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ON PRIMITIVE REPRESENTATIONS BY UNIMODULAR QUADRATIC $\mathbf{Z}_2$-LATTICES

HIDEYO SASAKI

ABSTRACT. Necessary and sufficient conditions are given for the primitive representations of an odd quadratic $\mathbf{Z}_2$-lattice M with rank m by an odd unimodular quadratic $\mathbf{Z}_2$-lattice L with rank n for $m + m(1) = n$, where $m(1)$ is the rank of a sublattice $M(1)$ of M for a Jordan decomposition $M = M(0) \perp M(1)$ with $M(0)$ unimodular and $B(M(1), M(1)) \subseteq 2\mathbf{Z}_2$ for the associated bilinear form B.

1. INTRODUCTION

D.G.James gives necessary and sufficient conditions for the primitive representations of quadratic lattices by unimodular lattices in his paper [J1][J2]. In this paper we give a similar result to James' on primitive representations over $\mathbf{Z}_2$. Terminology and notations are followed in [K]. We write $\mathbf{Z}_2$ and $\mathbf{Q}_2$ for the ring of 2-adic integers and the field of 2-adic numbers, respectively. Let V be a regular quadratic space over $\mathbf{Q}_2$, with finite dimension $n \geq 3$, with quadratic form $Q : V \longrightarrow \mathbf{Q}_2$ and associated bilinear form B with $B(x,x) = Q(x)$, $x \in V$. Assume V supports a unimodular $\mathbf{Z}_2$-lattice L; thus $L = \{x \in V | B(x, L) \subseteq \mathbf{Z}_2\}$ is self dual. The lattice L is called *even* when $Q(x) \in 2\mathbf{Z}_2$ for all $x \in L$; otherwise L is *odd.*

Let M be an $\mathbf{Z}_2$-lattice on a second quadratic space W over $\mathbf{Q}_2$ of dimension m. Assume that M is integral, i.e. $B_W(M, M) \subseteq \mathbf{Z}_2$. A quadratic space V *represents* a quadratic space W over $\mathbf{Q}_2$ is if there is an injective isometry $\phi : W \to V$ over $\mathbf{Q}_2$. In the same way a *representation* of M by L over $\mathbf{Z}_2$ is an injective isometry $\psi : M \to L$ over $\mathbf{Z}_2$. The representation (over $\mathbf{Z}_2$) is *primitive* if $\psi(M)$ is a direct summand of L.

Let $M = M(0) \perp M(1)$ be a Jordan splitting, where $M(0)$ is unimodular and
$B(M(1), M(1)) \subseteq 2\mathbf{Z}_2$, the sublattices $M(0)$ and $M(1)$ are not uniquely determined even up to isometry, although the ranks are invariants of M. Let $m(i)$ =rank $M(i)$, $i = 0, 1$, so that $m = m(0) + m(1)$. The $\mathbf{Z}_2$-lattice M is *even* if and only if $M(0)$ is even.

$\mathbf{Z}_2^*$ denotes the group of units in $\mathbf{Z}_2$. dM and dL denote the discriminants of M and L, respectively. $S(V)$ denotes the Hasse invariant of a quadratic space V over $\mathbf{Q}_2$.

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For a regular quadratic lattice L over $\mathbf{Z}_2$, $n(L)$ denotes the *norm ideal* of L. A lattice (with rank$L > 1$) is *almost unimodular* if it has a Jordan splitting $K \perp \mathbf{Z}_2 x$ where K is unimodular and $Q(x) \in 2\mathbf{Z}_2^*$.

H denotes a hyperbolic even unimodular plane over $\mathbf{Z}_2$ and A an anisotropic binary even unimodular lattice over $\mathbf{Z}_2$.

James gives necessary and sufficient conditions for primitive representations of a $\mathbf{Z}_2$-lattice M by a unimodular $\mathbf{Z}_2$-lattice L for M even, and both M and L odd with $m + m(1) < n$. In this paper we give necessary and sufficient conditions for primitive representations of an odd $\mathbf{Z}_2$-lattice M by an odd unimodular $\mathbf{Z}_2$-lattice L for $m + m(1) = n$. We show Main Theorem (Theorem 2.1) in the next section.

2. Main Theorem

Theorem 2.1. *Let L be an odd unimodular $\mathbf{Z}_2$-lattice with rank n and M an odd $\mathbf{Z}_2$-lattice with rank m. Assume that $m + m(1) = n$. Then L primitively represents M if and only if:*

There exist splittings $M = \widehat{M} \perp \widehat{M}^{\perp}$, with $\widehat{M}$ be either unimodular or almost uni-modular and $n(\widehat{M}^{\perp}) \subseteq 2\mathbf{Z}_2$, and $L = \widehat{L} \perp (\perp_{rank \widehat{M}^{\perp}} H)$ such that $\mathbf{Q}_2\widehat{L}$ represents $\mathbf{Q}_2\widehat{M}$ over $\mathbf{Q}_2$.

Before proving the theorem we need several lemmas. These lemmas are easily deducible from [K].

Lemma 2.2. *Let $\alpha \in \mathbf{Z}_2^*$ with $\alpha \equiv -1 \bmod 4$, and let $\mathbf{Z}_2$-lattice M be isometric to $\langle\epsilon\rangle \perp \langle 2\eta\rangle \perp \langle 2\epsilon\eta\alpha\rangle (\epsilon, \eta \in \mathbf{Z}_2^*)$. Then*

$$\begin{aligned} S(\mathbf{Q}_2 M) &= -(2,\alpha)(2,\epsilon)(\epsilon,\eta) \\ &= \begin{cases} (2,\epsilon)(\epsilon,\eta) & (\alpha \equiv 3 \bmod 8) \\ -(2,\epsilon)(\epsilon,\eta) & (\alpha \equiv -1 \bmod 8), \end{cases} \end{aligned}$$

where $(*,*)$ denotes the Hilbert symbol over $\mathbf{Q}_2$.

Lemma 2.3. *Let $M \cong \langle\epsilon\rangle \perp \langle 2\eta\rangle, \epsilon, \eta \in \mathbf{Z}_2^*$. If $(\epsilon, 2)(\epsilon, \eta) = 1$ then M represents 1 over $\mathbf{Z}_2$.*

Lemma 2.4. *$\langle\epsilon_1\rangle \perp \langle\epsilon_2\rangle \perp \langle 2\eta\rangle$ represents 1 over $\mathbf{Z}_2$, where $\epsilon_1, \epsilon_2, \eta \in \mathbf{Z}_2^*$.*

Proof of Theorem 2.1. Note that $\mathbf{Z}_2$-lattice $\widehat{M}$ in the Theorem is isometric to $M(0)$ or $M(0) \perp \mathbf{Z}_2 x$ with $x \in M(1)$ and $Q(x) \in 2\mathbf{Z}_2^*$.

First we prove the "if" part. By the assumption there exist sublattices $\widehat{M}$, $\widehat{L}$ of M, L, respectively such that the condition in Theorem is hold. If we can take $\widehat{M} = M(0)$ then $\mathbf{Q}_2\widehat{M} \cong \mathbf{Q}_2\widehat{L}$ since rank $\widehat{M}$ =rank $\widehat{L}$. Then $\widehat{M} \cong \widehat{L}$ since both $\widehat{M}$ and $\widehat{L}$ are odd unimodular. Since $\perp_{rank \widehat{M}^{\perp}} H$ primitively represents $\widehat{M}^{\perp}$, L primitively represents M. So we assume that $\widehat{M} = M(0) \perp \mathbf{Z}_2 x$ with $x \in M(1)$ and $Q(x) \in 2\mathbf{Z}_2^*$. Note that rank$\widehat{L}$ =rank$\widehat{M} + 1$.

We use an induction on $m(0)$. First assume $m(0) = 1$. Then we have rank$\widehat{L} = 3$, rank$\widehat{M} = 2$, and $\widehat{M}$ is isometric to $\langle \epsilon \rangle \perp \langle 2\eta \rangle, \epsilon, \eta \in \mathbf{Z}_2$.

By scaling we may assume $\widehat{L} \cong \langle 1 \rangle \perp E$, where E is a binary even unimodular $\mathbf{Z}_2$-lattice. Then we have $d\widehat{L} = -1$ and $S(\mathbf{Q}_2\widehat{L}) = -1$ (if $E = H$), or $d\widehat{L} = 3$ and $S(\mathbf{Q}_2\widehat{L}) = 1$ (if $E = A$).

Note that $\mathbf{Q}_2\widehat{L}$ represents $\mathbf{Q}_2\widehat{M}$ if and only if $\mathbf{Q}_2\widehat{L} \cong \mathbf{Q}_2\widehat{M} \perp \langle d\widehat{L}d\widehat{M} \rangle$. Then we have $S(\mathbf{Q}_2(\widehat{M} \perp \langle d\widehat{L}d\widehat{M} \rangle)) = -(2, d\widehat{L})(2, \epsilon)(\epsilon, \eta) = S(\mathbf{Q}_2\widehat{L})$ by Lemma 2.2 since $d\widehat{L} \equiv -1 \bmod 4$, and we obtain $(2, \epsilon)(\epsilon, \eta) = 1$. Then $\widehat{M} \cong \langle \epsilon \rangle \perp \langle 2\eta \rangle$ represents 1 by Lemma 2.3 and we have $\widehat{M} \cong \langle 1 \rangle \perp \langle 2\eta' \rangle, \eta' \in \mathbf{Z}_2^*$. Then $\langle 1 \rangle$ in $\widehat{M}$ is primitively represented by $\langle 1 \rangle$ in $\widehat{L}$ and so is $\langle 2\eta' \rangle$ by E by Theorem 1 in [J1]. Then $\widehat{L}$ primitively represents $\widehat{M}$ and so is for L and M, respectively. Now we proved for $m(0) = 1$.

Suppose $m(0) > 1$. Since an odd unimodular $\mathbf{Z}_2$-lattice has an orthogonal $\mathbf{Z}_2$-basis, we have $\widehat{L} = \perp_i \langle \epsilon_i \rangle, \epsilon_i \in \mathbf{Z}_2^*$. Then by scaling, we may assume

$$\widehat{L} \cong \langle 1 \rangle \perp \widehat{L}',$$

where $\widehat{L}'$ is an odd unimodular $\mathbf{Z}_2$-lattice with rank$\widehat{L}' =$ rank$\widehat{L} - 1$.

By the assumption, $\widehat{M}$ has an orthogonal component $\langle \epsilon_1 \rangle \perp \langle \epsilon_2 \rangle \perp \langle 2\eta \rangle (\epsilon_1, \epsilon_2, \eta \in \mathbf{Z}_2^*)$. By Lemma 2.4 this orthogonal component represents 1 and we can write $\widehat{M}$ as

$$\widehat{M} \cong \langle 1 \rangle \perp \widehat{M}',$$

where $\widehat{M}'$ is an odd $\mathbf{Z}_2$-lattice with rank$\widehat{M}'$ =rank$\widehat{M} - 1$ and $\widehat{M}'(1) \cong \langle 2\delta \rangle, \delta \in \mathbf{Z}_2^*$.

If $\mathbf{Q}_2\widehat{L}$ represents $\mathbf{Q}_2\widehat{M}$ then $\mathbf{Q}_2\widehat{L}'$ represents $\mathbf{Q}_2\widehat{M}'$, however, by the assumption of induction, $\widehat{L}'$ primitively represents $\widehat{M}'$. Thus $\widehat{L}$ primitively represents $\widehat{M}$ so is for L and M. Thus we have proved for $\widehat{M} \cong M(0) \perp \langle 2\eta \rangle$ and proved the "if" part.

Next we prove the "only if" part. We assume that L primitively represents M in the following. We use an induction on $m(0)$.

Suppose $m(0) = 1$. By scaling, we may assume

$$L \cong \langle 1 \rangle \perp E',$$

where E' is an even unimodular $\mathbf{Z}_2$-lattice. M is isometric to $\langle \epsilon \rangle \perp M(1)$ $(\epsilon \in \mathbf{Z}_2^*)$.

It is obtained that

$$\begin{aligned} (1) \qquad L &\cong \langle 1 \rangle \perp (\perp_{m(1)-1} H) \perp E \\ (2) \qquad &\cong \langle \epsilon \rangle \perp \langle \begin{pmatrix} 1-\epsilon & 1 \\ 1 & \gamma \end{pmatrix} \rangle \perp (\perp H) \perp E \end{aligned}$$

for some $\gamma \in \mathbf{Z}_2^*$, and $E \cong H$ or A.

If $\epsilon \equiv 1 \bmod 8$ then splitting off $\langle 1 \rangle$ from M and L in (1)(2), we may assume that $E = H$ or there exists $x \in M(1)$ such that $Q(x) \in 2\mathbf{Z}_2^*$ by applying Theorem 1 and Proposition 4 in [J1]. Put $\widehat{M} = \widehat{L} = \langle 1 \rangle$ if $E = H$, or $\widehat{M} = \langle 1 \rangle \perp \langle Q(x) \rangle$ and $L = \langle 1 \rangle \perp E$ if there exists $x \in M(1)$ such that $Q(x) \in 2\mathbf{Z}_2^*$. Then it is easily shown that these $\widehat{M}$ and $\widehat{L}$ satisfy the condition in Theorem.

Suppose that $\epsilon \not\equiv 1 \bmod 8$. Splitting off $\langle \epsilon \rangle$ from M and L in (2), $M(1)$ is primitively represented by $\langle \epsilon \rangle^\perp$ in $L \cong \langle \epsilon \rangle \perp \langle \begin{pmatrix} 1-\epsilon & 1 \\ 1 & \gamma \end{pmatrix} \rangle \perp (\perp H) \perp E$. Noting that $1-\epsilon \notin 8\mathbf{Z}_2$, there exists $x_1 \in M(1)$ with $Q(x_1) \equiv 1-\epsilon \bmod 8$ by applying Theorem 8 in [J1]. Then $\langle \epsilon \rangle \perp \langle Q(x_1) \rangle$ is a submodule of M and since $\epsilon + Q(x_1) \equiv 1 \bmod 8$, M is isometric to $\langle 1 \rangle \perp M(1)$. So we can apply the case of $\epsilon \equiv 1 \bmod 8$. Thus we have proved "only if" part for $m(0) = 1$.

Suppose $m(0) > 1$. Then we may assume that there exists a decomposition $M = \langle \epsilon \rangle \perp M'$, where M' is an odd $\mathbf{Z}_2$-lattice with rank $m-1$. Then splitting $\langle \epsilon \rangle$ off from M and L, since $L' = (\langle \epsilon \rangle^\perp$ in $L)$ is an odd unimodular with rank $n-1$ by the assumption, there exist submodules $\widehat{M'}$ and $\widehat{L}'$ of M' and L', respectively such that the conditions in Theorem are hold by applying the assumption of induction on $m(0)$.

Put $\widehat{L} = \langle \epsilon \rangle \perp \widehat{L}'$ and $\widehat{M} = \langle \epsilon \rangle \perp \widehat{M'}$. Then these lattices satisfy the conditions in Theorem for L and M.

Thus we have proved for $m(0) > 1$ and complete the proof of the "only if" part. □

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Hideyo Sasaki
The Graduate School of Science and Technology
Kobe University
1-1 Rokkodai-cho Nada-ku
Kobe 657-8501, Japan
e-mail address: hsasaki@math.kobe-u.ac.jp

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