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SEMIRING-VALUED QUASIMETRICS ON THE SET OF SUBMODULES OF A MODULE

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A nonempty set S on which we have operations of addition and multiplication defined, is a *semiring* if and only if the following conditions are satisfied:

(1) $(S,+)$ is a commutative monoid with identity element 0;

(2) $(S,\cdot)$ is a monoid with identity element 1;

(3) Multiplication distributes over addition from either side;

(4) $0s = 0 = s0$ for all $s \in S$;

(5) $1 \neq 0$.

The following are examples of semirings:

- $(\mathbb{N},+,\cdot)$, where $\mathbb{N}$ is the set of natural numbers;
- $(\mathbb{R}\cup\{\infty\},\oplus,\odot)$, where $\mathbb{R}$ is the set of real numbers and where $a\oplus b = min\{a,b\}$ while $a\odot b = a+b$;
- $(ideal(R),+,\cdot)$, where $ideal(R)$ is the set of all two-sided ideals of a ring R and the operations are the usual addition and multiplication of ideals;
- $(R-fil,\cap,\cdot)$, where $R-fil$ is the set of all topologizing filters of left ideals of a noncommutative ring R and multiplication is given by the Gabriel product. [1]

See the detailed introduction to semiring theory given in [3], for details.

Semirings have proven to be important tools in such diverse areas as graph theory, discrete dynamical systems, formal language theory, automata theory, optimization, and theoretical computer science. Again, see [3] for further details.

A semiring S is *partially ordered* if and only if there exists a partial order relation $\leq$ defined on S satisfying the condition that for all $s,s',s'' \in S$ we have:

(1) If $s \leq s'$ then $s + s'' = s' + s''$;

(2) If $s \leq s'$ and $0 \leq s''$ then $ss'' \leq s's''$ and $s''s \leq s''s'$.

A partially-ordered semiring S is *positive* if and only if $0 \leq s$ for all $s \in S$. A semiring S is [*complete*] *lattice-ordered* if, in addition, it has the structure of a [complete] lattice $(S, \vee, \wedge)$ satisfying $s + s' = s \vee s'$ for all $s, s' \in S$. The last three examples given above are complete lattice-ordered semirings. All of these notions are extensively considered in [3].

Let S be a semiring. A *left S-semimodule* is a commutative monoid $(U, +)$ with additive identity 0_U for which we have a "scalar multiplication" function $S \times U \to U$, denoted by $(s, u) \mapsto su$, which satisfies the following conditions for all elements s and s' of S and all elements u and u' of U:

(1) $(ss')u = s(s'u)$;

(2) $s(u + u') = su + su'$;

(3) $(s + s')u = su + s'u$;

(4) $1u = u$;

(5) $r0_U = 0_U = 0u$.

Right S-semimodules are defined in an analogous manner. In this note, we will apply the theory of semimodules over semirings to study the structure of the set of submodules of a module over a ring.

Now let R be a ring, which is not necessarily commutative. Let M be an R-module, and let $sub(M)$ be the set of all submodules of M. There are several methods of naturally defining topologies on the set $sub(M)$. Thus, for example, if $\mathcal{X}$ is a class of left R-modules containing (0) and closed under taking isomorphisms and extensions then $\mathcal{X}$ defines a topology on $sub(M)$ by saying that a nonempty subset $\mathcal{A}$ of $sub(M)$ is $\mathcal{X}$-closed if and only if $N' \in \mathcal{A}$ whenever there exists a member N of $\mathcal{A}$ containing N' such that $N/N' \in \mathcal{X}$. Note that the topologies thus defined form a *compatible class* in the following sense: if M_1 is a submodule of M_2 then $sub(M_1) \subseteq sub(M_2)$ and the topology defined by $\mathcal{X}$ on $sub(M_1)$ is just the restriction of the topology defined by $\mathcal{X}$ on $sub(M_2)$.

Another approach to defining such topologies is via semiring-valued quasimetrics and pseudometrics. Let S be a partially-ordered semiring and let U be an S-semimodule. A function $\rho\colon U \times U \to S$ is a *quasimetric* on U with values in S if and only if the following conditions are satisfied:

(1) $\rho(u,u) = 0$ for all $u \in U$;

(2) $\rho(u,u'') \leq \rho(u,u') + \rho(u',u'')$ for all $u,u',u'' \in U$.

If the additional condition

(3) $\rho(u,u') = \rho(u',u)$ for all $u,u' \in U$

is satisfied, then ρ is a *pseudometric* on U. All topological spaces can be defined using quasimetrics and pseudometrics with values in suitable semirings.

If the semiring S is a complete lattice-ordered semiring and if $\theta\colon (S,+) \to (S,\cdot)$ is a function satisfying

(i) $\theta(0) = 1$; and

(ii) $\theta(s)\theta(s') \leq \theta(s+s')$ for all $s,s' \in S$

then, for any left S-semimodule U, we can define a quasimetric ρ_θ on U with values in S by setting

$$\rho_\theta(u,u') = \bigwedge\{s \in S \mid u \leq \theta(s)u'\}$$

for all $u,u' \in U$. Indeed, since lattice-ordered semirings are positive [3, Proposition 19.13] we immediately see that $\rho_\theta(u,u) = 0$ for all $u \in U$. Moreover, if $u,u',u'' \in U$ and if $\rho_\theta(u,u') = s_1$ while $\rho_\theta(u',u'') = s_2$ then

$$u \leq \theta(s_1)u' \leq \theta(s_1)\theta(s_2)u'' \leq \theta(s_1+s_2)u''$$

so $\rho_\theta(u,u'') \leq \rho_\theta(u,u') + \rho_\theta(u',u'')$. Moreover, we can also define a pseudometric δ_θ on U with values in S by setting $\delta_\theta(u,u') = \rho_\theta(u,u') + \rho_\theta(u',u)$. Note that if $\alpha\colon U_1 \to U_2$ is a homomorphism of left S-semimodules and if $u,u' \in U_1$ then $\rho_\theta(u\alpha,u'\alpha) \leq \rho_\theta(u,u')$. Moreover, we have equality if α is monic.

Quasimetrics and pseudometrics for right S-semimodules are defined analogously. Let S be a complete lattice-ordered semiring and let U be an S-semimodule. Given a quasimetric $\rho: U \times U \to S$ and a nonempty subset E of S closed under taking meets, we define

$$W_{\rho,e}(u) = \{u' \in U \mid \rho(u, u') \leq e\}$$

for all $e \in E$ and all $u \in U$. The family of all subsets of U of this form is closed under taking finite intersections and so forms a basis for a topology on U. Compare this construction with that given in [2].

EXAMPLE 1. *Quasimetrics of this type on semimodules over the semiring* $\mathbb{R} \cup \{\infty\}$ *defined above, using the function* $\theta: s \mapsto e^s$, *have been studied in* [5].

EXAMPLE 2. *Let* R *be a ring and denote by* $ideal(R)$ *the complete lattice-ordered semiring of all (two-sided) ideals of* R, *as defined above. Let* M *be a left* R*-module and, as above, let* $sub(M)$ *denote the set of all submodules of* M. *Then* $(sub(M), +)$ *is a left* $ideal(R)$*-semimodule, with the product of an ideal of* R *and a submodule of* M *being defined in the standard manner. If* $H, I \in ideal(R)$ *set* $(H : I) = \{r \in R \mid rI \subseteq H\}$. *Then* $(H : (0)) = R$ *for all* $H \in ideal(R)$ *and* $(H : I)(H : I') \subseteq (H : I) \cap (H : I') \subseteq (H : I + I')$ *for all* $H, I, I' \in ideal(R)$. *This means that each ideal* H *of* R *defines a function* $\theta_H: I \mapsto (H : I)$ *from* $ideal(R)$ *to itself which satisfies conditions (i) and (ii) above and so defines a quasimetric* ρ_H *on* $sub(M)$ *given as follows:*

$$\rho_H(N, N') = \bigcap\{I \in ideal(R) \mid N \subseteq (H : I)N'\}.$$

Note that these quasimetrics are compatible in the following sense: if M_1 *is a submodule of* M_2 *then* $sub(M_1) \subseteq sub(M_2)$ *and* ρ_H *on* $sub(M_1)$ *is merely the restriction of* ρ_H *as defined on* $sub(M_2)$.

EXAMPLE 3. *Let R be a ring and denote by $R-fil$ the set of all topologizing filters of left ideals of R. As noted above, $(R-fil, \cap, \cdot)$ is a complete lattice-ordered semiring in which the induced order is reverse inclusion. If $\kappa_1, \kappa_2 \in R-fil$ then the right residual $\kappa_1^{-1}\kappa_2$ is the element of $R-fil$ defined by*

$$\kappa_1^{-1}\kappa_2 = \bigcap\{\kappa \in R-fil \mid \kappa_1\kappa \supseteq \kappa_2\}.$$

By Proposition 3.6 and Proposition 4.14 of [1] *we see that*

$$(\kappa_1 \cap \kappa_2)^{-1}\kappa = \kappa_1^{-1}\kappa \vee \kappa_2^{-1}\kappa \subseteq (\kappa_1^{-1}\kappa)(\kappa_2^{-1}\kappa)$$

for all $\kappa_1, \kappa_2, \kappa \in R-fil$. Moreover, the $\cap$-neutral element of $R-fil$ is the filter $\eta[0]$ of all left ideals of R and for any $\kappa \in R-fil$ we have $\eta[0]^{-1}\kappa = \{R\}$, which is just the neutral element of $R-fil$ with respect to multiplication (i.e. the Gabriel product). Thus we see that each $\kappa \in R-fil$ defines a function $\theta_\kappa\colon R-fil \to R-fil$ satisfying conditions (i) and (ii) above.

Let M be a left R-module and let $sub(M)$ again denote the lattice of all submodules of M. Following Example 13.13 of [3], *we note that $(sub(M), \cap)$ can be considered as a right semimodule over $R-fil$ where, for each $N \in sub(M)$ and each $\kappa \in R-fil$, we let $N\kappa$ be the κ-purification of N in M. That is to say, an element $m \in M$ belongs to $N\kappa$ if and only if there exists a left ideal I of R belonging to κ and satisfying $Im \subseteq N$. As above, we thus have a quasimetric ρ_κ defined as follows: if N and N' are submodules of M then*

$$\rho_\kappa(N, N') = \bigvee\{\kappa_1 \in R-fil \mid N \supseteq N'(\kappa_1^{-1}\kappa)\}$$

Again, these quasimetrics are compatible.

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