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COMMUTATIVE GROUP ALGEBRAS OF ABELIAN Σ-GROUPS

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ABSTRACT. Let G be an abelian group with first p-Ulm subgroup G^1, p-torsion part G_p and socle $G[p]$. The class of all abelian Σ-groups, which was introduced by Irwin-Walker (1961), is very large and possesses a key role in the abelian group theory. The first main result is actually a new explicit criterion for one arbitrary primary abelian group to be a Σ-group, namely: A p-primary abelian group A is a Σ-group if and only if $A[p]=\bigcup_{n=1}^{\infty} A_n$, $A_n \subseteq A_{n+1}$ and $A_n \cap A^{p^n}=A^1[p]$ for each $n \in \mathbb{N}$. In particular, some classical facts in this way are confirmed.

As applications, suppose FG is the group algebra over the field F of characteristic $p>0$ and $S(FG)$ is the group of all normed p-units in FG. The second central result, however, is the following: $S(FG)$ is a Σ-group if and only if G_p is a Σ-group, provided F is perfect. Besides, G_p a Σ-group and F perfect imply that $S(FG)/G_p$ is a Σ-group. As a final, it is shown that if $FH \cong FG$ as F-algebras for any group H and such that G_p is a Σ-group, then H_p is a Σ-group and moreover their high subgroups are isomorphic. In particular, G a p-mixed Σ-group whose G/G_p is reduced and the F-isomorphism $FH \cong FG$ does imply that H is a p-mixed Σ-group and even more, G and H have isomorphic high subgroups.

INTRODUCTION

As usual, throughout the rest of this article, G is an abelian group written multiplicatively with first (p-)Ulm subgroup (i.e. first Ulm subgroup with respect to p) G^1, with p-component G_p and with socle $G[p]$. Also everywhere in the text, F will denote a field of characteristic $p \neq 0$, and R denotes an unitary commutative ring of the same characteristic p. For R such a ring, RG designates the group algebra of G over R with a group of all normalized units $V(RG)$ and p-component $V_p(RG) = S(RG)$.

For A a subgroup of G, we define $I(RG; A)$ as a relative augmentation ideal of RG with respect to A. All other notations and the terminology are standard and follow essentially the classical monographies and books [10,18,21] and [19,20,26].

Our global aim here is to study the commutative group rings of abelian Σ-groups and more specially, the questions about the isomorphism between group algebras of such groups and the structure of the group of all normalized p-elements in such algebras. For this purpose we organize our work as follows: In the first paragraph, we will obtain a new simplified but more

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convenient for us major necessary and sufficient condition, when an arbitrary abelian p-group can be a Σ-group. Besides, we shall next summarize some significant facts very needed for our good presentation.

In the second paragraph, we investigate some specific conjectures for the group algebras of abelian Σ-group, applying the criterion founded by us above. The main multiplicities are selected in two sections, where are considered the modular and semisimple cases, respectively.

In the third paragraph, we conclude with some several left-open questions which immediately arise and which are of some interest and importance. We list only those which we believe are major.

And so, we start with

I. A CRITERION FOR AN ABELIAN GROUP TO BE A Σ-GROUP

First and foremost, we recall the definition for a Σ-group. Well, a subgroup H_G of G is said to be a high subgroup if H_G is maximal in G with respect to $H_G \cap G^1 = 1$. If H_G is a high subgroup of G, then H_G is pure in G, G/H_G is divisible and H_G contains a basic subgroup of G (see [15,16]). An arbitrary abelian group G is called a Σ-group if all of its high subgroups are direct sums of cyclic groups. If one high subgroup of G is a direct sum of cyclics, then all high subgroups of G are isomorphic and so G is a Σ-group (cf. [16,17,27]). This group class listed above was posed by Irwin-Walker [15,16] on 1961 and it possesses a paramount role in the abelian group theory. Besides, it is quite large and properly contains the class of all abelian groups such that their first Ulm factor is a direct sum of cyclics (in particular, as a consequence, all simply presented torsion groups are itself Σ-groups). Really, an example due to Megibben [24] demonstrates that there exists a p-torsion Σ-group G so that G^1 is homogeneous of order p and with power $\aleph_1$, and G/G^1 is unbounded torsion complete (whence it is not a direct sum of cyclics).

After the above discussion, we are in position to state and prove the following assertion described only in the terms of socles and heights.

Theorem (CRITERION). *The abelian p-group G is a Σ-group if and only if $G[p] = \bigcup_{n=1}^{\infty} G_n$, where $G_n \subseteq G_{n+1}$ and for every $n \in N$ is fulfilled that $G_n \cap G^{p^n} = G^1[p]$.*

Proof. By definition, G is a Σ-group if and only if some of its high subgroups is a direct sum of cyclics. Moreover H_G is a high subgroup of G if and only if H_G is pure in G and $G[p] = H_G[p] \times G^1[p]$ (cf. [28,15,16]). That is why applying for H_G the Kaplansky's form of the well-known and documented classical criterion of Kulikov for direct sum of cyclic groups, namely $H_G[p] = \bigcup_{n=1}^{\infty} H_n$, $H_n \subseteq H_{n+1}$ and $H_n \cap H_G^{p^n} = 1$ (see [18,12]), we deduce $G[p] = \bigcup_{n=1}^{\infty} (H_n \times G^1[p])$. Say, $G_n = G^1[p] \times H_n$, whence $G_n \subseteq G_{n+1}$ and $G_n \cap G^{p^n} =$

$(G^1[p] \times H_n) \cap (G^1[p] \times H_G^{p^n}[p]) = G^1[p] \times (H_n \cap H_G^{p^n}) = G^1[p]$, where we have applied the fact that $H_G^{p^n}$ is a high subgroup of G^{p^n} (cf. [15,16,2]). Thus the necessity holds.

Now, assume that the sufficiency is true. Furthermore $H_G[p] = \bigcup_{n=1}^{\infty} (G_n \cap H) = \bigcup_{n=1}^{\infty} H_n$, putting $H_n = H_G \cap G_n$. Apparently $H_n \subseteq H_{n+1}$, and besides we calculate $H_n \cap H_G^{p^n} = G_n \cap H_G^{p^n} = G_n \cap (G^{p^n} \cap H_G) = (G_n \cap G^{p^n}) \cap H_G = G^1 \cap H_G = 1$. Therefore by virtue of the Kaplansky criterion for direct sum of cyclic groups mentioned above, we may conclude that H_G is a direct sum of cyclics. Thus by definition, G must be a Σ-group, as claimed. The proof is finished. □

Next, we shall confirm some principal well-known and documented group-theoretic facts concerning the characteristic properties of Σ-groups, namely

– A pure subgroup of a primary Σ-group is a Σ-group (see [15,16]). A factor-group modulo pure subgroup of a Σ-group need not be a Σ-group. The converse is also impossible.

Indeed, let it is given that C is a pure subgroup in the Σ-group G. Then using the above necessary and sufficient condition we can write $G[p] = \bigcup_{n=1}^{\infty} G_n$, $G_n \subseteq G_{n+1}$ and $G_n \cap G^{p^n} = G^1[p]$. In this way we obtain $C[p] = \bigcup_{n=1}^{\infty} (G_n \cap C)$. Moreover, it is elementary to compute that $G_n \cap C \cap C^{p^n} = G_n \cap C^{p^n} = G_n \cap (G^{p^n} \cap C) = (G_n \cap G^{p^n}) \cap C = G^1[p] \cap C = C^1[p]$. Applying again the above criterion we get the claim.

– Direct sums of primary Σ-groups is a primary Σ-group (cf. [15,16]).

In fact, it is enough to apply our criterion together with some standard set – theoretical and group-theoretical observations that we leave to the reader.

– G is a Σ-group $\iff$ G^{p^n} is a Σ-group $(n \in N)$ (see [2]).

= Every p-group G with the property G/G^1 is a direct sum of cyclics is a Σ-group (in particular so is each simply presented group) (cf. [15,16]).

Well, using the standard form of the classical Kulikov's necessary and sufficient condition for direct sums of cyclic groups, we derive $G = \bigcup_{n=1}^{\infty} G_n$, $G_n \subseteq G_{n+1}$ and $G_n \cap G^{p^n} = G^1$. Thus we have automatically seen that our criterion for Σ-groups is immediately fulfilled.

= We shall say that one abelian group is a PT-group if it is a pure subgroup of a totally projective = simply presented (p-)torsion group (cf. [13]). In particular, PT-groups are all simply presented groups and their generalizations, so called S-groups of Warfield [14,13] and the more general A-groups of Hill [13]. Besides, all of these classes are Σ-groups since as we have seen a pure subgroup of a Σ-group (the simply presented groups are so) is a Σ-group; or by other way, their first Ulm factor is itself a direct sum of cyclics.

= The torsion subgroup of one Warfield group is an S-group (see [14]) whence it is a Σ-group.

= The $p^{\omega+n}$-projective p-group ($n \in N$) need not be a Σ-group.

This is so because every subgroup of $p^{\omega+n}$-projective group is themselves $p^{\omega+n}$-projective [25], while this is not the case with the Σ-groups [24,27,11].

= An unbounded cotorsion (in other terms, coperiodical) group (in particular an algebraically compact group) need not be absolutly a Σ-group (cf. [27]). More precise, each cotorsion Σ-group is bounded [27].

= An unbounded torsion-complete (or more generally, quasi-complete) p-group strongly is not a Σ-group (see [10]). More precisely, every quasi-complete (p-)torsion Σ-group is bounded [10].

= A torsion pure-complete Σ-group is a direct sum of cyclics.

This is so since we have no elements of infinite heights.

= All summable p-groups with countable lengths are Σ-groups.

And so, utilizing the classical Honda's criterion [10] which is a strong generalization of the presented above Kaplansky's criterion, we can write $G[p] = \bigcup_{n=1}^{\infty} G_n$, $G_n \subseteq G_{n+1}$ and G_n are height-finite with heights as computed in G for each $n \in N$. But therefore G_n has elements with bounded finite heights (for example at n), i.e. $G_n \cap G^{p^n} \subseteq G^1[p]$, as required in our criterion.

= The σ-summable p-groups need not be Σ-groups [4].

= in some instances the torsion reduced Σ-groups can be simply presented, summable or σ-summable, but this is a problem of some other approach might work (see, for example, [24,21,11]).

However, we mentione the nontrivial fact:

Proposition. *Let G be an abelian p-group of length $\omega+m$, where $m \in N_0$. Then G is a Σ-group if and only if it is summable.*

Proof. From our criterion it follows that $G[p] = \bigcup_{n<\omega} G_n$, $G_n \subseteq G_{n+1}$ and $G_n \cap G^{p^n} = G^{p^\omega}[p]$ for all naturals n where $G^{p^{\omega+m}} = 1$. It is a routine matter to see that G_n has only elements with a finite number of finite and infinite heights. Finally the Honda's criterion [10] yields the assertion. □

We continue with a significant paragraph entitled

II. Group algebras of abelian Σ-groups

We shall classify the results here of two sections, namely

1. Modular group algebras of abelian Σ-groups. A paramount role in this theory play the hypotheses for the *isomorphism* [19,20] and for the *direct factor* [19,20,9] (in particular, for the *structure*). First and foremost we shall examine the isomorphism conjecture.

AN ISOMORPHISM QUESTION

In the general situation, the isomorphism problem for group algebras of primary abelian groups asks the following: Does it follow that $FG \cong FH$ as F-algebras for some (p-)groups G and H will imply that $G \cong H$? The question has an affirmative answer for some large classes of abelian groups; see for example [1,23]. The general solution to this mystery is probably very very difficult, hence we can restate it thus: Suppose $G \in \mathcal{K}$, any class of abelian (p-)groups. Then whether $FG \cong FH$ as F-algebras implies $H \in \mathcal{K}$ and even more, is $G \cong H$? Here, we will answering in the affirmative only the first half of the above reformulation for the class of all abelian Σ-groups with p-torsion elements. Well, the main theorem which will be attached now, however, is announced in [6] and states as follows:

Theorem (ISOMORPHISM). *Presume G is so that G_p is a Σ-group. Then $FH \cong FG$ as F-algebras for any group H yields that H_p is a Σ-group and even more, that the high subgroups of G_p and H_p are isomorphic.*

Before proving the statement we need some technical preliminaries, starting with

Proposition (INVARIANTS). *The F-group algebra FG determines invariantly the group $1 + I(F^{p^n}G^{p^n}; G^1[p])$.*

Proof. In fact, it is no harm in assuming that $FG = FH$ for any group H. Therefore $F^{p^n}G^{p^n} = (FG)^{p^n} = (FH)^{p^n} = F^{p^n}H^{p^n}$. On the other hand analogously $F^1G^1 = F^1H^1$ and hence owing to [4] we derive $S(F^1G^1)[p] = 1{+}I(F^1G^1; G^1[p]) = 1{+}I(F^1H^1; H^1[p]) = S(F^1H^1)[p]$, i.e. $I(F^1G^1; G^1[p]) = I(F^1H^1; H^1[p])$. Finally we yield $1 + I(F^{p^n}G^{p^n}; G^1[p]) = 1 + F^{p^n}G^{p^n}. I(F^1G^1; G^1[p])1 + F^{p^n}H^{p^n}.I(F^1H^1; H^1[p]) = 1 + I(F^{p^n}H^{p^n}; H^1[p])$. So, the proof is completed. □

Next, we come to the

Lemma (INTERSECTION). *Suppose $1 \in P \leq R$ and $A, B \leq G$. Then*

$$(1 + I(RG; A)) \cap S(PB) \subseteq 1 + I(PB; B \cap A).$$

Proof. It is given with all details in [4]. □

Now, we are ready to attack

Proof of the theorem. Applying our criterion stated and proved in paragraph I, we may write $G[p] = \bigcup_{n=1}^{\infty} G_n$, where $G_n \subseteq G_{n+1}$ and $G_n \cap G^{p^n} = G^1[p]$. Without loss of generality we can assume that $FG = FH$ and that F is perfect. Therefore, employing [4] we establish $S(FG)[p] = S(FH)[p] = \bigcup_{n=1}^{\infty} S(FG; G_n)$ where $S(FG; A) \overset{def}{=} 1 + I(FG; A)$ whenever $A \leq G$, whence we have $H[p] = \bigcup_{n=1}^{\infty} (S(FG;G_n) \cap H[p])$. It is evident that $S(FG;G_n) \cap H[p] \subseteq S(FG;G_{n+1}) \cap H[p]$. Moreover, invoking to the listed above previous facts, we compute $S(FG;G_n) \cap H[p] \cap H^{p^n} = S(FG;G_n) \cap H[p] \cap S^{p^n}(FH) = S(FG;G_n) \cap H[p] \cap S^{p^n}(FG) = S(FG;G_n) \cap S(FG^{p^n}) \cap H[p] = S(FG^{p^n};G_n \cap G^{p^n}) \cap H[p] = S(FG^{p^n};G^1[p]) \cap H[p] = S(FH^{p^n};H^1[p]) \cap H[p] = H^1[p]$, as desired. Thus by virtue of the group criterion, H_p must be a Σ-group, as claimed.

Well, G_p and H_p are both Σ-groups, hence their high subgroups are direct sums of cyclics. But from a monumental result of W.May [22,1], G_p and H_p have equal functions of Ulm-Kaplansky. Furthermore [16] ensure that equal Ulm-Kaplansky invariants have their respective high subgroups. That is why they are isomorphic [10], as stated. The proof is finished after all. □

We begin with direct consequences, namely

Corollary. *Let G be a Σ-group and $FH \cong FG$ as F-algebras for any group H. Then G_p and H_p have isomorphic high subgroups.*

Proof. Clearly G a Σ-group implies that G_p is one also. Thus the main theorem will be applied to complete the proof in general. □

Corollary. *Let G be a p-torsion Σ-group and $FH \cong FG$ as F-algebras for some group H. Then H is a p-torsion Σ-group, as G and H have isomorphic high subgroups.*

The next is also valuable

Corollary. *Let G be an abelian group whose G_p is simply presented. Then the F-isomorphism $FH \cong FG$ for any group H gives that H_p and G_p have isomorphic high subgroups.*

Remark. *The last consequence ensure a positive light on the long-standing and very difficult problem raised by W.May (1988) in [23] which asks whether $G_p \cong H_p$ (see [4], too)?*

Corollary. *Let G be a Warfield group and the F-isomorphism $FH \cong FG$ be valid for some group H. Then H_p and G_p have isomorphic high subgroups.*

Proof. Since G_p is an S-group (cf. [14]) whence a Σ-group, we observe that the central theorem is applicable, completing the proof. □

Remark. *When G is a p-local Warfield group, W.May showed in [23] that $G \cong H$.*

We continue the study with more serious results for group rings of mixed Σ-groups. What must be proved is the following.

Theorem (ISOMORPHISM). *Suppose G is a p-mixed reduced Σ-group and G/G_p is reduced. If A is such a group that $FA \cong FG$ as F-algebras, then A and G have isomorphic high subgroups and A is a p-mixed reduced Σ-group, too.*

Proof. It is evident that A must be p-mixed and besides reduced [7]. On the other hand [22] guarantees that $G/G_p \cong A/A_p$. From the hypothesis follows that H_G is a direct sum of cyclics, whence the same holds for $(H_G)_p$ [10] which, however, is a high subgroup of G_p, i.e. $(H_G)_p = H_{G_p}$ (see [15,16]) and therefore G_p is a Σ-group. Applying [17, Theorem 2], G splits and $H_{G_p} \times G/G_p \cong H_G$. Certainly, A/A_p is free and thus A splits [10]. Employing the technique of [17, Theorem 2], $H_A \cong H_{A_p} \times A/A_p$. By virtue of the previous theorem we have $H_{G_p} \cong H_{A_p}$, hence in other words A_p is a Σ-group. Finally this leads us to the fact that $H_A \cong H_G$, and the last means that A is itself a Σ-group. This finishes the proof. □

Remark. *By what we have used above, $G \cong G_p \times G/G_p$ and $A \cong A_p \times A/A_p$, where $G/G_p \cong A/A_p$. Consequently the isomorphism $G \cong A$ depends on the fact whether or not $G_p \cong A_p$.*

The next shows that the restriction "reduced" can be dropped.

Corollary (ISOMORPHISM). *Let G be a p-mixed Σ-group whose G/G_p is reduced. Then if for an arbitrary group A the group algebras FA and FG are F-isomorphic, A and G have isomorphic high subgroups and A is a p-mixed Σ-group.*

Proof. According to [7], $FA_r \cong FG_r$, where A_r and G_r are the corresponding reduced parts of A and G. Moreover G_r is a p-mixed Σ-group and $G_r/(G_r)_p = G_r/(G_r \cap G_p) \cong G_rG_p/G_p \subseteq G/G_p$ is reduced. The observation that the last theorem is applicable ensure $H_{A_r} \cong H_{G_r}$, i.e. $H_A \cong H_G$, as claimed. The proof is completed. □

Recall that as usual G_t will designate the torsion part, i.e. the maximal torsion subgroup of G. We generalize the above idea to the next

Theorem (ISOMORPHISM). *Suppose G is a reduced Σ-group for which G/G_t is reduced. If A is arbitrary so that $KA \cong KG$ as K-algebras over all fields K, then A and G have isomorphic high subgroups and A is also a reduced Σ-group.*

Proof. According to [22] we have, $G/G_t \cong A/A_t$. By application of [17, Theorem 2], $H_G \cong (H_G)_t \times G/G_t \cong H_{G_t} \times G/G_t$ is a direct sum of cyclics. Furthermore A/A_t is free and so A is splitting [10]. Let us now K_p be a field of char$K_p = p > 0$. On an other hand $H_{G_t} = \coprod_p H_{G_p}$ is a direct sum of cyclics. From the first major theorem we obtain $H_{G_p} \cong H_{A_p}$ and thus $H_{G_t} \cong \coprod_p H_{A_p} \cong H_{A_t}$. Further it is easily seen that A_t is reduced since G_t is, and hence so is A. Consequently we can apply the method in [17, Theorem 2] to verify that $H_A \cong H_{A_t} \times A/A_t$. Finally we derive $H_G \cong H_A$, finishing the proof after all. □

Remark. *From the listed above procedure,* $G \cong G_t \times G/G_t$ *and* $A \cong A_t \times A/A_t$. *But* $G/G_t \cong A/A_t$. *Therefore the eventual existence of the isomorphism* $G \cong A$ *is equivalent to that between* G_t *and* A_t, *i.e. between* G_p *and* A_p *for all primes* p.

The other hot section is, however, the following

A STRUCTURE QUESTION

The next technical matter is one of the keys for the further investigations.

Proposition. *Suppose* $A \leq G$ *and* R *is with no nilpotents. Then*

$$(S(RA)G_p)[p] = S(RA)[p]G[p].$$

Proof. Taken x in the left-hand side. Hence we can write $x = g\sum_i r_i a_i$, where $g \in G_p$, $r_i \in R$, $a_i \in A$ plus the condition $g^p \sum_i r_i^p a_i^p = 1$ where $\sum_i r_i = 1$; $1 \leq i \leq n$. Without loss of generality we may presume that the following relations hold: $a_1^p = \cdots = a_s^p \neq a_{s+1}^p = \cdots = a_l^p \neq a_{l+1}^p \neq \cdots \neq a_n^p \neq a_1^p$. So, we deduce $g_1^p a_1^p = 1$, $r_1 + \cdots + r_s = 1$; $r_{s+1} + \cdots + r_l = 0$; $r_{l+1} = \cdots = r_n = 0$. Further we automatically obtain $ga_1 \in G[p]$ and thus we have $x = ga_1(r_1 + r_2a_2a_1^{-1} + \cdots + r_sa_sa_1^{-1} + r_{s+1}a_{s+1}a_1^{-1} + \cdots + r_la_la_1^{-1})$ where the last element obviously lies in $G[p]S(RA)[p]$ because $r_1 + r_2a_2a_1^{-1} + \cdots + r_la_la_1^{-1} \in S(RA)[p]$. This verifies the equality. □

A direct consequence is the following

Corollary. *For any ordinal number* α *and* R *without nilpotents is fulfilled*

$$(S(R^{p^\alpha}G^{p^\alpha})G_p)[p] = S(R^{p^\alpha}G^{p^\alpha})[p]G[p].$$

Further, the following assertion is crucial (cf. [5]).

Lemma. *The abelian group* G_p *is balanced in* $S(RG)$, *i.e. in other words* $G_p \cap S^{p^\alpha}(RG) = G_p^{p^\alpha}$ *and* $(S(RG)/G_p)^{p^\alpha} = S^{p^\alpha}(RG)G_p/G_p = S(R^{p^\alpha}G^{p^\alpha})G_p/G_p$ *for each ordinal* α.

We proceed by proving now our important goal partially mentioned in [8], namely:

Theorem (STRUCTURE). *Presume that* G *is an abelian group and* R *is a perfect ring with trivial zero divisors. Then* $S(RG)$ *is a* Σ*-group if and only if* G_p *is a* Σ*-group. Moreover,* G_p *a* Σ*-group implies that* $S(RG)/G_p$ *is a* Σ*-group.*

Proof. The necessity holds by the above discussion in paragraph I, since the last lemma gives that G_p is pure in $S(RG)$. For the sufficiency, we shall use the necessary and sufficient condition for a Σ-group, given by us in the first paragraph. In fact, write $G[p] = \bigcup_{n=1}^{\infty} G_n$, where $G_n \subseteq G_{n+1}$ and $G_n \cap G^{p^n} = G^1[p]$ for all positive integers n. Hence owing to [4], $S(RG)[p] = 1 + I(RG; G[p]) = \bigcup_{n=1}^{\infty}(1 + I(RG; G_n))$. Next, we will construct subgroups M_n of $S(RG)[p]$ so that the choosen groups satisfies the conditions: $S(RG)[p] = \bigcup_{n=1}^{\infty} M_n$, $M_n \subseteq M_{n+1}$ and $M_n \cap S(RG^{p^n}) = S(RG^1)[p]$. Really, these groups can be constructed thus: $M_n = \langle \sum_i r_i^{(n)} g_i^{(n)},\ 1 + \sum_{j,k} \alpha_{jk}^{(n)} a_{jk}^{(n)}(1 - b_j^{(n)}) \mid r_i^{(n)}, \alpha_{jk}^{(n)} \in R,\ \sum_i r_i^{(n)} = 1;\ g_i^{(n)}, b_j^{(n)} \in G_n,\ a_{jk}^{(n)} \in G$ with the property that $a_{jk}^{(n)} = 1$ or $\text{height}_p(a_{jk}^{(n)^{\varepsilon}} b_l^{(n)^{\delta\varepsilon'}})$ and $\text{height}_p(a_{jk}^{(n)^{\varepsilon}} a_{lk}^{(n)^{\delta\varepsilon'}})$ are $\leq n-1$ or $\geq \omega$, where $0 \leq \varepsilon \leq 2p$, $0 \leq \varepsilon' \leq 2p$ and $\delta = 1$ or $\delta = -1$ when $j \neq l$, for each fixed index $j, k, l, n \in N\rangle$. Foremost, it is our opinion to be proved that the choosing groups M_n are correct. In fact, the restrictions on the heights of the elements and their products are possible, because the sums, in which these elements are members, are finite, and the degrees ε and ε' are also a finite number. And so, every element in M_n is of the form $x_{1n}^{\varepsilon_1} \ldots x_{tn}^{\varepsilon_t}$, where x_{in} are of the above kind and $0 \leq \varepsilon_i \leq p$, $1 \leq i \leq t \in N$. The group M_{n+1} can be constructed by the same token, too. That is why, clearly $M_n \subseteq M_{n+1}$ and $M_n \subseteq 1 + I(RG; G_n)$. Next, we will show that

(a) $S(RG)[p] = \bigcup_{n=1}^{\infty} M_n$

(b) $M_n \cap S(RG^{p^n}) = S(RG^1)[p]$ for every $n \in N$.

Indeed, if x is arbitrary from $S(RG)[p]$, then x lies in $1 + I(RG; G_s)$ for some $s \in N$. Therefore x can be written as a finite sum $1 + \sum_m \sum_s f_{ms} g_{ms}(1 - g_s)$, where $f_{ms} \in R$, $g_{ms} \in G$ and $g_s \in G_s$. Since the conditions on p-heights can be satisfied evidently, then certainly we deduce $x \in M_n$ for some $n \in N$, whence $x \in \bigcup_{n=1}^{\infty} M_n$ and thus the relation "$\subseteq$" holds. The converse is clear, which proves (a).

To prove (b), it is a routine matter to establish that the left hand-side contains the right hand-side, because by [4] we have $S(RG^1)[p] = S(RG^1; G^1[p]) \subseteq M_n$ for all $n \in N$ owing to the construction of M_n along with the fact that $G^1[p] \subseteq G_n$ for each $n \in N$.

Conversely, for the other relation we take an arbitrary intersection $M_n \cap S(RG^{p^n})$. Hence it is enough to show that its elements have infinite heights (as computed in $S(RG)$), which is equivalent to prove that each element of M_n has height $\leq n-1$ or $\geq \omega$ calculated in $S(RG)$. Well, select an arbitrary element $y_n \in M_n$. Hence $y_n = x_{1n}^{\varepsilon_1} \ldots x_{tn}^{\varepsilon_n}$, where $x_{1n}, \ldots, x_{tn}$ possesse the present form (i.e. are generating elements) and $t \in N$. Write

$x_{1n} = r_1^{(n)} c_1^{(n)} + \cdots + r_k^{(n)} c_k^{(n)}$ in a canonical form, $k \in N$. Without harm, we may presume that the following dependences are fulfilled: $c_1^{(n)} \in G_n$, $r_1^{(n)} = 1$; $c_3^{(n)} \in c_2^{(n)} G_n$, $r_2^{(n)} + r_3^{(n)} = 0$, $c_2^{(n)} \in G|G_n$; $c_4^{(n)} \in c_s^{(n)} G_n, \ldots, c_{s-1}^{(n)} \in c_s^{(n)} G_n$, $s \geq 5$, $r_4^{(n)} + \cdots + r_s^{(n)} = 0$, $c_s^{(n)} \in G|G_n$ and etc to the final. Thus it is not difficult to be seen that x_{1n} together with its nontrivial degrees have heights $\leq n-1$ or $\geq \omega$, according to the fact that $x_{1n} = 1 - (1 - c_1^{(n)}) + r_2^{(n)} c_2^{(n)} (1 - b_2^{(n)}) + r_4^{(n)} c_4^{(n)} (1 - c_s^{(n)} c_4^{(n)^{-1}}) + \cdots + r_{s-1}^{(n)} c_{s-1}^{(n)} (1 - c_s^{(n)} c_{s-1}^{(n)^{-1}}) + \cdots$, where $b_2^{(n)} \in G_n$ and to the construction of these elements in M_n, namely more specially that $\text{height}_p(c_j^{(n)^{\eta}} c_l^{(n)^{\delta\eta'}}) \leq n-1$ or $\geq \omega$ for every $1 \leq j$, $l \leq k$ and $0 \leq \eta$, $\eta' \leq 2\varepsilon_1$ where $\delta = 1$ or $\delta = -1$ whenever $j \neq l$. The real inconvenience, however, is when we have a multiplication of two or more basic elements. In this case, we restrict our consideration on two elements since the general situation follows by means of a standard induction on the number of the elements from the basis or by similar arguments as below. And so, write $x_{2n} = \alpha_1^{(n)} d_1^{(n)} + \cdots + \alpha_k^{(n)} d_k^{(n)}$, where $\alpha_1^{(n)} = 1$, $d_1^{(n)} \in G_n$ and analogous relations as to the above are valid. Although that we have not zero divisors in the coefficient ring it is not easy, however, to prove that in the canonical form of $x_{1n}^{\varepsilon_1} x_{2n}^{\varepsilon_2}$ there exist elements of G with nonzero coefficients which have heights $\geq \omega$ and $\leq n-1$ when they are finite eventually; thus $\text{height}_p(x_{1n}^{\varepsilon_1} x_{2n}^{\varepsilon_2})$ is $\leq n-1$ or $\geq \omega$ and we are done. Indeed, in this canonical form we will examine the p-heights of the products $c_1^{(n)^{\varepsilon_1}} d_1^{(n)^{\varepsilon_2}}$, $c_2^{(n)^{\varepsilon_1}} d_1^{(n)^{\varepsilon_2}}$ and $c_2^{(n)^{\varepsilon_1}} d_2^{(n)^{\varepsilon_2}}$. So, the first element $c_1^{(n)^{\varepsilon_1}} d_1^{(n)^{\varepsilon_2}} \in G_n$ and then it has height $\leq n-1$ or $\geq \omega$. For the second we have, $c_2^{(n)^{\varepsilon_1}} d_1^{(n)^{\varepsilon_2}} = c_2^{(n)^{\varepsilon_1}} c_1^{(n)^{-\varepsilon_1}} c_1^{(n)^{\varepsilon_1}} d_1^{(n)^{\varepsilon_2}}$. If $\text{height}_p(c_1^{(n)^{\varepsilon_1}} d_1^{(n)^{\varepsilon_2}}) \leq n-1$ there is nothing to prove since in the canonical form of $x_{1n}^{\varepsilon_1} x_{2n}^{\varepsilon_2}$ must to exist an element of G_n with nonzero coefficient, say for example $c_1^{(n)^{\varepsilon_1}} d_1^{(n)^{\varepsilon_2}}$. Suppose now it is $\geq \omega$. By hypothesis $c_2^{(n)^{\varepsilon_1}} c_1^{(n)^{-\varepsilon_1}}$ has height $\geq \omega$ or $\leq n-1$. If the first claim holds, $c_2^{(n)^{\varepsilon_1}} d_1^{(n)^{\varepsilon_2}}$ possesses height $\geq \omega$, and in the remaining case height $\leq n-1$. Further, we consider $c_2^{(n)^{\varepsilon_1}} d_2^{(n)^{\varepsilon_2}} = c_2^{(n)^{\varepsilon_1}} d_1^{(n)^{\varepsilon_2}} d_2^{(n)^{\varepsilon_2}} d_1^{(n)^{-\varepsilon_2}}$. If now $c_2^{(n)^{\varepsilon_1}} d_1^{(n)^{\varepsilon_2}}$ has height $\leq n-1$ and nonzero coefficient, the situation is very good and so we are done. But now, if it has height $\geq \omega$, the desired element has height $\geq \omega$ or $\leq n-1$ because by hypothesis such height has eventually $d_2^{(n)^{\varepsilon_2}} d_1^{(n)^{-\varepsilon_2}}$. And so, if $c_2^{(n)^{\varepsilon_1}} d_2^{(n)^{\varepsilon_2}}$ has height $\leq n-1$ or $\geq \omega$ and nonzero coefficient we are done as usually. All other cases from this type are similar.

But, let us now $c_2^{(n)^{\varepsilon_1}} d_1^{(n)^{\varepsilon_2}}$ and $c_1^{(n)^{\varepsilon_1}} d_2^{(n)^{\varepsilon_2}}$ have height $\leq n-1$ and zero coefficients, and $c_2^{(n)^{\varepsilon_1}} d_2^{(n)^{\varepsilon_2}}$ has nonzero coefficient. Because not there is zero divisors, we routine observe that relations between the group members with zero coefficients must to exist; without loss of generality presume that $c_1^{(n)^{\varepsilon_1}} d_2^{(n)^{\varepsilon_2}} = c_2^{(n)^{\varepsilon_1}} d_1^{(n)^{\varepsilon_2}}$ plus this that $\alpha_2^{(n)^{\varepsilon_2}} + r_2^{(n)^{\varepsilon_1}} = 0$. Consequently $c_2^{(n)^{\varepsilon_1}} d_2^{(n)^{\varepsilon_2}} = c_1^{(n)^{\varepsilon_1}} d_1^{(n)^{-\varepsilon_2}} d_2^{(n)^{2\varepsilon_2}} = c_1^{(n)^{\varepsilon_1}} d_1^{(n)^{\varepsilon_2}} (d_1^{(n)^{-1}} d_2^{(n)})^{2\varepsilon_2}$ has height $\leq n-1$ or $\geq \omega$ since the height of $c_1^{(n)^{\varepsilon_1}} d_1^{(n)^{\varepsilon_2}}$ is $\geq \omega$ (otherwise every is proved). Further, we consider $c_3^{(n)^{\varepsilon_1}} d_3^{(n)^{\varepsilon_2}} \in c_2^{(n)^{\varepsilon_1}} d_2^{(n)^{\varepsilon_2}} G_n$ by construction, and hence the arguments about the p-heights and ring coefficients are analogous as the above preceding. All other possible cases may be reduced or are identical to the present two main steps. By what we have just shown above we deduce that (b) is true, concluding the proof of the first half.

Next, the present equality (a) leads us to $(S(RG)/G_p)[p] = S(RG)[p]G_p/G_p = \bigcup_{n=1}^{\infty} (M_n G_p/G_p)$. Further we will compute that

(c) $(M_n G_p) \cap S(RG^{p^n}) \subseteq G_p S(RG^1)[p]$ for all $n \in N$.

And so, given x in the left hand-side. Therefore we may write $x = gy_n$, where $g \in G_p$ and $y_n \in M_n$. But we write $y_n = \sum_t r_t^{(n)} c_t^{(n)}$, where $r_t^{(n)} \in R$, $c_t^{(n)} \in G$, $t \in N$ and in this canonical form we will assume that $c_1^{(n)} \in G_n \subseteq M_n$. Further we derive $gc_t^{(n)} \in G^{p^n}$. Thus $x = gc_1^{(n)} \sum_t r_t^{(n)} c_t^{(n)} c_1^{(n)^{-1}} \in G_p(M_n \cap S(RG^{p^n}))$ and invoking step (b) we conclude $x \in G_p S(RG^1)[p]$, as claimed.

As a final, according to (b), to the modular law and to the Proposition plus the Lemma, we calculate $(M_n G_p/G_p) \cap (S(RG)/G_p)^{p^n} = (M_n G_p/G_p) \cap (S(RG^{p^n})G_p/G_p) = [(M_n G_p) \cap (S(RG^{p^n})G_p)]/G_p = G_p[(M_n G_p) \cap S(RG^{p^n})]/G_p = G_p S(RG^1)[p]/G_p = (S(RG^1)G_p)[p]G_p/G_p = (S(RG^1)G_p/G_p)[p] = (S(RG)/G_p)^1[p]$ because G_p is pure in $S(RG)$, whence in $S(RG^1)G_p \subseteq S(RG)$ as well. That is why our criterion for a Σ-group is applicable to obtain the claim. The theorem is verified. □

Remark. *Probably there are and other sets of subgroups different from $\{M_n\}_{n=1}^{\infty}$ with the properties (a), (b) and (c). For example, we can take all possible products of $c_j^{(n)}$'s for $1 \leq j \leq k$ to have* heights $\leq n-1$ *or* $\geq \omega$ *and etc the same procedure for every other generating element.*

The following are also well to be documented.

Corollary. *Assume that G is a p-group and R is perfect with no zero divisors. Then $V(RG)$ is a Σ-group if and only if G is a Σ-group. Moreover, the Σ-group G implies that the same is $V(RG)/G$.*

Remark. *In the proof of this consequence we may take $0 \leq \varepsilon \leq$* order$(a_{jk}^{(n)})$ *by making use of the same method.*

Corollary. *Let G be a Σ-group and R be perfect without zero divisors. Then $S(RG)$ and $S(RG)/G_p$ are both Σ-groups.*

Proof. As we have seen, G_p is a Σ-group immediately, finishing the proof. □

Corollary. *Suppose G is a group whose G_p is simply presented and R is perfect with no zero divisors. Then $S(RG)$ and $S(RG)/G_p$ are both Σ-groups.*

Corollary. *Presume that G is a Warfield group and R is perfect with elementary zero divisors. Then $S(RG)$ and $S(RG)/G_p$ are both Σ-groups.*

Proof. As we have mentioned above, G_p is an S-group [14] and thus it is a Σ-group. The proof is over. □

Remark. *The last two claims give a partial positive (negative) light of a question raised in the sense of May, which asks what is the structure of $V(FG)$ and whether the direct factor conjecture holds, provided F is perfect and G is a p-local Warfield group. We note that in this case $V(FG) = GS(FG)$ (cf. [23,4]) and so $V(FG)/G \cong S(FG)/G_p$.*

Remark. *An other valuable observation is that the first main isomorphism theorem in this article can be retrieved automatically from the main structural result.*

We conclude the investigation on Σ-groups and their group rings with important paragraph entitled

2. **Semisimple group algebras of abelian Σ-groups.** Let G be an abelian p-group and K be a field of the first kind with characteristic $\neq p$. The letters $U_p(KG)$ and $S(KG)$ designate the unit p-group and the normalized unit p-group in the group ring KG, respectively. Besides, $U_p(K)$ is the group of all invertible p-elements in K.

The author feels that of some interest and importance is the following

Theorem (STRUCTURE).

(i) *$S(KG)$ is a Σ-group if and only if G/G^1 is a direct sum of cyclics.*

(ii) *$U_p(KG)$ is a Σ-group if and only if G/G^1 is a direct sum of cyclics.*

Proof. (i) Using a result of T.Mollov (cf. [3]), we may write the formula

$$S(KG) \cong S^1(KG) \times S(K(G/G^1)),$$

where $S^1(KG)$ is the maximal divisible subgroup of $S(KG)$.

Now, presume that $S(KG)$ is a Σ-group. Hence so is $S(K(G/G^1))$ which is, however, separable (see [3]). Consequently $S(K(G/G^1))$ is a direct sum of cyclics, whence G/G^1 is a direct sum of cyclics, too.

Conversely, G/G^1 a direct sum of cyclics quarantees that the same is valid for $S(K(G/G^1))$ (cf. [3]). Therefore the present above formula leads us to this that $S(KG)$ must be a direct sum of cocyclic groups, hence it is a Σ-group. This proves the first half.

(ii) The next dependence is well-known

$$U_p(KG) = S(KG) \times U_p(K).$$

Moreover $U_p(K)$ is known to be cyclic. Furthermore $U_p(KG)$ is a Σ-group if and only if $S(KG)$ is. Thus we can apply the step (i) to finish the proof in general after all. □

We begin with

III. Concluding discussion

We close the paper with some several remarks, suggestions plus left-open questions and problems. And so, first and foremost we recall that if F_pG and F_pH are F_p-isomorphic over the prime field with characteristic p, then $G[p]$ and $H[p]$ are isometric [1]. So, by making use of our group criterion, if G_p is a Σ-group, then the same holds for H_p. But our theorem in this way, however, is more general since the coefficient field is arbitrary, while the above is finite. The conjecture of whether G_p and H_p are isomorphic is interesting to be known. In this direction the following is of some interest and importance: It follows from [5] that two primary abelian Σ-groups are not isomorphic only when there exists an isometry between their socles. Probably some other explicit numerical invariants that characterize the high subgroups are needed. If the last is so, then probably we will be done. After this, it is a major fact to know whether the direct factor problem is fulfilled, i.e. does it follow that the Σ-group G_p is a direct factor of $S(FG)$ or more generally, is then $S(FG)/G_p$ a simply presented group provided F is perfect? In this way, $S(FG)/G_p$ is a σ-summable Σ-group provided G_p is simply presented of length cofinal with ω, owing to the results in [4] and this article. Moreover, the direct factor problem probably will be positive answered if the isomorphism conjecture stated above when two Σ-groups are isomorphic, is true.

And as a final, we ask the following: What are the structures of $S(KG)$ and $U_p(KG)$ presuming that G is a Σ-group with p-elements only and K is a field of the first kind? In this aspect we observe that the high subgroup of $S(KG)$ is well-known and it is isomorphic to $S(K(G/G^1))$. Next, we also ask: Suppose G is a p-torsion Σ-group and $KH \cong KG$ as K-algebras over the first kind field K with $\mathrm{char}K \neq p$. Then whether H is also a Σ-group and even more, what is the complete system of invariants for KG in this direction?

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