2t+1} \vee P''_{t,t} \vee P''_{t-1,t-1},\ vPP'_{t,2t+1,4t+3,t,2t+2} \vee P''_{t,t},\ \Sigma^4 \vee vPP'_{t,2t+1,4t+3,t,2t+2} \vee P''_{t,t}$
according as $n = 3, 4, \cdots, 7,\ 8t,\ 8t+1,\ \cdots,\ 8t+7$ $(t \geq 1)$.

Proof. By virtue of Proposition 2.3 we may assume that $m = 1$. According to Proposition 2.4, $\Sigma^{-4s-3} L_3^{4s+2}$ is quasi KO_*-equivalent to the S-dual DL^{4s}. By dualizing Theorem 2.8 and using Corollary 1.8 we can immediately obtain our result when $n \equiv 2 \bmod 4$. Using Proposition 2.9 in place of Proposition 2.7 we can show our result by a similar discussion to Theorem 2.8 when $n \not\equiv 2 \bmod 4$. □

Let us denote by $M'M_m$ the small spectrum constructed as the cofiber of the map $h'_M \eta : \Sigma^2 \to M'_m$, in which $h'_M : \Sigma^1 \to M'_m$ satisfies $j'_M h'_M = i$ for the bottom cell collapsing $j'_M : M'_m \to \Sigma^1 SZ/2^m$ (cf. [17, (1.7)]). Notice that the spectra $M'M_m$ and $\Sigma^4 M'M_m$ have the same quasi KO_*-type (see [19, Theorem 5.3] or [15, (4.4)]). Using Theorems 2.8 and 2.10, Propositions 2.3, 2.4 and 2.6 and Lemma 2.5 we can show the following result.

Theorem 2.11. *The suspended stunted mod 8 lens space $\Sigma^{1-4m} L_{4m-2}^{4m+n-4}$ is quasi KO_*-equivalent to the following small spectrum:*

Σ^7, $\Sigma^0\vee\Sigma^7$, $\Sigma^7 M_3'$, $\Sigma^3 M' M_3$, $\Sigma^2 P' M_{2,4}'$, $\Sigma^4\vee\Sigma^2 P' M_{2,4}'$, $\Sigma^3 M_{4t+1}'\vee P_{2t,2t}''\vee P_{t-1,t-1}''\vee P_{t-1,t-1}''$, $\Sigma^3 M' M_{4t+1}\vee P_{2t,2t}''\vee P_{t-1,t-1}''\vee P_{t-1,t-1}''$, $\Sigma^6 P' P M_{2t+1,t,4t+2,2t,t-1}'\vee P_{t-1,t-1}''$, $\Sigma^0\vee\Sigma^6 P' P M_{2t+1,t,4t+2,2t,t-1}'\vee P_{t-1,t-1}''$, $\Sigma^7 M_{4t+3}'\vee P_{2t+1,2t+1}''\vee P_{t,t}''\vee P_{t-1,t-1}''$, $\Sigma^3 M' M_{4t+3}\vee P_{2t+1,2t+1}''\vee P_{t,t}''\vee P_{t-1,t-1}''$, $\Sigma^2 P' P M_{2t+2,t,4t+4,2t+1,t-1}'\vee P_{t,t}''$, $\Sigma^4\vee\Sigma^2 P' P M_{2t+2,t,4t+4,2t+1,t-1}'\vee P_{t,t}''$ *according as* $n=2,3,\cdots,7,8t,8t+1,\cdots,8t+7$ $(t\geq 1)$.

Proof. By virtue of Proposition 2.3 we may assume that $m=1$. According to Proposition 2.4, $\Sigma^{-4s-3}L_2^{4s}$ and $\Sigma^{-4s-3}L_2^{4s+2}$ are quasi KO_*-equivalent to the S-duals DL_3^{4s+1} and DL^{4s+1}, respectively. Therefore we dualize Theorems 2.8 and 2.10 to obtain our result when n is even. On the other hand, we can show our result by a similar discussion to Theorem 2.8 when n is odd. □

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