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STRUCTURE OF MINIMAL NON-COMMUTATIVE ZERO-INSERTIVE RINGS

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INTRODUCTION

Throughout this paper, rings are associative rings but need not have an identity. The field with p^n elements is denoted by $GF(p^n)$ for a prime p and a natural number n.

It is known that: (1) a minimal non-commutative ring contains 4 elements and up to isomorphism there are two such rings, $\begin{bmatrix} GF(2) & GF(2) \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} GF(2) & 0 \\ GF(2) & 0 \end{bmatrix}$, and (2) a minimal non-commutative ring with identity contains 8 elements and up to isomorphism there is only one such ring, $\begin{bmatrix} GF(2) & GF(2) \\ 0 & GF(2) \end{bmatrix}$.

A ring R is called *zero-insertive* [7] if for all $a, b, r \in R$, $ab = 0$ implies $arb = 0$, and R is called *zero-commutative* [2, 7] if for all $a, b \in R, ab = 0$ implies $ba = 0$. One notes that every zero-commutative ring is zero-insertive. However, the rings $\begin{bmatrix} GF(2) & GF(2) \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} GF(2) & 0 \\ GF(2) & 0 \end{bmatrix}$ are zero-insertive but neither of them is zero-commutative. (Zero-insertive rings with identity which are not zero-commutative are given in Example 7.) We conclude that $\begin{bmatrix} GF(2) & GF(2) \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} GF(2) & 0 \\ GF(2) & 0 \end{bmatrix}$ are the two minimal non-commutative zero-insertive rings. These two rings do not have an identity.

In this paper, the structure of minimal non-commutative zero-insertive rings with identity is obtained. The main Theorem 8 states that a minimal non-commutative zero-insertive ring with identity has 16 elements and up to isomorphism there are five such rings. Since only one of these five rings is zero-commutative, it follows that a minimal non-commutative zero-commutative ring with identity contains 16 elements and up to isomorphism there is only one such ring. Although we are unable to have a complete list of minimal non-commutative zero-commutative rings without identity, we

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show that such rings have 16 elements and they are nilpotent with characteristic 2 or 4.

1. Preliminaries

A ring R is *normal* if each idempotent of R lies in the center.

Lemma 1. *Every zero-commutative ring R is normal.*

Proof. Let $e = e^2 \in R$ and $r \in R$. Since $e(r-er) = 0$, we have $(r-er)e = 0$, so $re = ere$. Similarly, $er = ere$. Hence $re = er$. □

A ring R is called *right (left) duo* if every right (left) ideal of R is an ideal, and it is called *duo* if it is both right and left duo.

Lemma 2. *Every duo ring R is normal.*

Proof. Let $e = e^2 \in R$ and $r \in R$. Since eR is an ideal, we have $re \in eR$, so $re = ea$ for some $a \in R$. Hence $ere = e^2a = ea = re$. Similarly, $ere = er$, hence $re = er$. □

Lemma 3. *Every zero-insertive ring R with identity is normal.*

Proof. Let $e = e^2 \in R$. Since $e(1-e) = 0$, we have $er(1-e) = 0$ for each $r \in R$, so $er = ere$. Similarly, $re = ere$. Hence $re = er$. □

If R is a ring and $a \in R$, we let $\mathrm{r}(a) = \{r \in R | ar = 0\}$ and $\mathrm{l}(a) = \{r \in R | ra = 0\}$ be the *right annihilator* and the *left annihilator* of a in R, respectively. Applying [7, Lemma 1] we see that a ring R is zero-insertive if and only if $\mathrm{r}(a)$ (equivalently $\mathrm{l}(a)$) is an ideal for each $a \in R$. Hence we have the following two corollaries.

Corollary 4. *Every right (left) duo ring is zero-insertive.*

Corollary 5 ([3]). *Every right (left) duo ring with identity is normal.*

In view of Lemma 2 and Corollary 5, we give the following example.

Example 6. *The ring* $\begin{bmatrix} GF(p) & GF(p) \\ 0 & 0 \end{bmatrix}$ *is right duo, hence zero-insertive by Corollary 4, but it is not normal, so it is neither zero-commutative nor left duo by Lemmas 1 and 2. This shows that the condition "with identity" in both Lemma 3 and Corollary 5 is essential. Similarly,* $\begin{bmatrix} GF(p) & 0 \\ GF(p) & 0 \end{bmatrix}$ *is left duo* *and zero-insertive, but it is not normal and hence neither zero-commutative nor right duo.*

It was proved in [11, Proposition 3] that a finite right duo ring with identity is left duo. The condition "with identity" is necessary by the above example.

A ring R is called a *minimal non-commutative zero-insertive (zero-commutative) ring* if R has the smallest order $|R|$ among the non-commutative zero-insertive (zero-commutative) rings.

Recall that any finite ring, with or without identity, is a direct sum of rings of prime power order. Consequently, a finite indecomposable ring has prime power order. For a finite ring R, we let $|R|$ denote the order of R, and let $X(R)$ denote the characteristic of R. If $R = \prod_{i \in I} R_i$ is a product of rings then R is zero-insertive (zero-commutative) if and only if each R_i is zero-insertive (zero-commutative). It follows that mimimal non-commutative zero-insertive (zero-commutative) rings are indecomposable, so they have prime power orders. It was proved in [5] that a finite ring R is commutative if $|R|$ has square free factorization. It was proved in [10] that if R is a non-commutative ring with $|R| = p^2$ then $R \cong \begin{bmatrix} GF(p) & GF(p) \\ 0 & 0 \end{bmatrix}$ *or* $\begin{bmatrix} GF(p) & 0 \\ GF(p) & 0 \end{bmatrix}$. From Example 6 we can conclude that a minimal non-commutative zero-insertive ring has 4 elements and up to isomorphism $\begin{bmatrix} GF(2) & GF(2) \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} GF(2) & 0 \\ GF(2) & 0 \end{bmatrix}$ are the two such rings.

2. Main Result

In this section we consider minimal non-commutative zero-insertive (zero-commutative) rings with identity. In [4] Eldridge proved that: (1) a finite ring R with identity is commutative if $|R|$ has cube free factorization, and (2) if R is a non-commutative ring with identity and $|R| = p^3$ then $R \cong \begin{bmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{bmatrix}$, which is not normal hence not zero-insertive by Lemma 3. Consequently, a minimal non-commutative zero-insertive (zero-commutative) ring with identity has at least 16 elements.

Next we construct five distinct non-commutative zero-insertive rings with identity, each of which has 16 elements, so they are minimal non-commutative zero-insertive rings with identity.

Example 7. *We let $R[x,y]$ denote the polynomial ring over a ring R with non-commutative indeterminates x and y, and the ideal of $R[x,y]$ generated by $S \subseteq R[x,y]$ is denoted by $< S >$. Let $R_1 = GF(2)[x,y]/ < x^3, y^3, yx, x^2 - xy, y^2 - xy >$. Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$, and $R_2 = \mathbb{Z}_4[x,y]/ < x^3, y^3, yx, x^2 - xy, x^2 - \bar{2}, y^2 - \bar{2}, \bar{2}x, \bar{2}y >$. Let R_3 be the ring of all matrices of the form $\begin{bmatrix} a & b \\ 0 & a^2 \end{bmatrix}$ over the field $GF(4)$.Let $R_4 = GF(2)[x,y]/ < x^3, y^2, yx, x^2 - xy >$ and $R_5 = \mathbb{Z}_4[x,y]/ < x^3, y^2, yx, x^2 - xy, x^2 - \bar{2}, \bar{2}x, \bar{2}y >$. It is easy to see that each R_i is a non-commutative ring with identity and $|R_i| = 16$. In [12] we proved that R_1, R_2 and R_3 are the 3 non-isomorphic minimal non-commutative duo rings with identity. By Corollary 4 and the above discussion, R_1, R_2 and R_3 are minimal non-commutative zero-insertive rings with identity. We note that both R_4 and R_5 are also zero-insertive, so they are also minimal non-commutative zero-insertive rings with identity. Since $X(R_4) = 2$ and $X(R_5) = 4$, we have $R_4 \ncong R_5$.Now neither R_4 nor R_5 is duo, so we conclude that $R_i \ncong R_j$ for all $i \neq j$ in $\{1, 2, 3, 4, 5\}$.*

In the above example, we gave five non-isomorphic minimal non-commutative zero-insertive rings with identity. The next theorem asserts that they are all the such rings, up to isomorphism. We let $J(R)$ denote the Jacobson radical of a ring R.

Theorem 8. *A minimal non-commutative zero-insertive ring with identity is a local ring with 16 elements, and if R is such a ring then $R \cong R_i$ for some $i = 1, 2, 3, 4, 5$, where the R_i's are the rings in Example 7.*

Proof. Since zero-insertive rings with identity are normal rings by Lemma 3 and finite normal rings with identity are direct sums of local rings, we see that minimal non-commutative zero-insertive rings with identity, being indecomposable, must be local rings. Hence, we have the first assertion by Example 7.

To prove the second one, we assume that R is a local non-commutative zero-insertive ring with identity and $|R| = 16$.

If $R/J(R) \cong GF(8)$, $|J(R)| = 2$ and $J(R)$ is a vector space over the field $R/J(R)$ with 8 elements, which is impossible. So we have that either $R/J(R) \cong GF(2)$ or $R/J(R) \cong GF(4)$.

If $R/J(R) \cong GF(2)$ then $|J(R)| = 8$. We claim that $2j = 0$ for all $j \in J(R)$. To prove this, we may assume that $X(R) = 4$, then $J(R)$ contains a non-zero central element 2 so our claim follows from the non-commutativity of R. Now $J(R)$ is a non-commutative nilpotent algebra of dimension 3 over $GF(2)$, so it follows from [8, Theorem 2.3.6] that $J(R)$ has a basis $\{a, b, c\}$ such that $cJ(R) = J(R)c = 0$,$a^2 = ab = c$,$ba = 0$, and $b^2 = 0$ or c. Assume $b^2 = c$. Then $R \cong R_1$ if $X(R) = 2$. If $X(R) = 4$, we view $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ as a subring of R, then since $\bar{2} \in J(R)$ we have $c = \bar{2}$, and thus $R \cong R_2$. Similarly, if $b^2 = 0$ then $R \cong R_4$ or $R \cong R_5$.
Now let $R/J(R) \cong GF(4)$ then $|J(R)| = 4$. We have that $J(R)^2 = 0$, for if $J(R)^n = 0$ and $J(R)^{n-1} \neq 0$, then $J(R)^{n-1}$ is a vector space over $R/J(R) \cong GF(4)$ and so $|J(R)^{n-1}| \geq 4$. Since R is non-commutative, it follows from [10, Theorem2] that $X(R) = 2$. Hence $R \cong R_3$ by [10, Theorem 3]. □

We note that the ring R_3 in Example 7 is zero-commutative, but none of R_1, R_2, R_4 and R_5 in Example 7 is zero-commutative.

Corollary 9. *A minimal non-commutative zero-commutative ring with identity has 16 elements and up to isomorphism there is only one such ring, the ring R_3 in Example 7.*

3. Zero-Commutative Rings without Identity

The structure of minimal non-commutative zero-insertive rings is given at the end of Section 1, and the structure of minimal non-commutative zero-insertive (zero-commutative) rings with identity is given in Theorem 8 (Corollary 9). In this final section we consider minimal non-commutative zero-commutative rings without identity.

It is known that there are 52 non-isomorphic rings of order 8 (see [1], [9]), and we note that none of them is non-commutative zero-commutative. Next we give three nilpotent non-commtative zero-commutative rings with 16 elements, so they are minimal non-commtative zero-commutative rings without identity.

An additive group $(S, +)$ is of $(n_1, ..., n_t)$*-type* if $(S, +) = S_1 \oplus \ldots \oplus S_t$ where S_i is a cyclic group of order n_i .

Example 10. *Let $\sigma : GF(4) \longmapsto GF(4)$ be the automorphism via $a \longmapsto a^2$. Let $GF(4)[x,\sigma]$ be the skew polynomial ring, i.e., $GF(4)[x,\sigma] = GF(4)[x]$ as abelian groups and the multiplication is given as an extension of: $(ax^i)(bx^j) = a\sigma^i(b)x^{i+j}$ where $ax^i, bx^j \in GF(4)[x,\sigma]$. Let $R =< \bar{x} >= \{a\bar{x} + b\bar{x}^2 | a, b \in GF(4)\}$ be the ideal of $GF(4)[x,\sigma]/ < x^3 >$ generated by $\bar{x}$. Then R is a non-commutative zero-commutative ring with $|R| = 16$, and $R^3 = 0$.*

Let $S =< \bar{x}, \bar{y} >$ be the ideal of $\mathbb{Z}_4[x,y]/ < x^2, y^2, xy - \bar{2}x, yx - \bar{2}y >$ generated by $\{\bar{x}, \bar{y}\}$. Then $(S,+) = (\bar{x}) \oplus (\bar{y})$ is of $(4,4)$-type, so $|S| = 16$. We note that S is a non-commutative zero-commutative ring with $S^3 = 0$.

Let $T =< \bar{x}, \bar{y} >$ be the ideal of $\mathbb{Z}_4[x,y]/ < x^2, y^2, xy - \bar{2}x, \bar{2}y >$ generated by $\{\bar{x}, \bar{y}\}$. Then $(T,+) = (\bar{x}) \oplus (\bar{y}) \oplus (\bar{y}\bar{x})$ is of $(4,2,2)$-type, so $|T| = 16$. We note that T is a non-commutative zero-commutative ring with $T^3 = 0$.

Since $(R,+)$ is of $(2,2,2,2)$-type, $(S,+)$ is of $(4,4)$-type,and $(T,+)$ is of $(4,2,2)$-type, we see that $R \not\cong S \not\cong T \not\cong R$.

The rings R, S, and T in the above example are minimal non-commtative zero-commutative rings without identity, moreover they are nilpotent. More general results are given as follows.

Proposition 11. *An indecomposable finite normal ring R without identity is nilpotent.*

Proof. (1) Let $e = e^2 \in R$. Since e lies in the center, both eR and $\mathrm{r}(e)$ are ideals of R and $R = eR \oplus \mathrm{r}(e)$. Since R is an indecomposable ring without identity, we have $e = 0$.

(2) Let $0 \neq a \in R$. Since R is a finite ring, we have $a^i = a^j$ for some $j > i$. Then $a^i = a^j = a^i a^{j-i} = a^i(a^{j-i})^2 = \ldots = a^i a^i a^k$ for some $k \geq 1$. It follows that a^{i+k} is an idempotent, and so $a^{i+k} = 0$ by (1).

We have proved that R is a nil ring. Now it is well-known that a finite nil ring is nilpotent. □

Corollary 12. *Let R be an indecomposable finite ring without identity. If R is zero-commutative, then R is nilpotent.*

Proof. By Lemma 1 and Proposition 11. □

We can not give the structure of minimal non-commutative zero-commutative rings without identity, but our concluding theorem gives some properties of such rings.

Theorem 13. *If R is a minimal non-commutative zero-commutative ring without identity, then $|R| = 16$ and R is nilpotent with $X(R) = 2$ or 4.*

Proof. We have $|R| = 16$ by Example 10 and the discussion preceding it. Using Corollary 9 we see that R must be indecomposable, so R is nilpotent by Corollary 12.

Since R is not commutative, $X(R) \neq 16$.

Suppose $X(R) = 8$. Then $(R, +)$ must be of $(8, 2)$-type. Let $(R, +) = (a) \oplus (b)$, where $|a| = 8$ and $|b| = 2$. ($|\cdot|$ denotes the order of an element in a group.) Since R is non-commutative and zero-commutative, we have $0 \neq ab \neq ba \neq 0$. Hence $|ab| = |ba| = 2$. Since a is nilpotent, we have $ab \neq b \neq ba$. So we only have $ab = 4a$ and $ba = 4a + b$, or $ab = 4a + b$ and $ba = 4a$. In the first case we obtain: $aba = 4a^2$ and $aba = 4a^2 + ab$, so $ab = 0$, which is a contradiction. The second case also derives a contradiction by similarity. Therefore $X(R) \neq 8$. □

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