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ON TEST MODULES FOR INJECTIVITY

Dedicated to Professor Yukio Tsushima for his sixtieth birthday

SHOJI MORIMOTO

Throughout this note, R means an associative ring with identity and modules mean unitary left R-modules. We fix a module M. As is well-known, R. Baer showed that R is a test module for injectivity [3]. After that, G. Azumaya proved that if R can be embedded in M^n for some n, every M-injective module is injective [2, Theorem 14]. The main purpose of this note is to give equivalent conditions for M to be a test module for injectivity by means of the notion of weakly t_M-injective modules (Theorem 3).

In this paper, the category of all modules is denoted by *R-mod* and the injective hull of a module A by $E(A)$. For the terminologies and basic properties of preradicals and torsion theories, we refer to [7]. For each preradical r, we denote the *r-torsion class* (resp. *r-torsionfree class*) by $T(r)$(resp. $F(r)$). Also the left linear topology corresponding to a left exact preradical r is denoted by $L(r)$. Now, for two preradicals r and s, we shall say that *r is larger than s* if $r(A) \supseteq s(A)$ for all modules A. Also for a preradical r, we put $\tilde{r}(A) = r(E(A)) \cap A$ for all modules A. Then $\tilde{r}$ is the smallest left exact preradical larger than r.

Let r be a preradical for *R-mod*. We call a module Q *weakly r-injective* (resp. *r-injective*) if the functor $Hom_R(-, Q)$ preserves the exactness of all exact sequences of modules $O \to A \to B \to C \to O$ with $B \in T(r)$(resp. $C \in T(r)$).

Lemma 1. *Let r be an idempotent preradical for R-mod. Then a module H is weakly r-injective if and only if it is weakly $\tilde{r}$-injective.*

Proof. Let H be a weakly r-injective module. Since $\tilde{r}(E(H)) = r((E(H)) \cap E(H) = r(E(H))$ and $r(E(H)) = r(H)$, $\tilde{r}(H) = r(E(H)) \cap r(H)$. Hence $\tilde{r}(E(H)) = \tilde{r}(H)$, that is, H is weakly $\tilde{r}$-injective.

A module Q is called *M-injective* if every homomorphism of any submodule of M into Q can be extended to a homomorphism of M into Q. Also t_M denotes the smallest one of those preradicals r such that $r(M) = M$. As is well-known, t_M is idempotent.

Proposition 1. *For a module Q, the following conditions are equivalent :*

(1) *Q is M-injective.*

(2) *Q is weakly t_M-injective.*

(3) *Q is weakly $\tilde{t}_M$-injective.*

Proof. The implication (2) $\Rightarrow$ (3) is clear by Lemma 1 and (3) $\Rightarrow$ (1) is obvious by $M \in T(\tilde{t}_M)$. (1) $\Rightarrow$ (2). Let $O \to A \to B \to C \to O$ be an exact sequence of modules with $B \in T(t_M)$. Then B is a homomorphic images of $M^{(\Lambda)}$ for some index set Λ. Since Q is M-injective, it is $M^{(\Lambda)}$-injective and so it is B-injective by [1, 16.13 Proposition]. Hence Q is weakly t_M-injective.

Combining [6, Theorem 2.1] with Proposition 1, we have a generalization of Azumaya [2, Theorem 15].

Theorem 1. *Let Q be an M-injective module and Q' a submodule of Q. If $\alpha(M) \subseteq Q'$ for all $\alpha \subseteq Hom_R(M, Q)$, then Q' is M-injective. Moreover if Q' is essential in Q, then the converse holds.*

Proof. Since $\alpha(M) \subseteq Q'$ for all $\alpha \in Hom_R(M, Q)$, $t_M(Q) \subseteq t_M(Q')$ and so $t_M(Q) = t_M(Q')$. Thus $t_M(Q') = t_M(Q) = t_M(E(Q)) \supseteq t_M(E(Q')) \supseteq t_M(Q')$, that is Q' is weakly t_M-injective, that is, it is M-injective. Next suppose that Q' is an M-injective module and is essential in Q. Then $t_M(Q') = t_M(E(Q')) = t_M(E(Q)) = t_M(Q)$. Thus $\alpha(M) \subseteq Q'$ for all $\alpha \in Hom_R(M, Q)$.

In [2], an exact sequence $O \longrightarrow A \stackrel{j}{\longrightarrow} B$ called *M-monomorphism* if $B = h(M) + j(A)$ for some $h \in Hom_R(M, B)$. We call an exact sequence $O \longrightarrow A \stackrel{j}{\longrightarrow} B$ of modules *M-coindependent* if $B = j(A) + \alpha(M^{(\Lambda)})$ for some index set Λ and $\alpha \in Hom_R(M^{(\Lambda)}, B)$.

Since every M-monomorphism $O \longrightarrow A \stackrel{j}{\longrightarrow} B$ is M-coindependent, we can obtain a generalization of Azumaya [2, Theorem 11] by combining Proposition 1 with [4, Theorem 2.5].

Theorem 2. *For a module Q, the following conditions are equivalent :*

(1) *Q is M-injective.*

(2) *For every M-coindependent sequence $O \longrightarrow A \stackrel{j}{\longrightarrow} B$ and every homomorphism $f : A \to Q$, there exists a homomorphism $g : B \to Q$ such that $g \circ j = f$.*

(3) *For every M-coindependent sequence $O \longrightarrow A \stackrel{j}{\longrightarrow} B$ with $j(A)$ essential in B and every homomorphism $f : A \to Q$, there exists a homomorphism $g : B \to Q$ such that $g \circ j = f$.*

(4) *Every M-coindependent sequence $O \to Q \stackrel{j}{\longrightarrow} Q'$ splits.*

(5) *Every M-coindependent sequence $O \to Q \stackrel{j}{\longrightarrow} Q'$ with $j(Q)$ essential in Q' splits.*

Now, we shall rewrite Proposition 1 by using a generator Q of $T(\tilde{t}_M)$. Since $t_Q = \tilde{t}_M$, we have

Lemma 2. *Let Q be a generator of $T(\tilde{t}_M)$. Then the following conditions are equivalent :*

(1) *Every M-injective module is injective.*

(2) *Every Q-injective module is injective.*
(3) *Every $Q^{(\Lambda)}$-injective module is injective for all index setsΛ.*

We call a module M a *test module* (for injectivity) if every M-injective module is injective. Following [8, Theorem 2.1], we call a module P *locally projective* if for each element $x \in P$, there exist $x_1, x_2, \cdots, x_n \in P$ and $f_1, f_2, \cdots, f_n \in Hom_R(P, R)$ such that $x = \sum_{i=1}^{n} f_i(x)x_i$, or equivalently, $t_P(A) = t_P(R)A$ for all modules A. Also we call a preradical r *pseudo-cohereditary* if every homomorphic image of $A/(A \cap r(E(A)))$ is in $F(r)$ for all modules A.

Theorem 3. *Let Q be a module with $t_Q = \tilde{t}_M$. Then the following conditions are equivalent :*
(1) *M is a test module.*
(2) *Q is a test module.*
(3) *Q is locally projective and $R = (R) + I$ for all essential left ideals I of R.*
(4) *$t_Q(R) = Re$ for some idempotent element e of R and every module A such that $eA = O$ is injective.*
(5) *$t_Q(R) = Re$ for some idempotent element e of R and R/Re is a semisimple artinian ring.*
(6) *$t_Q(R) = Re$ for some central idempotent element e of R and $R = t_Q(R) + I$ for all essential left ideals I of R.*

Proof. (1) $\Longleftrightarrow$ (2) is clear by Lemma 2. (3) $\Rightarrow$ (2). Let H be a weakly t_Q-injective module. Since Q is locally projective, t_Q is pseudo-cohereditary, that is, H is t_Q-injective from [4, Theorem 3.3]. Let I be an essential left ideal of R. Since $R = t_Q(R) + I$, $I \in L(t_Q)$ and so $Z \leq t_Q$, where Z is the singular torsion functor. Thus $G \leq t_Q$, that is, every t_Q-injective module is injective, where G is the Goldie torsion functor. Hence Q is a test module. Also if we take t_Q as r of [5, Theorem 2.6], then (4), (5) and (6) are equivalent to (2), (3) and (6) of [5, Theorem 2.6] respectively. Moreover (2) is equivalent to (2) of [5, Theorem 2.6] and so (2) is equivalent to (4). (6) $\Rightarrow$ (3) is clear.

We call a module A *cofaithful* if R can be embeded in $A^{(\Lambda)}$ for some index set Λ.

Corollary 1. *Every cofaithful module is a test module.*

Corollary 2. *If R is an artinian ring, then every faithful module is a test module.*

Now, we shall give an example of a test module. Also , in general, a nonzero direct summand of a test module is not necessarily a test module (Example 1 (2)).

Example 1. Let R be the 2×2 upper triangular matrix ring over a field K.

(1) We put $M = \begin{pmatrix} O & K \\ O & K \end{pmatrix}$. Then $t_M(R) = \begin{pmatrix} O & K \\ O & K \end{pmatrix}$ and $t_M(R) = t_M(E(R)) \cap R = \begin{pmatrix} K & K \\ K & K \end{pmatrix} \cap R = R$. Hence $\bar{t}_M = 1$, that is, M is a test module.

(2) We put $S = \begin{pmatrix} K & O \\ O & O \end{pmatrix}$ and $M = \begin{pmatrix} K & K \\ O & O \end{pmatrix}$. Then $t_S(R) = t_M(R) = M$. Since R/M is flat as a right R-module, t_M is left exact. Also M is essential in R. However, $t_M(R) + M = M \neq R$, that is, M is not a test module by (3) of Theorem 3. Hence S is not a test module.

Next we give an example of a test module which is not cofaithful. We call a preradical t *centrally splitting* if there is a central idempotent e of R such that $t(A) = eA$ for all modules A.

Example 2. We put $R = \mathbb{Z}/12\mathbb{Z}$ and $M = 3\mathbb{Z}/12\mathbb{Z}$, where $\mathbb{Z}$ is the ring of rational integers. Since M is not faithful, it is not cofaithful. Also t_M is left exact, $L(t_M) = \{R,\ 2\mathbb{Z}/12\mathbb{Z},\ 4\mathbb{Z}/12\mathbb{Z}\}$ and $L(Z) = \{R,\ 2\mathbb{Z}/12\mathbb{Z}\}$. Thus t_M is larger than Z. Moreover, $R = 3\mathbb{Z}/12\mathbb{Z} \oplus 4\mathbb{Z}/12\mathbb{Z}$, that is, t_M is centrally splitting. Hence M is a test module by (6) of Theorem 3.

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Shoji MORIMOTO
Hagi Koen Gakuin
15,Higashitamachi, Hagi
Yamaguchi, JAPAN 758-0047

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