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ON INJECTIVE MODULES AND LOCALLY NILPOTENT ENDOMORPHISMS OF INJECTIVE MODULES

OKYEON YI

ABSTRACT. First, injective modules are one of the most popular objects in homological algebra. In most cases, base rings are commutative and Noetherian so that the testing the injectivity of a given module is an important topic. Bear's criterion for injective modules over any ring gives a big tool to classify injective modules. Every morphism from an ideal I of R should be extended to the whole ring R to be an injective module R-module. In this paper, we can show that the Baer's test can be reduced from all ideals of R to all prime ideals of R to test the injectivity of a given R-module M if the base ring R is commutative and Noetherian.

Second, the Enochs' Theorem can be extended to an arbitrary sequence $\{f_i\}$ of endomomorphisms of an injective left module over a left Noetherian ring R. Moreover if the ring is left Noetherian and if a diagram of the minimal injective resolution of ${}_RM$ is commutative, then the locally nilpotence of f implies the locally nilpotence of other maps in the diagram.

1. A new criterion to injective modules

Proposition 1. (**Baer Criterion**) *An R-module E is injective if and only if every map $f : I \longrightarrow E$, where I is a left ideal of R can be extended to R.*

Proof. Theorem 3.20 [1]. □

If the ring is commutative and Noetherian, then we can show that the test can be weaken to all prime ideals of the ring R instead of all ideals in the ring R.

Theorem 1. *Let R be a commutative Noetherian ring and M be an R-module with a submodule A. For every $x \in M$ with $x \notin A$, there exists an $r \in R$ such that*

$$P_{rx} = \{r' \in R | r'(rx) \in A\}$$

is a prime ideal.

Proof. Consider the collection C_x of ideals P_{rx} with $rx \notin A$ then $P_{1\cdot x} \in C_x$. Since R is Noetherian there exists a maximal element P_{rx} in C_x for some $r \in R$. Since $rx \notin A$, $1 \notin P_{rx}$. Suppose $st \in P_{rx}$ with $s \notin P_{rx}$. Then $srx \notin A$. So P_{srx} is in C_x. If $y \in P_{rx}$ then $yrx \in A$ so $y(srx) = s(yrx) \in A$

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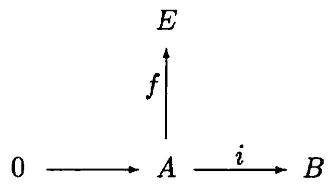
and so $P_{rx} \subset P_{srx}$. Hence by the maximality of P_{rx}, $P_{srx} = P_{rx}$. But $t(srx) = strx \in A$. So $t \in P_{srx} = P_{rx}$. Therefore P_{rx} is a prime ideal. $\square$

Theorem 2. *Let R be a commutative Noetherian ring. An R-module E is injective if and only if every map $f : I \longrightarrow E$, where I is a prime ideal in R, can be extended to R.*

Proof. Suppose E is injective, then the hypothesis is just a special case of the definition of injective modules. Suppose we have the diagram



Let $\mathcal{G}$ consist of all pairs (A', g'), where $A \subset A' \subset B$ and $g' : A' \to E$ extends f. Note that $\mathcal{G} \neq \phi$, for $(A, f) \in \mathcal{G}$. Partially order $\mathcal{G}$ by saying $(A_1, g_1) \leq (A_2, g_2)$ if $A_1 \subset A_2$ and g_2 extends g_1. By Zorn's lemma, there is a maximal pair (A_0, g_0) in $\mathcal{G}$. Assume that $A_0 \neq B$. Let $x \in B - A_0$. Then there exists an $r \in R$ such that $P_{rx} = \{r' \in R | r'(rx) \in A_0\}$ is a prime ideal. Define

$$\begin{array}{rcl} h : P_{rx} & \longrightarrow & E \\ r' & \longmapsto & g_0(r'rx) \end{array}$$

By hypothesis there exists $h' : R \to E$ extending h. Define $A_1 = A_0 + R(rx)$ and define

$$\begin{array}{rcl} g_1 : A_1 & \to & E \\ a_0 + s(rx) & \mapsto & g_0 a_0 + sh'(1) \end{array}$$

where $s \in R$.
First, g_1 is well-defined: if $a_0 + s(rx) = a_0' + s'(rx)$ then $(s - s')(rx) = a_0' - a_0 \in A_0$ so $s - s' \in P_{rx}$. Therefore $g_0((s - s')(rx))$ and $h((s - s')(rx))$ are defined and we have

$$\begin{array}{rcl} g_0(a_0' - a_0) & = & g_0((s - s')(rx)) = h((s - s')) \\ & = & h'(s - s') = (s - s')h'(1) \end{array}$$

Thus $g_0(a_0') - g_0(a0) = sh'(1) - s'h'(1)$ and $g_0(a_0') + s'h'(1) = g_0(a_0) + sh'(1)$. Second, g_1 extends g_0, for $g_1(a_0) = g_0(a_0)$ for all $a_0 \in A_0$. The pair (A_1, g_1) lies in $\mathcal{G}$ and is larger than the maximal pair (A_0, g_0), a contradiction. Therefore $A_0 = B$ and E is injective. This completes the proof of the theorem. $\square$

2. Locally Nilpotent Endomorphisms over Injective Modules

Definition 1. *Given any module M and $f \in End(M)$ we say f is locally nilpotent on M if for every $x \in M$, there exists $n \geq 1$ such that $f^n(x) = 0$.*

Definition 2. *An essestial extension of a module M is a module E containing M such that every nonzero submodule of E meets M (i.e., if $S \subset E$ and $S \neq 0$, then $S \bigcap M \neq 0$). We denote by $M \subset' E$.*

Lemma 1. *If E_1 and E_2 are injective envelopes of M, then any linear map $g : E_1 \longrightarrow E_2$ such that $g|_M$ is an automorphism of M is an isomorphism from E_1 to E_2.*

Proof. Since $M \subset' E_1$ is essential and $g|_M$ is injective the fact $0 = \ker(g|_M) = \ker g \bigcap M$ implies $\ker g = 0$ i.e., $g(E_1) \cong E_1$. Hence $g(E_1)$ is an injective submodule of E_2. So $E_2 = g(E_1) \oplus S$ for some S in E_2 with $g(E_1) \bigcap S = 0$. Since $M \subset g(E_1)$, we get $M \bigcap S = 0$. So the fact $M \subset' E_2$ implies $S = 0$. So $E_2 = g(E_1)$. Hence g is an isomorphism of E_1 to E_2. □

The following proposition first occured in Mathis' Thesis at University of Chicago in 1958 but only a special case. It was his Theorem 3.4 on page 520 of his article. But his result only applied if the ring R is commutative and if the map $f : E \to E$ is a multiplication by an element r of R. The more general result follows from Proposition 4.2 [4]. Remark 2 of page 200 of that paper shows how to get Mathis' result from that theorem. But the result Proposition 4.2 [4] also implies in a similar manner that if M has E as an injectice envelope and if R is left Noetherian then $f : E \to E$ such that $f(M) = 0$ is locally nilpotent on E.

Proposition 2. *If R is left Noetherian, E is an injective left R-module, and $f \in End(E)$ is such that $ker(f) \subset' E$, then f is locally nilpotent on E.*

Proof. Proposition 4.2 [4]. □

Example) If R is commutative, $r \in R$ and E is an R-module, then $E \xrightarrow{r} E (x \mapsto rx)$ is linear. Hence if E is injective and R is commutative Noetherian and if $ker(r) = K \subset' E$, then we can apply the Proposition 2 and get for any $x \in E$, $r^n x = 0$ for some $n \geq 1$.

Example) $Z(p^1) \subset' Z(p^\infty)$. But $p(Z(p^1)) = 0$. So every element in $Z(p^\infty)$ has order a power of p.

If we only assume $ker(f^2) \subset' E$ in the above, then f^2 is locally nilpotent in E and hence f itself is locally nilpotent on E. Similarly if $ker(f^n) \subset' E$ then f is locally nilpotent on E. Also note that $ker(f) \subseteq ker(f^2) \subseteq ker(f^3) \subseteq \cdots$ for any endomorphism $f : E \longrightarrow E$.

Theorem 3. *If R is left Noetherian, E is an injective left R-module, and $f \in End(E)$ is such that $\bigcup_{n=1}^{\infty} ker(f^n) \subset' E$, then f is locally nilpotent on E .*

Proof. Let $K = \cup_{n=1}^{\infty} ker(f^n) \subset' E$. Define

$$\begin{array}{rcl}\psi : E \oplus E \oplus \cdots & \longrightarrow & E \oplus E \oplus \cdots \\ (x_1, x_2, \cdots) & \longmapsto & (x_1, x_2 - f(x_1), x_3 - f(x_2), \cdots)\end{array}$$

Then easily ψ is a homomorphism. If $(x_1, x_2, \cdots)$ in $K \oplus K \oplus \cdots$, then there exist $a_i, b_i \geq 1$ such that $f^{a_i}(x_i) = 0$ and $f^{b_i}(x_{i-1}) = 0$ So $f^{m_i}(x_i - f(x_{i-1})) = 0$ for some $m_i \geq max\{a_i, b_i\}$ for each i. So ψ maps $K \oplus K \oplus \cdots$ into $K \oplus K \oplus \cdots$. And $\psi|_{K \oplus K \oplus \cdots}$ is injective since its kernel is 0. Now let $(y_1, y_2, \cdots) \in K \oplus K \oplus \cdots$. Since each $y_i \in K = \bigcup_{n=1}^{\infty} Ker(f^n)$ there exists $t_i \in \{1, 2, \cdots\}$ such that $f^{t_i}(y_i) = 0$. Given $n \in \{1, 2, \cdots\}$ choose m such that

$$m \geq max\{t_n, t_{n-1} - 1, t_{n-2} - 2, \cdots, t_1 - (n-1)\}.$$

Then

$$\begin{array}{rl} & f^m(y_n + f(y_{n-1}) + f^2(y_{n-2}) + \cdots + f^{n-1}(y_1)) \\ = & f^m(y_n) + f^{m+1}(y_{n-1}) + f^{m+2}(y_{n-2}) + \cdots + f^{m+n-1}(y_1) \\ = & 0 \end{array}$$

so for every $n \in \{1, 2, \cdots\}$

$$y_n + f(y_{n-1}) + f^2(y_{n-2}) + \cdots + f^{n-1}(y_1) \in K$$

Then

$$\begin{array}{rl} & \psi(y_1, y_2 + f(y_1), \cdots, y_n + f(y_{n-1}) + f^2(y_{n-2}) + \cdots + f^{n-1}(y_1), \cdots) \\ = & (y_1, y_2 + f(y_1) - f(y_1), y_3 + f(y_2) + f^2(y_1) - f(y_2) - f^2(y_1), \cdots) \\ = & (y_1, y_2, \cdots, y_n, \cdots). \end{array}$$

So $\psi|_{K \oplus K \oplus \cdots}$ is onto. Hence $\psi|_{K \oplus K \oplus \cdots}$ is an isomorphism on $K \oplus K \oplus \cdots$ and by the Lemma 1 ψ is an automorphism of $E \oplus E \oplus \cdots$. Let $x \in E$ and consider $(x, 0, 0, \cdots)$. Then $\psi(x_1, x_2, \cdots) = (x, 0, 0, \cdots)$ for some $(x_1, x_2, \cdots) \in E \oplus E \oplus \cdots$. Then $x_1 = x, x_2 - f(x_1) = 0, x_3 - f(x_2) = 0, \cdots, x_n - f(x_{n-1}) = 0, \cdots$. So $x_n = f(x_{n-1}) = f^2(x_{n-2}) = \cdots = f^{n-1}(x_1) = f^{n-1}(x)$. But for some n, $x_{n+1} = 0$ i.e., $f^n(x) = 0$. Therefore f is a locally nilpotent on E. □

Corollary 1. *If R is a left Noetherian ring and $f : E \longrightarrow E$ is an endomorphism of an injective left R-module such that $\cup_{n=1}^{\infty} ker(f^n) \subset' E$ then $\cup_{n=1}^{\infty} ker(f^n) = E$.*

Proof. By Theorem 3, $\cup_{n=1}^{\infty} ker(f^n) \subset' E$ implies f is locally nilpotent on E, so we easily obtain $\cup_{n=1}^{\infty} ker(f^n) = E$. □

Now we consider an arbitrary sequence $f_0, f_1, f_2, \cdots$ in $End(E)$.

Lemma 2. *Let R be a left Noetherian ring. Let E be an injective left R-module and let $f_0, f_1, f_2, \cdots$ be a sequence of elements in $End(E)$. Consider those $x \in E$ such that $f_n \circ f_{n-1} \circ \cdots f_1 \circ f_0(x) = 0$ for some n(depending on x). If K is the set of such x, then K is a submodule of E.*

Proof. Obvious. □

Theorem 4. *Let R be a left Noetherian ring, E be an injective left R-module and $f_0, f_1, f_2, \cdots$ be a sequence of elements in $End(E)$. Let $K = \{x \in E | f_n \circ f_{n-1} \circ \cdots \circ f_0(x) = 0$ for some $n \geq 1\}$. If $f_i(K) \subset K$ for all $i \geq 0$ and $K \subset' E$ then $K = E$.*

Proof. Consider the direct sum $K \oplus K \oplus \cdots$ of a countable number of Ks. Then $E \oplus E \oplus \cdots$ is easily an essential extension of $K \oplus K \oplus \cdots$. Since R is left Noetherian, $E \oplus E \oplus \cdots$ is injective, so is an injective envelope of $K \oplus K \oplus \cdots$. So define a map

$$\begin{aligned} \phi : E \oplus E \oplus E \oplus \cdots &\longrightarrow E \oplus E \oplus E \oplus \cdots \\ (x_1, x_2, x_3, \cdots) &\mapsto (x_1, x_2 - f_0(x_1), x_3 - f_1(x_2), \cdots) \end{aligned}$$

Then ϕ is a homomorphism, and $\phi|_{K \oplus K \oplus \cdots}$ is an injection since $\ker\phi|_{K \oplus K \oplus \cdots} = 0$. And if $(y_1, y_2, y_3, \cdots) \in K \oplus K \oplus \cdots$, then let

$$\begin{aligned} x_1 &= y_1 \\ x_2 &= y_2 + f_0(x_1) \\ x_3 &= y_3 + f_1(x_2) \\ &\cdots \end{aligned}$$

Then

$$\begin{aligned} \phi(x_1, x_2, \cdots) &= (x_1, x_2 - f_0(x_1), x_3 - f_1(x_2), \cdots) \\ &= (y_1, y_2 + f_0(x_1) - f_0(x_1), y_3 + f_1(x_2) - f_1(x_2), \cdots) \\ &= (y_1, y_2, y_3, \cdots) \end{aligned}$$

So ϕ is onto on $K \oplus K \oplus \cdots$. Hence by the Lemma 1, $\phi|_{K \oplus K \oplus \cdots}$ is an isomorphism of $K \oplus K \oplus \cdots$. So ϕ is an isomorphism of $E \oplus E \oplus \cdots$ and in particular ϕ is onto. Let $x \in E$ and consider $(x, 0, 0, \cdots)$. Then $\phi(x_1, x_2, x_3, \cdots) = (x, 0, 0, \cdots)$ for some $(x_1, x_2, x_3, \cdots) \in E \oplus E \oplus \cdots$. Then $x_1 = x, x_2 - f_0(x_1) = 0, x_3 - f_1(x_2) = 0$ and so on. So $x_n = f_{n-2}(x_{n-1})$ for all $n \geq 2$. But for some n, $x_{n+2} = 0$. Then

$$\begin{aligned} 0 &= x_{n+2} \\ &= f_n(x_{n+1}) \\ &= f_n(f_{n-1}(x_n)) \\ &= f_n(f_{n-1}(f_{n-2}(x_{n-1}))) \\ &= \cdots \\ &= (f_n \cdot f_{n-1} \cdot \cdots \cdot f_0)(x) \end{aligned}$$

So $x \in K$ for all $x \in E$. □

Theorem 5. *If R is left Noetherian, M is a left R-module and if the diagram of minimal injective resolution of M is commutative and f is locally* *nilpotent on M then each f^n is also locally nilpotent on $E^n(M)$.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\epsilon} & E^0(M) & \xrightarrow{d_0} & E^1(M) & \longrightarrow & \cdots \\ & & \downarrow f & & \downarrow f^0 & & \downarrow f^1 & & \\ 0 & \longrightarrow & M & \xrightarrow{\epsilon} & E^0(M) & \xrightarrow{d_0} & E^1(M) & \longrightarrow & \cdots \end{array}$$

Proof. Let $x \in E^0(M)$ then since $M \subset' E^0(M)$, there exists $r \in R$ such that $rx \in M$. Since the first square is commutative, for all $y \in M$, $f(y) = f^0(y)$. The fact that f is locally nilpotent on M implies $(f^0)^n(rx) = f^n(rx) = 0$ for some $n \geq 1$. So $rx \in ker(f^0)^n$ for some n. Therefore we can say $\cup_{n=1}^{\infty} ker((f^0)^n) \subset' E^0(M)$. Hence by the Theorem 4, f^0 is locally nilpotent on $E^0(M)$.

Given $x \in E^1(M)$, there exists $r \in R$ such that $rx \in E^0(M)/M$ since the sequence is a minimal injective resolution. So $rx = y + M$ for some $y \in E^0(M)$. Then $(f^1 \circ d^0)(y) = (d^0 \circ f^0)(y)$. Since f^0 is locally nilpotent on $E^0(M)$, $(f^0)^n(y) = 0$ for some n . So $(f^0)^n(y) + M = 0 + M$. Hence

$$\begin{aligned} (f^1)^n(rx) &= (f^1)^n(y + M) \\ &= (f^1)^n(d^0(y)) \\ &= (f^1)^{n-1}(f^1 d^0(y)) \\ &= (f^1)^{n-1}(d^0 f^0(y)) \\ &= (f^1)^{n-2}(f^1 d^0 f^0(y)) \\ &= (f^1)^{n-2}(d^0 (f^0)^2(y)) \\ &= \cdots \\ &= d^0(f^0)^n(y) = 0 \end{aligned}$$

So $rx \in ker(f^1)^n$ for some n. Therefore $\cup_{n=1}^{\infty} ker(f^1)^n \subset' E^1(M)$. By the Theorem 4 f^1 is locally nilpotent on $E^1(M)$. Similarly f^i is locally nilpotent on $E^i(M)$ for each $i \geq 2$. □

REFERENCES

[1] J. Rotman, *An introduction to homological algebra*, Academic Press Inc., New York, 1979.

[2] H.Bass, *Finitistic dimensions and a homological generalization of semi-primary rings*, Trans. AMS. 39(1960),466-488

[3] H.Bass, *On the ubiquity of Gorenstein rings*, Math. Z. 82(1963), 8-28

[4] E. Enochs, *Injective and flat covers, envelopes and resolvents*, Israel J of Math. v39 (1981), 189-209.

[5] E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. 220, (1995), 611-633.

[6] J. -L. Koszul, *Sur les modules de représentation des algébres de Lie résolubles*, Amer. J. Math. 76(1954), 535-554.

[7] D. G. Northcott, *Injective envelopes and inverse polynomials*, J. London Math. Soc. (2), (1974), 290-296.

Yi, Okyeon
Department of Mathematics
Korea University
136-701, Seoul, Korea
e-mail address: oyyi00@semi.korea.ac.kr

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