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A REMARK ON LOCALIZATION OF INJECTIVE MODULES

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1. Introduction

Let A be a commutative ring and S a multiplicatively closed subset of A. In [D], Dade studies the conditions under which the localization $M[S^{-1}]$ of any injective A-module M is an injective $A[S^{-1}]$-module. In this paper we shall investigate the Dade's result [D, Theorem 13] in case that A is not necessarily commutative and a Gabriel topology $\mathcal{F}$ instead of S respectively. That is, we shall study the conditions under which the localization $M_{\mathcal{F}}$ of any injective A-module M is an injective $A_{\mathcal{F}}$-module. See [S, VI §5] for Gabriel topologies and [S, VI §6 p.151] for relations with commutative rings case.

Let A be a ring can be non-commutative, M a right A-module and $\mathcal{F}$ a Gabriel topology of right ideals on the ring A. Then we have the localization of M at $\mathcal{F}$, that is, the direct limit

$$M_{\mathcal{F}} = \varinjlim \mathrm{Hom}_A\Big(\mathfrak{a}, M/t(M)\Big) \quad \text{for } \mathfrak{a} \in \mathcal{F},$$

where $t(M)$ is the $\mathcal{F}$-torsion submodule of M. See [S, IX §1] for $M_{\mathcal{F}}$.

2. The injectiveness of localized modules

As in the introduction, by $M_{\mathcal{F}}$ we denote $\varinjlim \mathrm{Hom}_A\Big(\mathfrak{a}, M/t(M)\Big)$ for $\mathfrak{a} \in \mathcal{F}$, and in this section we denote

$$\varinjlim \mathrm{Hom}_A(\mathfrak{a}, M) \quad \text{for } \mathfrak{a} \in \mathcal{F}$$

by $M_{(\mathcal{F})}$. Then we have two canonical homomorphisms:

$$\varphi_M : \quad M \cong \mathrm{Hom}_A(A, M) \to M_{(\mathcal{F})}$$

and

$$\psi_M : \quad M \cong \mathrm{Hom}_A(A, M) \to M_{\mathcal{F}}.$$

For these canonical homomorphisms, [S, Lemma 1.2, p.196] tells us that

Lemma 1. $\ker \psi_M = \ker \varphi_M = t(M)$.

By First Isomorphism Theorem and Lemma 1, we have $\psi_M(M) \cong M/\ker \psi_M = M/t(M)$. Therefore $\psi_M(M)$ is an $\mathcal{F}$-torsion-free A-module, and we have

$$M_{\mathcal{F}} = (M/t(M))_{(\mathcal{F})} \cong (\psi_M(M))_{(\mathcal{F})} \cong (\psi_M(M))_{\mathcal{F}}.$$

Hence by applying [S, Proposition 2.7, p.203] to $\psi_M(M)$ and using the isomomorphism $(\psi_M(M))_{\mathcal{F}} \cong M_{\mathcal{F}}$, we have

Proposition 2. *For every A-module M, the following properties are equivalent:*

(a) *$M_{\mathcal{F}}$ is injective over $A_{\mathcal{F}}$.*
(b) *$M_{\mathcal{F}}$ is injective over A.*

Proposition 2 is a generalization of [S, Proposition 2.7, p.203], but Porposition 2 does not need an assumption on M in [S, Proposition 2.7, p.203], that is, M is not needless to be trosion-free as an A-module.

Lemma 3. *Let A be a ring and $\mathcal{F}$ a right Gabriel topology on A. Then for an injective right A-module M, the following are equivalent:*

(a) $\mathrm{Ext}^1_A(I, t(M)) = 0$ *for any right I ideal of A.*
(b) *$M/t(M)$ is an injective A-module.*

When these equivalent conditions hold, $M_{\mathcal{F}}$ is an injective $A_{\mathcal{F}}$-module.

Proof. By the natural exact A-sequence $0 \to I \to A \to A/I \to 0$ we obtain a long exact sequence:

$$\mathrm{Hom}_A(I, t(M)) \to \mathrm{Ext}^1_A(A/I, t(M)) \to \mathrm{Ext}^1_A(A, t(M))$$

$$\to \mathrm{Ext}^1_A(I, t(M)) \to \mathrm{Ext}^2_A(A/I, t(M)) \to \mathrm{Ext}^2_A(A, t(M)).$$

Since $\mathrm{Ext}^1_A(A, t(M)) = \mathrm{Ext}^2_A(A, t(M)) = 0$, we have isomorphism $\mathrm{Ext}^1_A(I, t(M)) \cong \mathrm{Ext}^2_A(A/I, t(M))$. Similarly by natural exact A-sequence $0 \to t(M) \to M \to M/t(M) \to 0$ we obtain exact isomorphism $\mathrm{Ext}^1_A(A/I, M/t(M)) \cong \mathrm{Ext}^2_A(A/I, t(M))$ since M is A-injective. Therefore $\mathrm{Ext}^1_A(I, t(M)) \cong \mathrm{Ext}^1_A(A/I,\ M/t(M))$ holds. Hence the conditioins (a) and (b) are equivalent. Since $\psi_M(M)$ is an $\mathcal{F}$-torsion-free A-module, by [S, Proposition 2.7, p.203] $\psi_M(M)$ is an injective $A_{\mathcal{F}}$-module when $\psi_M(M)$ is an injective A-module. Therefore when the conditions (a) and (b) hold, $M_{\mathcal{F}}$ is an injective $A_{\mathcal{F}}$-module since we have the isomorphism $\psi_M(M) \cong M/t(M)$ □.

As a particular case of Lemma 3, we get [S, Lemma 2.6, p.202], and obviously right hereditary rings satisfy the assumption in Lemma 3.

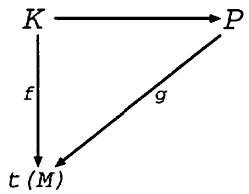
3. On the Dade's result

In this section, using lemmas in section 2 we have a result relating to Dade's result [D, Theorem 13].

Lemma 4. *Let I be a right ideal of A. Then* $\mathrm{Ext}^1_A(I, t(M)) = 0$ *for any injective A-module M if and only if there exists an exact A-sequence $0 \to K \hookrightarrow P \to I \to 0$ (P is projective) which satisfies the following condition:*

(1) *If L is any A-submodule of K such that K/L is an $\mathcal{F}$-torsion module, then there exists an A-submodule N of P such that P/N is an $\mathcal{F}$-torsion module and $N \cap K = L$.*

Proof. Assume that $\mathrm{Ext}^1_A(I, t(M)) = 0$ for any injective A-module M. Let $0 \to K \hookrightarrow P \to I \to 0$ (P is projective) be any exact A-sequence, L be any A-submodule of K such that K/L is an $\mathcal{F}$-torsion module and take any injective A-module M such that $M \supseteq K/L$. Considering natural homomorphism $f\colon K \to K/L$, since K/L is a torsion module and $t(M)$ is a maximal torsion submodule, so $f \in \mathrm{Hom}_A(K, t(M))$. By the exact A-sequence $0 \to K \to P \to I \to 0$ we have an exact sequence $\mathrm{Hom}_A(P, t(M)) \to \mathrm{Hom}_A(K, t(M)) \to \mathrm{Ext}^1_A(I, t(M))$. Since $\mathrm{Ext}^1_A(I, t(M)) = 0$ by the assumption, we have a homomorphism $g \in \mathrm{Hom}_A(P, t(M))$ such that $g|_K = f$. Put $N = \ker g$. Then g induces a monomorphism $P/N \to t(M)$. Since $t(M)$ is a torsion module, so P/N is a torsion module. Since $L = \ker f$, $N = \ker g$ and $g|_K = f$, we have $L = \ker(g|_K) = K \cap N$.

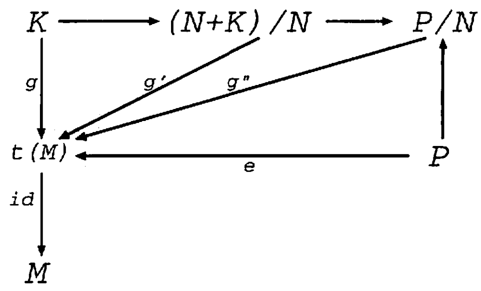


Conversely, assume that there exists an exact A-sequence $0 \to K \to P \to I \to 0$ (P is projective) which satifies the condition (1). Then for any injective A-module M we obtain an exact sequence

$$\mathrm{Hom}_A(P, t(M)) \xrightarrow{h'} \mathrm{Hom}_A(K, t(M)) \xrightarrow{h''} \mathrm{Ext}^1_A(I, t(M)) \to 0.$$

For any $f \in \mathrm{Ext}^1_A(I, t(M))$, there exists a homomorphism $g \in \mathrm{Hom}_A(K, t(M))$ such that $h''(g) = f$. Put $L = \ker g$. Then g induces a monomorphism $K/L \to t(M)$. Since $t(M)$ is a torsion module, so is K/L. Therefore by the condition (1) we have an A-submodule N of P such that P/N is a torsion module and $N \cap K = L$. Thus $g\colon K \to t(M)$ induces a homomorphism: $K/N \cap K = K/L \to t(M)$.

By noting $K/N \cap K \cong (N + K)/N \subseteq P/N$ (see Second Isomorphism Theorem), that is, g induces a homomorphism $g'\colon (N + K)/N \to t(M)$. Since M is an injective A-module, so g' can be extended to $g''\colon P/N \to M$, and since P/N is a torsion module, so $g''(P/N)$ is also a torsion module, that is, $g''(P/N) \subseteq t(M)$. Therefore composing g'' with the natural epimorphism $P \to P/N$, we have a homomorphism $e\colon P \to t(M)$, which satisfies $e|_K = g$, i.e. $h'(e) = g$. By the exactness we can obtain $f = h''(g) = h''h'(e) = 0$. Hence $\mathrm{Ext}^1_A(I, t(M)) = 0$ holds. □



Notes that proof of Lemma 4 implies that one short exact A-sequence

(2) $0 \to K \hookrightarrow P \to I \to 0$ (P is projective)

of I satisfies (1) if and only if all such (2) do, too.

By Lemma 3 and Lemma 4, we have the following main result in this paper.

Theorem 5 (See [D, Theorem 13]). *If a ring A and a Gabriel topology $\mathcal{F}$ of right ideals of A satisfy the following condition* (3), *then the localization $M_{\mathcal{F}}$ of any injective A-module M is injective over $A_{\mathcal{F}}$.*

(3) *For any right ideal I of A, there exists a short exact A-sequence* (2) *such that the condition* (1) *is satisfied.*

Note that there are differences between Theorem 5 and [D, Theorem 13]. For example, the condition (3) of Theorem 5 is stronger than the condition of [D, Theorem 13], and for commutative noetherian rings, every Gabriel topology satisfies the condition (3) by [S, Proposition 4.5, p.170]. We can give an example that the converse of Theorem 5 does not hold: Let A be a quasi-Frobenius serial ring with Kupisch series e_1A, e_2A and admissible sequence 2, 2. (See: [AF, Section 32]) Let $P_i = e_iA$ and $S_i = P_i/radP_i$ for $i = 1, 2$. Then P_i are projective and injective modules of composition length 2. Let $\mathcal{T}$ be the hereditary torsion class *generated* by S_1 and $\mathcal{F}$ the corresponding Gabriel topology on A. Then it is routine to check that the localization of any injective A-module is injective. However $P_2/t(P_2) \cong S_2$ is not injective. Therefore, by Lemma 3, the converse of Theorem 5 does not hold.

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