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A GENERATION OF THE HOPF CONSTRUCTION

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Introduction. Let Γ be a co-Hopf space. For any space X, we define the Γ-*suspension space* of X by $\Gamma X = \Gamma \wedge X$. For any map $f : X \to Y$, a map $\Gamma f : \Gamma X \to \Gamma Y$ is induced. It is called the Γ-*suspension map* of f. If $\Gamma = S^1$(1-sphere), then $\Gamma X = \Sigma X$ is the usual suspension space and $\Gamma f = \Sigma f : \Sigma X \to \Sigma Y$ is the usual suspension map.

It is known that there are various definitions of the Hopf construction. For example I. M. James provides one of the definitions by using the difference element $d(\cdot, \cdot)$ in [4] and M. Arkowitz and P. Silberbush study six different types of elements which are called the Hopf-type constructions in [1]. K. A. Hardie and A. V. Jansen define the Hopf construction $c(f,g) \in [\Sigma^{m+n+1}W, \Sigma Z]$ (we call it the Hopf construction with a space W) based on the definition by James in [2] when there is a pairing $\mu : S^m \times S^n \to Z^W$ with axes $f : S^m \to Z^W$ and $g : S^n \to Z^W$. On the other hand, N. Oda defines the Γ-Hopf construction in [6]. If $\Gamma = S^1$, then it is one of the Hopf-type constructions. In this paper, we introduce a concept of the skew pairing $\mu_W : (X \times Y) \wedge W \to Z$ with axes $f : X \wedge W \to Z$ and $g : Y \wedge W \to Z$ and we define the Γ-*Hopf construction with a space* W by

$$J_\Gamma^W(\mu_W) = \Gamma\mu_W \circ (v \wedge 1_W) \in [\Gamma(X \wedge Y) \wedge W, \Gamma Z]$$

for any skew pairing μ_W, where v is the element in Proposition 1.1. This generalizes the Γ-Hopf construction by Oda. Throughout this paper, the space W is any space except otherwise stated explicitly.

In §1, we begin by studying fundamental properties of the Γ-Hopf construction with a space W. We next study the Hopf invariant. Let Z be a connected CW-complex and $H : [\Sigma A, \Sigma Z] \to [\Sigma A, \Sigma(Z \wedge Z)]$ the Hopf invariant. Then we have the following results.

Theorem 1.11 *Let $v \in [\Sigma(X \wedge Y), \Sigma(X \times Y)]$ be the element of Proposition 1.1 and $\beta : \Sigma(X \times Y) \wedge W \to \Sigma Z$ any map. If W is a co-Hopf space, then we have*

$$H(\beta \circ (v \wedge 1_W)) = H(\beta) \circ (v \wedge 1_W).$$

Theorem 1.13 *Suppose that there is a skew pairing $\mu_W : (X \times Y) \wedge W \to Z$. If W is a co-Hopf space, then $H(J_\Sigma^W(\mu_W)) = 0$.*

In §2, we define an element $c(\alpha)$ for any skew pairing $\alpha : (X \times Y) \wedge W \to Z_\infty$ as follows

$$c(\alpha) = \phi(\alpha) \circ (v \wedge 1_W) \in [\Sigma(X \wedge Y) \wedge W, \Sigma Z].$$

The element $c(\alpha)$ is a generalization of the Hopf construction with a space W. Let Z_∞ be the reduced product space of a connected CW-complex Z. Then we have the following result.

Theorem 2.2 *If we are given two skew pairings $\alpha, \beta : (X \times Y) \wedge W \to Z_\infty$ with the same axes $f : X \wedge W \to Z_\infty$ and $g : Y \wedge W \to Z_\infty$, then the following relation holds.*

$$c(\alpha) \dot{-} c(\beta) = c(\alpha \underset{\cdot}{-} \beta).$$

For any maps $\alpha : X \wedge W \to Z_\infty$ and $\beta : Y \wedge W \to Z_\infty$, we define two skew pairings $M, \overline{M} : (X \times Y) \wedge W \to Z_\infty$ by

$$M = (\alpha \circ (p_1 \wedge 1_W)) \underset{\cdot}{+} (\beta \circ (p_2 \wedge 1_W)),$$

and

$$\overline{M} = (\beta \circ (p_2 \wedge 1_W)) \underset{\cdot}{+} (\alpha \circ (p_1 \wedge 1_W)).$$

Using the isomorphism $\phi : [A, Z_\infty] \to [\Sigma A, \Sigma Z]$ (cf. (2.3) of [9]), we have the following results.

Theorem 2.5 *For any maps $\alpha : X \wedge W \to Z_\infty$ and $\beta : Y \wedge W \to Z_\infty$, we have*

$$c(M) = 0 \quad \text{and} \quad c(\overline{M}) = \dot{-} [\phi(\alpha), \phi(\beta)]^W_\Sigma .$$

Here $[\cdot, \cdot]^W_\Sigma$ denotes the generalized Hardie-Jansen product (cf. [8]).

Let

$$\tau : X \wedge Y \wedge W \wedge W \to X \wedge W \wedge Y \wedge W$$

be the natural homeomorphism interchanging the second and third factors of the smash products. Let $q : X \times Y \to X \wedge Y$ be the identification map. Let $\chi : W \to W \wedge W$ be the reduced diagonal map.

Theorem 2.6 *We define a skew pairing $M = (i \circ f \circ (p_1 \wedge 1_W)) \underset{\cdot}{+} (i \circ g \circ (p_2 \wedge 1_W))$ for any maps $f : X \wedge W \to Z$ and $g : Y \wedge W \to Z$. Then we have*

$$H(\phi(M)) = \Sigma(f \wedge g) \circ \Sigma\tau \circ (1_{\Sigma X \wedge Y} \wedge \chi) \circ (\Sigma q \wedge 1_W)$$

and hence

$$H(\phi(M)) \circ (v \wedge 1_W) = \Sigma(f \wedge g) \circ \Sigma\tau \circ (1_{\Sigma X \wedge Y} \wedge \chi).$$

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1. Γ-Hopf construction with a space W for a skew pairing

We work in the category of compactly generated Hausdorff spaces with nondegenerate base point $*$ [10]. All maps and homotopies are base point preserving. For any spaces X and Y, the space $X \times Y$ is the product space and $X \vee Y$ is the wedge sum (one-point union). The space $X \vee Y$ is the subset of the space $X \times Y$. Let $X \wedge Y$ be the smash product which is the identification space $X \times Y / X \vee Y$. Let $[A, Z]$ be the set of homotopy classes of maps from A to Z. For any elements α and β in $[A, Z]$, we define the following maps: if A is a co-Hopf space with co-multiplication $\nu : A \to A \vee A$, then we define

$$\alpha \dot{+} \beta = \nabla_Z \circ (\alpha \vee \beta) \circ \nu : A \to Z$$

where $\nabla_Z : Z \vee Z \to Z$ is the folding map; or if Z is a Hopf space with multiplication $\mu : Z \times Z \to Z$, then we define

$$\alpha \underset{\cdot}{+} \beta = \mu \circ (\alpha \times \beta) \circ \Delta_A : A \to Z$$

where $\Delta_A : A \to A \times A$ is the diagonal map.

We shall recall the definition of pairings in [5]. A map $\mu : X \times Y \to Z$ is said to be a *pairing* with *axes* $f : X \to Z$ and $g : Y \to Z$ if the following diagram is homotopy commutative:

$$\begin{array}{ccc} X \times Y & \xrightarrow{\ \mu\ } & Z \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle \nabla_Z} \\ X \vee Y & \xrightarrow[f \vee g]{} & Z \vee Z \end{array}$$

where $j : X \vee Y \to X \times Y$ is the inclusion map.

Now let Γ be a co-grouplike space, namely an associative co-Hopf space with an inverse. Then ΓX is also a co-grouplike space. Let $j_1 : X \to X \vee Y$ and $j_2 : Y \to X \vee Y$ be the inclusions, and let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ the projections. We define a map

$$\rho = \Gamma(j_1 \circ p_1) \dot{+} \Gamma(j_2 \circ p_2) : \Gamma(X \times Y) \to \Gamma(X \vee Y).$$

Let $q : X \times Y \to X \wedge Y$ be the identification map. Then we have the following results.

Proposition 1.1. (Proposition 2.1, Theorem 2,4 and Proposition 2.6 in [6]) *There is a unique elemenet $v \in [\Gamma(X \wedge Y), \Gamma(X \times Y)]$ such that*

(I) $v \circ \Gamma q \dot{+} \Gamma j \circ \rho = 1_{\Gamma(X \times Y)}$ *in* $[\Gamma(X \times Y), \Gamma(X \times Y)]$.

And these ρ and v satisfy the following relations:

(II) $\rho \circ \Gamma j = 1_{\Gamma(X \vee Y)}$ *in* $[\Gamma(X \vee Y), \Gamma(X \vee Y)]$.

(III) $\Gamma q \circ v = 1_{\Gamma(X \wedge Y)}$ *in* $[\Gamma(X \wedge Y), \Gamma(X \wedge Y)]$.

(IV) $\rho \circ v = *$ *in* $[\Gamma(X \wedge Y), \Gamma(X \vee Y)]$.

Suppose that there is a pairing $\mu : X \times Y \to Z$. Then the Γ-Hopf construction is the element

$$J_\Gamma(\mu) \in [\Gamma(X \wedge Y), \Gamma Z]$$

defined by $J_\Gamma(\mu) = \Gamma\mu \circ v$ in [6]. If $\Gamma = S^1$(1-sphere), then the Γ-Hopf construction is one of the Hopf-type constructions in [1].

Let $t_{X\wedge W} : W \wedge X \to X \wedge W$ be the switching map defined by $t_{X\wedge W}(w \wedge x) = x \wedge w$ for any elements $x \in X$ and $w \in W$, where $x \wedge w$ is an elements in $X \wedge W$. It is a natural homeomorphim. The inverse map of $t_{X\wedge W}$ is denoted by $(t_{X\wedge W})^{-1} = t_{W\wedge X}$. For any map $f : X \to Y$and the Γ-suspension map $\Gamma t_{X\wedge W} : \Gamma(W \wedge X) \to \Gamma(X \wedge W)$, we have

$$\Gamma t_{Y\wedge W} \circ (\Gamma W f) = (\Gamma f \wedge 1_W) \circ \Gamma t_{X\wedge W}.$$

Let us define $\hat{\rho} = \Gamma W(j_1 \circ p_1) \dotplus \Gamma W(j_2 \circ p_2) : \Gamma W(X \times Y) \to \Gamma W(X \vee Y)$. Then we have a unique element $\hat{v} \in [\Gamma W(X \wedge Y), \Gamma W(X \times Y)]$ such that

$$\hat{v} \circ \Gamma W q \dotplus \Gamma W j \circ \hat{\rho} = 1_{\Gamma W(X \times Y)}$$

by Proposition 1.1. (We use ΓW instead of Γ.)

Lemma 1.2. *For the elements*

$$v \in [\Gamma(X \wedge Y), \Gamma(X \times Y)] \quad \text{and} \quad \hat{v} \in [\Gamma W(X \wedge Y), \Gamma W(X \times Y)],$$

the following diagram is homotopy commutative.

$$\begin{array}{ccc} \Gamma W \wedge (X \wedge Y) & \xrightarrow{\hat{v}} & \Gamma W \wedge (X \times Y) \\ \Big\downarrow{\scriptstyle \Gamma t_{(X\wedge Y)\wedge W}} & & \Big\downarrow{\scriptstyle \Gamma t_{(X\times Y)\wedge W}} \\ \Gamma(X \wedge Y) \wedge W & \xrightarrow{v \wedge 1_W} & \Gamma(X \times Y) \wedge W \end{array}$$

Proof. We have $\Gamma t_{(X\wedge Y)\wedge W} \circ \Gamma W q = (\Gamma q \wedge 1_W) \circ (\Gamma t_{(X\times Y)\wedge W})$ and

$$\begin{aligned} &\Gamma t_{(X\times Y)\wedge W} \circ \Gamma W j \circ \hat{\rho} \\ &= (\Gamma j \wedge 1_W) \circ \Gamma t_{(X\vee Y)\wedge W} \circ (\Gamma W(j_1 \circ p_1) \dotplus \Gamma W(j_2 \circ p_2)) \\ &= (\Gamma j \wedge 1_W) \circ \{(\Gamma t_{(X\vee Y)\wedge W} \circ \Gamma W(j_1 \circ p_1)) \\ &\qquad\qquad \dotplus (\Gamma t_{(X\vee Y)\wedge W} \circ \Gamma W(j_2 \circ p_2))\} \\ &= (\Gamma j \wedge 1_W) \circ \{(\Gamma((j_1 \circ p_1) \wedge 1_W) \circ \Gamma t_{(X\times Y)\wedge W}) \\ &\qquad\qquad \dotplus (\Gamma((j_2 \circ p_2) \wedge 1_W) \circ \Gamma t_{(X\times Y)\wedge W})\} \\ &= (\Gamma j \wedge 1_W) \circ \{\Gamma((j_1 \circ p_1) \wedge 1_W) \dotplus \Gamma((j_2 \circ p_2) \wedge 1_W)\} \circ \Gamma t_{(X\times Y)\wedge W} \\ &= (\Gamma j \wedge 1_W) \circ (\rho \wedge 1_W) \circ \Gamma t_{(X\times Y)\wedge W}. \end{aligned}$$

Therefore by the relation

$$\begin{aligned}&(\Gamma t_{(X\times Y)\wedge W})^{-1}\circ(v\wedge 1_W)\circ(\Gamma q\wedge 1_W)\circ(\Gamma t_{(X\times Y)\wedge W})\\ &\quad\dot{+}\ (\Gamma t_{(X\times Y)\wedge W})^{-1}\circ(\Gamma j\wedge 1_W)\circ(\rho\wedge 1_W)\circ(\Gamma t_{(X\times Y)\wedge W})\\ &\quad= (\Gamma t_{(X\times Y)\wedge W})^{-1}\circ 1_{\Gamma(X\times Y)\wedge W}\circ(\Gamma t_{(X\times Y)\wedge W}),\end{aligned}$$

we have

$$\{(\Gamma t_{(X\times Y)\wedge W})^{-1}\circ(v\wedge 1_W)\circ(\Gamma t_{(X\wedge Y)\wedge W})\}\circ\Gamma Wq\ \dot{+}\ \Gamma Wj\circ\hat{\rho} = 1_{\Gamma W(X\times Y)}.$$

Since $\hat{v}$ is a unique element which satisfies

$$\hat{v}\circ\Gamma Wq\ \dot{+}\ \Gamma Wj\circ\hat{\rho} = 1_{\Gamma W(X\times Y)},$$

we have $(\Gamma t_{(X\times Y)\wedge W})^{-1}\circ(v\wedge 1_W)\circ(\Gamma t_{(X\wedge Y)\wedge W}) = \hat{v}$. Hence we have the result.

Proposition 1.3. *If we are given a pairing $\mu : X\times Y\to Z$, then we have*

$$\Gamma t_{Z\wedge W}\circ J_{\Gamma W}(\mu) = \{J_\Gamma(\mu)\wedge 1_W\}\circ\Gamma t_{(X\wedge Y)\wedge W}.$$

Proof. By Lemma 1.2, we have

$$\begin{aligned}\Gamma t_{Z\wedge W}\circ J_{\Gamma W}(\mu) &= \Gamma t_{Z\wedge W}\circ\Gamma W\mu\circ\hat{v}\\ &= \Gamma(\mu\wedge 1_W)\circ\Gamma t_{(X\times Y)\wedge W}\circ\hat{v}\\ &= \Gamma(\mu\wedge 1_W)\circ(v\wedge 1_W)\circ\Gamma t_{(X\wedge Y)\wedge W}\\ &= \{(\Gamma\mu\circ v)\wedge 1_W\}\circ\Gamma t_{(X\wedge Y)\wedge W}\\ &= \{J_\Gamma(\mu)\wedge 1_W\}\circ\Gamma t_{(X\wedge Y)\wedge W}.\end{aligned}$$

Now we call a map $\mu_W : (X\times Y)\wedge W\to Z$ a *skew pairing* with *axes* $f : X\wedge W\to Z$ and $g : Y\wedge W\to Z$ when the following diagram is homotopy commutative:

$$\begin{array}{ccc}(X\times Y)\wedge W & \xrightarrow{\quad\mu_W\quad} & Z\\ \uparrow{\scriptstyle j\wedge 1_W} & & \uparrow{\scriptstyle\nabla_Z}\\ (X\vee Y)\wedge W\approx(X\wedge W)\vee(Y\wedge W) & \xrightarrow[f\vee g]{} & Z\vee Z\end{array}$$

If $W = S^0 = \{-1,1\}$, then a skew pairing μ_W is an ordinary pairing $\mu : X\times Y\to Z$.

Let $Z^W = \mathrm{map}^*(W,Z)$ be the space of base point preserving maps from W to Z with compact-open topology. Let W be a fixed space. For any space A and Z, let

$$\theta_W : [A\wedge W, Z]\longrightarrow[A, Z^W]$$

be the adjoint map. A map $\mu_W : (X\times Y)\wedge W\to Z$ is a skew pairing with axes $f : X\wedge W\to Z$ and $g : Y\wedge W\to Z$ if and only if a map $\theta_W(\mu_W) : X\times Y\to Z^W$ is an ordinary pairing with axes $\theta_W(f) : X\to Z^W$ and $\theta_W(g) : Y\to Z^W$.

Let ΩX denote the loop space, that is, $X^{S^1} = \mathrm{map}^*(S^1, X)$. Let us assume that Z is a connected CW-complex. Let Z_∞ be the reduced product space and

$\zeta : Z_\infty \to \Omega\Sigma Z$ the homotopy equivalence proved by James [3]. We define an isomorphism ϕ by

$$\phi : [A, Z_\infty] \xrightarrow{\zeta_*} [A, \Omega\Sigma Z] \xrightarrow{\theta_\Sigma^{-1}} [\Sigma A, \Sigma Z].$$

Suppose that there is a skew pairing $\mu_W : (S^m \times S^n) \wedge W \to Z$ with axes $f : \Sigma^m W \to Z$ and $g : \Sigma^n W \to Z$. Let $M : (S^m \times S^n) \wedge W \to Z_\infty$ be a skew pairing defined by $M = (i \circ f \circ (p_1 \wedge 1_W)) \dotplus (i \circ g \circ (p_2 \wedge 1_W))$. Then in [2], Hardie and Jansen define a Hopf construction $c(f, g) \in [\Sigma^{m+n+1} W, \Sigma X]$ (we call it the Hopf construction with a space W) by

$$c(f,g) = \left\{ \phi\Big(\theta_W^{-1}\big(d(\theta_W(M), \theta_W(i \circ \mu_W))\big)\Big) \,\middle|\, \begin{matrix} \mu_W : (S^m \times S^n) \wedge W \to Z \\ \text{with axes } f \text{ and } g \end{matrix} \right\}$$

where $d(\theta_W(M), \theta_W(i \circ \mu_W))$ is the difference element of $\theta_W(M)$ and $\theta_W(i \circ \mu_W)$ defined by James in [4].

When we are given a skew pairing $\mu_W : (X \times Y) \wedge W \to Z$, we give the following definition, using the element $v \in [\Gamma(X \wedge Y), \Gamma(X \times Y)]$ in Proposition 1.1.

Definition 1.4. If we are given a skew pairing $\mu_W : (X \times Y) \wedge W \to Z$, we define a Γ-*Hopf construction with a space* W by

$$J_\Gamma^W(\mu_W) = \Gamma\mu_W \circ (v \wedge 1_W) \in [\Gamma(X \wedge Y) \wedge W, \Gamma Z].$$

If $W = S^0$, then $J_\Gamma^W(\mu_W)$ is the Γ-Hopf construction $J_\Gamma(\mu)$. If we are given a pairing $\mu : X \times Y \to Z$, then we have $J_\Gamma^W(\mu \wedge 1_W) = J_\Gamma(\mu) \wedge 1_W$.

Lemma 1.5. (Theorem (8.7) and (8.8) of Chapter X in [10]) *Let X be a co-Hopf space. For any elements α and β in $[X, Y]$, and γ in $[A, B]$, we have*

$$\gamma \wedge (\alpha \dotplus \beta) = \gamma \wedge \alpha \dotplus \gamma \wedge \beta, \quad (\alpha \dotplus \beta) \wedge \gamma = \alpha \wedge \gamma \dotplus \beta \wedge \gamma.$$

Proposition 1.6. *If we are given a skew pairing $\mu_W : (X \times Y) \wedge W \to Z$, then we have*

(i) $J_\Gamma^W(\beta \circ \mu_W) = \Gamma\beta \circ J_\Gamma^W(\mu_W)$

(ii) $J_\Gamma^W(\mu_W \circ ((\gamma \times \delta) \wedge 1_W)) = J_\Gamma^W(\mu_W) \circ \Gamma(\gamma \wedge \delta) \wedge 1_W$

where $\beta : Z \to Z'$, $\gamma : X' \to X$ and $\delta : Y' \to Y$ are any maps.

Proof. (i) We have

$$J_\Gamma^W(\beta \circ \mu_W) = \Gamma(\beta \circ \mu_W) \circ (v \wedge 1_W) = \Gamma\beta \circ \Gamma\mu_W \circ (v \wedge 1_W) = \Gamma\beta \circ J_\Gamma^W(\mu_W).$$

(ii) Let $\rho' = \Gamma(j_1' \circ p_1') \dotplus \Gamma(j_2' \circ p_2')$ where $j_1' : X' \to X' \vee Y'$, $j_2' : Y' \to X' \vee Y'$ are the inclusions and $p_1' : X' \times Y' \to X'$, $p_2' : X' \times Y' \to Y'$ are the projections. Then there exists an element $v' \in [\Gamma(X' \wedge Y'), \Gamma(X' \times Y')]$ such that $v' \circ \Gamma q' \dotplus \Gamma j' \circ \rho' = 1_{\Gamma(X' \times Y')}$ by Proposition 1.1, where $j' : X' \vee Y' \to X' \times Y'$ is the inclusion map and

$q' : X' \times Y' \to X' \wedge Y'$ is the identification map. By Proposition 2.5 of [6], we have

$$\begin{aligned} J_\Gamma^W(\mu_W \circ ((\gamma \times \delta) \wedge 1_W)) &= \Gamma(\mu_W \circ ((\gamma \times \delta) \wedge 1_W)) \circ (v' \wedge 1_W) \\ &= \Gamma\mu_W \circ (\Gamma(\gamma \times \delta) \wedge 1_W) \circ (v' \wedge 1_W) \\ &= \Gamma\mu_W \circ (v \wedge 1_W) \circ (\Gamma(\gamma \wedge \delta) \wedge 1_W) \\ &= J_\Gamma^W(\mu_W) \circ \Gamma(\gamma \wedge \delta) \wedge 1_W. \end{aligned}$$

Proposition 1.7. *If we are given a skew pairing $\mu_W : (X \times Y) \wedge W \to Z$ with axes $f : X \wedge W \to Z$ and $g : Y \wedge W \to Z$, then we have*

$$J_\Gamma^W(\mu_W) \circ (\Gamma q \wedge 1_W) = \Gamma\mu_W \dot{-} (\Gamma f \circ (\Gamma p_1 \wedge 1_W) \dot{+} \Gamma g \circ (\Gamma p_2 \wedge 1_W)).$$

Proof. We have

$$\begin{aligned} &J_\Gamma^W(\mu_W) \circ (\Gamma q \wedge 1_W) \\ &\quad = \Gamma\mu_W \circ (v \wedge 1_W) \circ (\Gamma q \wedge 1_W) \\ &\quad = \Gamma\mu_W \circ (1_{\Gamma(X \times Y) \wedge W} \dot{-} (\Gamma j \circ \rho) \wedge 1_W) \\ &\quad = \Gamma\mu_W \dot{-} \Gamma\mu_W \circ (\Gamma j \wedge 1_W) \circ (\rho \wedge 1_W) \\ &\quad = \Gamma\mu_W \dot{-} \Gamma(\nabla_Z \circ (f \vee g)) \circ ((\Gamma(j_1 \circ p_1) \dot{+} \Gamma(j_2 \circ p_2)) \wedge 1_W) \\ &\quad = \Gamma\mu_W \dot{-} \{\Gamma(\nabla_Z \circ (f \vee g) \circ (j_1 \wedge 1_W) \circ (p_1 \wedge 1_W)) \\ &\qquad\qquad \dot{+} \Gamma(\nabla_Z \circ (f \vee g) \circ (j_2 \wedge 1_W) \circ (p_2 \wedge 1_W))\} \\ &\quad = \Gamma\mu_W \dot{-} (\Gamma f \circ (\Gamma p_1 \wedge 1_W) \dot{+} \Gamma g \circ (\Gamma p_2 \wedge 1_W)). \end{aligned}$$

Let $\rho_W = \Gamma((j_1 \wedge 1_W) \circ (p_1 \wedge 1_W)) \dot{+} \Gamma((j_2 \wedge 1_W) \circ (p_2 \wedge 1_W))$. We see $\rho_W = \rho \wedge 1_W$ by the definitions of ρ and ρ_W. Then by the same reason as Proposition 1.1, we have a unique element $v_W \in [\Gamma(X \wedge Y) \wedge W, \Gamma(X \times Y) \wedge W]$ which satisfies

$$v_W \circ (\Gamma q \wedge 1_W) \dot{+} (\Gamma j \wedge 1_W) \circ \rho_W = 1_{\Gamma(X \times Y) \wedge W}.$$

Lemma 1.8. *For the elements $v \in [\Gamma(X \wedge Y), \Gamma(X \times Y)]$ and $v_W \in [\Gamma(X \wedge Y) \wedge W, \Gamma(X \times Y) \wedge W]$, we have the following relation ;*

$$v \wedge 1_W = v_W.$$

Proof. From (I) of Proposition 1.1, we have

$$\begin{aligned} (v \circ \Gamma q \dot{+} \Gamma j \circ \rho) \wedge 1_W &= 1_{\Gamma(X \times Y)} \wedge 1_W \\ (v \circ \Gamma q) \wedge 1_W \dot{+} (\Gamma j \circ \rho) \wedge 1_W &= 1_{\Gamma(X \times Y) \wedge W} \\ (v \wedge 1_W) \circ (\Gamma q \wedge 1_W) \dot{+} (\Gamma j \wedge 1_W) \circ \rho_W &= 1_{\Gamma(X \times Y) \wedge W} \end{aligned}$$

Hence, $v_W \circ (\Gamma q \wedge 1_W) = (v \wedge 1_W) \circ (\Gamma q \wedge 1_W)$ in the homotopy set $[\Gamma(X \times Y) \wedge W, \Gamma(X \times Y) \wedge W]$. Since $(\Gamma q \wedge 1_W)^*$ is a monomorphism, we have the result.

Remark 1.9. If W is a co-grouplike space, then $v_W = \Gamma v'$ for some $v' \in [(X \wedge Y) \wedge W, (X \times Y) \wedge W]$.

Let $h_2' : (Z_2, Z) \to (Z \wedge Z, *)$ be the shrinking map and $h_2 : Z_\infty \to (Z \wedge Z)_\infty$ the combinatorial extension of h_2' in [3]. Then we define the Hopf invariant by $H = \phi \circ (h_2)_* \circ \phi^{-1}$:

$$\begin{array}{ccc} [A, Z_\infty] & \xrightarrow{(h_2)_*} & [A, (Z \wedge Z)_\infty] \\ \phi \downarrow & & \downarrow \phi \\ [\Sigma A, \Sigma Z] & \xrightarrow[H]{} & [\Sigma A, \Sigma(Z \wedge Z)]. \end{array}$$

We see this in (2.7) of [9] if Z is a sphere. In general, $H : [\Sigma A, \Sigma Z] \to [\Sigma A, \Sigma(Z \wedge Z)]$ is not a homomorphism, since h_2 is not a Hopf map. If A is a co-Hopf space, then the Hopf invariant H is a homomorphism. The following results are known.

Proposition 1.10. (cf. Proposition 2.2 in [9]) *Let $\alpha : A \to B$, $\beta : \Sigma B \to \Sigma Z$, $\gamma : \Sigma A \to \Sigma Z$ and $\delta : Z \to Z'$ be any maps. Then we have*
(i) $H(\beta \circ \Sigma\alpha) = H(\beta) \circ \Sigma\alpha$.
(ii) $H(\Sigma\delta \circ \gamma) = \Sigma(\delta \wedge \delta) \circ H(\gamma)$.

Theorem 1.11. *Let $v \in [\Sigma(X \wedge Y), \Sigma(X \times Y)]$ be the element of Proposition 1.1 and $\beta : \Sigma(X \times Y) \wedge W \to \Sigma Z$ any map. If W is a co-Hopf space, then we have*

$$H(\beta \circ (v \wedge 1_W)) = H(\beta) \circ (v \wedge 1_W).$$

Proof. Since W is a co-Hopf space, the Hopf invariant

$$H : [\Sigma(X \times Y) \wedge W, \Sigma Z] \longrightarrow [\Sigma(X \times Y) \wedge W, \Sigma(Z \wedge Z)]$$

is a homomorphism. Then we have

$$\begin{aligned} & H(\beta \circ (v \wedge 1_W)) \circ (\Sigma q \wedge 1_W) \\ &= H(\beta \circ (v \wedge 1_W) \circ (\Sigma q \wedge 1_W)) \\ &= H(\beta \circ (1_{\Sigma(X \times Y)} \dot{-} \Sigma j \circ \rho) \wedge 1_W)) \qquad \text{by (I) of Proposition 1.1.} \\ &= H(\beta \circ 1_{\Sigma(X \times Y) \wedge W}) \dot{-} H(\beta \circ ((\Sigma j \circ \rho) \wedge 1_W)) \\ &= H(\beta) \dot{-} H(\beta \circ (\Sigma j \circ (\Sigma(j_1 \circ p_1) \dot{+} \Sigma(j_2 \circ p_2)) \wedge 1_W)) \\ &= H(\beta) \dot{-} (H(\beta \circ (\Sigma(j \circ j_1 \circ p_1) \wedge 1_W)) \dot{+} H(\beta \circ (\Sigma(j \circ j_2 \circ p_2) \wedge 1_W))) \\ &= H(\beta) \dot{-} (H(\beta) \circ (\Sigma(j \circ j_1 \circ p_1) \wedge 1_W) \dot{+} H(\beta) \circ (\Sigma(j \circ j_2 \circ p_2) \wedge 1_W))) \\ &= H(\beta) \circ (1_{\Sigma(X \times Y) \wedge W} \dot{-} (\Sigma j \wedge 1_W) \circ ((\Sigma(j_1 \circ p_1) \wedge 1_W) \dot{+} (\Sigma(j_2 \circ p_2) \wedge 1_W))) \\ &= H(\beta) \circ (1_{\Sigma(X \times Y) \wedge W} \dot{-} (\Sigma j \wedge 1_W) \circ (\rho \wedge 1_W)) \\ &= H(\beta) \circ (v \wedge 1_W) \circ (\Sigma q \wedge 1_W). \end{aligned}$$

Since $(\Sigma q \wedge 1_W)^*$ is a monomorphism, we have the result.

Remark 1.12. If W is a co-grouplike space, then the proof is simplified making use of (i) of Proposition 1.10, since $v \wedge 1_W$ is a suspension element by Remark 1.9.

Theorem 1.13. *Suppose that there is a skew pairing* $\mu_W : (X \times Y) \wedge W \to Z$. *If* W *is a co-Hopf space, then* $H(J_\Sigma^W(\mu_W)) = 0$.

Proof. By Theorem 1.11, we have

$$H(J_\Sigma^W(\mu_W)) = H(\Sigma\mu_W \circ (v \wedge 1_W)) = H(\Sigma\mu_W) \circ (v \wedge 1_W) = 0.$$

Hence, we have the result.

2. A Hopf construction induced by a skew pairing

In this section we define a *generalized Hopf construction* $c(\alpha)$ induced by a skew pairing of the type $\alpha : (X \times Y) \wedge W \to Z_\infty$. Suppose that there is a skew pairing $\alpha : (X \times Y) \wedge W \to Z_\infty$. Then we define an element

$$c(\alpha) = \phi(\alpha) \circ (v \wedge 1_W) \in [\Sigma(X \wedge Y) \wedge W, \Sigma Z].$$

For a skew pairing $\mu_W : (X \times Y) \wedge W \to Z$ and the inclusion map $i : Z \to Z_\infty$, we have $c(i \circ \mu_W) = J_\Sigma^W(\mu_W)$. The following proposition is proved by the method similar to the proof of Proposition 1.7.

Proposition 2.1. *If we are given a skew pairing* $\alpha : (X \times Y) \wedge W \to Z_\infty$ *with axes* $f : X \wedge W \to Z_\infty$ *and* $g : Y \wedge W \to Z_\infty$, *then we have*

$$c(\alpha) \circ (\Sigma q \wedge 1_W) = \phi(\alpha) \dot{-} \{\phi(f) \circ (\Sigma p_1 \wedge 1_W) \dot{+} \phi(g) \circ (\Sigma p_2 \wedge 1_W)\}.$$

Theorem 2.2. *If we are given two skew pairings* $\alpha, \beta : (X \times Y) \wedge W \to Z_\infty$ *with the same axes* $f : X \wedge W \to Z_\infty$ *and* $g : Y \wedge W \to Z_\infty$, *then the following relation holds.*

$$c(\alpha) \dot{-} c(\beta) = c(\alpha \underset{\cdot}{-} \beta).$$

Proof. Since the space Z_∞ is a Hopf space, we have an exact sequence:

$$0 \longrightarrow [(X \wedge Y) \wedge W, Z_\infty] \overset{(q\wedge 1_W)^*}{\longrightarrow} [(X \times Y) \wedge W, Z_\infty] \overset{(j\wedge 1_W)^*}{\longrightarrow} [(X \vee Y) \wedge W, Z_\infty] \longrightarrow 0$$

by Lemma 1.3.5. in [11]. Then we have an element $\gamma \in [(X \wedge Y) \wedge W, Z_\infty]$ such that $\alpha \underset{\cdot}{-} \beta = \gamma \circ (q \wedge 1_W)$. Then by Proposition 2.1, we have

$$\begin{aligned}
&(c(\alpha) \dot{-} c(\beta)) \circ (\Sigma q \wedge 1_W)\\
&\quad = \{\phi(\alpha) \dot{-} (\phi(f) \circ (\Sigma p_1 \wedge 1_W) \dot{+} \phi(g) \circ (\Sigma p_2 \wedge 1_W))\}\\
&\quad\quad \dot{-} \{\phi(\beta) \dot{-} (\phi(f) \circ (\Sigma p_1 \wedge 1_W) \dot{+} \phi(g) \circ (\Sigma p_2 \wedge 1_W))\}\\
&\quad = \phi(\alpha) \dot{-} \phi(\beta)\\
&\quad = \phi(\alpha \underset{\cdot}{-} \beta)\\
&\quad = \phi(\gamma \circ (q \wedge 1_W)) = \phi(\gamma) \circ (\Sigma q \wedge 1_W)
\end{aligned}$$

and

$$\begin{aligned} c(\alpha \dot{-} \beta) \circ (\Sigma q \wedge 1_W) &= \phi(\alpha \dot{-} \beta) \circ (v \wedge 1_W) \circ (\Sigma q \wedge 1_W) \\ &= \phi(\gamma \circ (q \wedge 1_W)) \circ (v \wedge 1_W) \circ (\Sigma q \wedge 1_W) \\ &= \phi(\gamma) \circ ((\Sigma q \circ v) \wedge 1_W) \circ (\Sigma q \wedge 1_W) \\ &= \phi(\gamma) \circ (\Sigma q \wedge 1_W) \qquad \text{by (III) of Proposition 1.1.} \end{aligned}$$

Since $(\Sigma q \wedge 1_W)^*$ is a monomorphism, we have the result.

Corollary 2.3. *If we are given two skew pairings μ_W, $\mu'_W : (X \times Y) \wedge W \to Z$ with the same axes, then we have the following relation.*

$$J_\Sigma^W(\mu_W) \dot{-} J_\Sigma^W(\mu'_W) = c(i \circ \mu_W \dot{-} i \circ \mu'_W).$$

Proof. Since $c(i \circ \mu_W) = J_\Sigma^W(\mu_W)$ and $c(i \circ \mu'_W) = J_\Sigma^W(\mu'_W)$, we have the result.

Corollary 2.4. *If we are given two pairings μ, $\mu' : X \times Y \to Z$ with the same axes, then we have the following relation.*

$$J_\Sigma(\mu) \dot{-} J_\Sigma(\mu') = c(i \circ \mu \dot{-} i \circ \mu').$$

For any elements $\alpha \in [\Gamma X, Z]$ and $\beta \in [\Gamma Y, Z]$, Γ-Whitehead product $[\alpha, \beta]_\Gamma$ is defined in [7]. Now we recall that for any elements $\alpha \in [\Gamma X \wedge W, Z]$ and $\beta \in [\Gamma Y \wedge W, Z]$, the generalized Hardie-Jansen product $[\alpha, \beta]_\Gamma^W \in [\Gamma(X \wedge Y) \wedge W, Z]$ is defined by $\theta_W^{-1}([\theta_W(\alpha), \theta_W(\beta)]_\Gamma)$ in [8]. It is characterized by a relation

$$\begin{aligned} &[\alpha, \beta]_\Gamma^W \circ (\Gamma q \wedge 1_W) \\ &\quad = \alpha \circ (\Gamma p_1 \wedge 1_W) \dot{+} \beta \circ (\Gamma p_2 \wedge 1_W) \dot{-} \alpha \circ (\Gamma p_1 \wedge 1_W) \dot{-} \beta \circ (\Gamma p_2 \wedge 1_W) \end{aligned}$$

which is a commutator of $\alpha \circ (\Gamma p_1 \wedge 1_W)$ and $\beta \circ (\Gamma p_2 \wedge 1_W)$ by Corollary 1.14 in [8]. If $W = S^0$, then this product $[\alpha, \beta]_\Gamma^W$ is the Γ-Whitehead product.

For any maps $\alpha : X \wedge W \to Z_\infty$ and $\beta : Y \wedge W \to Z_\infty$, we define two skew pairings M, $\overline{M} : (X \times Y) \wedge W \to Z_\infty$ by

$$M = (\alpha \circ (p_1 \wedge 1_W)) \dot{+} (\beta \circ (p_2 \wedge 1_W)),$$

and

$$\overline{M} = (\beta \circ (p_2 \wedge 1_W)) \dot{+} (\alpha \circ (p_1 \wedge 1_W)).$$

Then the skew pairings M and $\overline{M}$ have the axes $\alpha : X \wedge W \to Z_\infty$ and $\beta : Y \wedge W \to Z_\infty$.

Theorem 2.5. *For any maps $\alpha : X \wedge W \to Z_\infty$ and $\beta : Y \wedge W \to Z_\infty$, we have*

$$c(M) = 0 \quad \text{and} \quad c(\overline{M}) = \dot{-} [\phi(\alpha), \phi(\beta)]_\Sigma^W.$$

Proof. By Proposition 2.1, we have

$$\begin{aligned} c(M)\circ(\Sigma q\wedge 1_W) &= \phi(M) \dot{-} (\phi(\alpha)\circ(\Sigma p_1\wedge 1_W) \dot{+} \phi(\beta)\circ(\Sigma p_2\wedge 1_W)) \\ &= \phi(M) \dot{-} \{\phi(\alpha\circ(p_1\wedge 1_W)) \dot{+} \phi(\beta\circ(p_2\wedge 1_W))\} \\ &= \phi(M) \dot{-} \phi\{(\alpha\circ(p_1\wedge 1_W)) \dot{+} (\beta\circ(p_2\wedge 1_W))\} \\ &= \phi(M) \dot{-} \phi(M) = 0. \end{aligned}$$

$$\begin{aligned} c(\overline{M})\circ(\Sigma q\wedge 1_W) &= \phi(\overline{M}) \dot{-} (\phi(\alpha)\circ(\Sigma p_1\wedge 1_W) \dot{+} \phi(\beta)\circ(\Sigma p_2\wedge 1_W)) \\ &= (\phi(\beta)\circ(\Sigma p_2\wedge 1_W) \dot{+} \phi(\alpha)\circ(\Sigma p_1\wedge 1_W)) \\ &\quad \dot{-} (\phi(\alpha)\circ(\Sigma p_1\wedge 1_W) \dot{+} \phi(\beta)\circ(\Sigma p_2\wedge 1_W)) \\ &= \phi(\beta)\circ(\Sigma p_2\wedge 1_W) \dot{+} \phi(\alpha)\circ(\Sigma p_1\wedge 1_W) \\ &\quad \dot{-} \phi(\beta)\circ(\Sigma p_2\wedge 1_W) \dot{-} \phi(\alpha)\circ(\Sigma p_1\wedge 1_W) \\ &= \dot{-} [\phi(\alpha),\phi(\beta)]^W_\Sigma \circ(\Sigma q\wedge 1_W). \end{aligned}$$

Since $(\Sigma q\wedge 1_W)^*$ is a monomorphism, we have the results.

We remark that for any maps $\alpha : S^m \to Z_\infty$ and $\beta : S^n \to Z_\infty$ (and hence $W = S^0$ in this case), I. M. James proves the relation $\phi(d(M,\overline{M})) = (\dot{-}1)^m[\phi(\alpha),\phi(\beta)]_\Sigma$ in Theorem 6.1 of [4]. Now let $f : X \to Z$ and $g : Y \to Z$ be any maps. By Theorem 2.5 ($W = S^0$, $\alpha = i\circ f$, $\beta = i\circ g$), we have $c(\overline{M}) = \dot{-}[\Sigma f,\Sigma g]_\Sigma$ for a pairing $\overline{M} = (i\circ g\circ p_2) \dot{+} (i\circ f\circ p_1)$. Hence $c(\overline{M})$ is the generalization of $\phi(d(M,\overline{M}))$.

We define a natural map $\sigma : (X\times Y)\wedge W \to (X\wedge W)\times(Y\wedge W)$ by

$\sigma((x,y)\wedge w) = (p_1(x,y)\wedge w, p_2(x,y)\wedge w) = (x\wedge w, y\wedge w)$ $(x\in X,\ y\in Y,\ w\in W)$.

Then the skew pairing $M = (\alpha\circ(p_1\wedge 1_W)) \dot{+} (\beta\circ(p_2\wedge 1_W))$ is expressed by

$$M = m\circ(\alpha\times\beta)\circ\sigma : (X\times Y)\wedge W \to Z_\infty.$$

Let $\chi : W \to W\wedge W$ be the reduced diagonal map and let

$$\tau : X\wedge Y\wedge W\wedge W \to X\wedge W\wedge Y\wedge W$$

be the natural homeomorphism interchanging the second and third factors of the smash products. We suppose that there is a skew pairing $\mu_W : (S^m\times S^n)\wedge W \to Z$ with axes $f : \Sigma^m W \to Z$ and $g : \Sigma^n W \to Z$. For the Hopf construction by Hardie and Jansen in [2], they prove the following relation (Theorem 2.4 in [2]):

$$H(c(f,g)) = \pm\Sigma(f\wedge g)\circ\Sigma^{m+n+1}\chi.$$

Here we determine the Hopf invariant of $\phi(M)$ for a skew pairing $M = (i\circ f\circ(p_1\wedge 1_W)) \dot{+} (i\circ g\circ(p_2\wedge 1_W))$.

Theorem 2.6. *We define a skew pairing* $M = (i\circ f\circ(p_1\wedge 1_W)) \dot{+} (i\circ g\circ(p_2\wedge 1_W))$ *for any maps* $f : X\wedge W \to Z$ *and* $g : Y\wedge W \to Z$. *Then we have*

$$H(\phi(M)) = \Sigma(f\wedge g)\circ\Sigma\tau\circ(1_{\Sigma X\wedge Y}\wedge\chi)\circ(\Sigma q\wedge 1_W)$$

and hence

$$H(\phi(M))\circ(v\wedge 1_W) = \Sigma(f\wedge g)\circ\Sigma\tau\circ(1_{\Sigma X\wedge Y}\wedge\chi).$$

Proof. The following diagram is commutative.

$$
\begin{array}{ccccccc}
(X\times Y)\wedge W & \xrightarrow{\sigma} & (X\wedge W)\times(Y\wedge W) & \xrightarrow{f\times g} & Z\times Z & \xrightarrow{i\times i} & Z_\infty\times Z_\infty \\
 & & & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\
\downarrow{\scriptstyle q\wedge 1_W} & & \downarrow{\scriptstyle q} & & Z_2 & \xrightarrow{i} & Z_\infty \\
 & & & & \downarrow{\scriptstyle h_2'} & & \downarrow{\scriptstyle h_2} \\
(X\wedge Y)\wedge W & \xrightarrow[\tau\circ(1_{X\wedge Y}\wedge\chi)]{} & (X\wedge W)\wedge(Y\wedge W) & \xrightarrow[f\wedge g]{} & (Z\wedge Z) & \xrightarrow[i]{} & (Z\wedge Z)_\infty
\end{array}
$$

Then by the diagram above, we have

$$
\begin{aligned}
H(\phi(M)) &= \phi(h_2\circ M) \\
&= \phi(h_2\circ((i\circ f\circ(p_1\wedge 1_W))\underset{\cdot}{+}(i\circ g\circ(p_2\wedge 1_W)))) \\
&= \phi(h_2\circ m\circ(i\times i)\circ(f\times g)\circ\sigma) \\
&= \phi(i\circ(f\wedge g)\circ\tau\circ(1_{X\wedge Y}\wedge\chi)\circ(q\wedge 1_W)) \\
&= \Sigma(f\wedge g)\circ\Sigma\tau\circ(1_{\Sigma X\wedge Y}\wedge\chi)\circ(\Sigma q\wedge 1_W)
\end{aligned}
$$

Corollary 2.7. *If $W = S^0$ in Theorem 2.6, then for $M = (i\circ f\circ p_1)\underset{\cdot}{+}(i\circ g\circ p_2)$ we have*

$$H(\phi(M)) = \Sigma(f\wedge g)\circ\Sigma q,\ \text{and hence}\ H(\phi(M))\circ v = \Sigma(f\wedge g).$$

Corollary 2.8. *Let $i_1 : X \to X\times Y$ and $i_2 : Y \to X\times Y$ be the inclusions. Then we have*

$$H(\Sigma j\circ\rho) = H(\Sigma(i_1\circ p_1)\dot{+}\Sigma(i_2\circ p_2)) = \Sigma(i_1\wedge i_2)\circ\Sigma q$$

Proof. We have

$$
\begin{aligned}
\Sigma j\circ\rho &= \Sigma j\circ(\Sigma(j_1\circ p_1)\dot{+}\Sigma(j_2\circ p_2)) \\
&= \Sigma(i_1\circ p_1)\dot{+}\Sigma(i_2\circ p_2) \\
&= \phi(i\circ i_1\circ p_1)\dot{+}\phi(i\circ i_2\circ p_2) \\
&= \phi((i\circ i_1\circ p_1)\underset{\cdot}{+}(i\circ i_2\circ p_2)).
\end{aligned}
$$

Then we see that $(i\circ i_1\circ p_1)\underset{\cdot}{+}(i\circ i_2\circ p_2) : X\times Y \to (X\times Y)_\infty$ is a pairing. Here we put $f = i_1$ and $g = i_2$ in Corollary 2.7. Then we have the result.

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