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WARPED PRODUCTS AND RIEMANNIAN MANIFOLDS ADMITTING A FUNCTION WHOSE GRADIENT IS OF CONSTANT NORM

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1. Introduction. Let (M, g) be a complete connected smooth Riemannian manifold of dimension m and $f\colon M \to \mathbb{R}$ a smooth function satisfying

$$\|\nabla f\| \equiv 1, \tag{1.1}$$

where ∇f denotes the gradient vector field of f. Then denoting by φ_s, $s \in \mathbb{R}$, the flow generated by ∇f, the trajectories $c\colon s \mapsto \varphi_s(p)$ are geodesics realizing the distance between levels of f. Namely, setting $Z := f^{-1}(0)$, the map $\Phi\colon X = \mathbb{R} \times Z \to M$ defined by $\Phi(s, z) := \varphi_s z$ is a diffeomorphism. In the previous paper [7], we studied the metrical structure of such manifolds under the condition that

$$\mathrm{Ric}_M(\nabla f, \nabla f) \geq -(m-1)\delta, \tag{1.2}$$

where δ is a nonnegative constant. Then for the Laplacian $\Delta f = -g^{ij}\nabla_i\nabla_j f$ of f we have

$$|\Delta f| \leq (m-1)\delta. \tag{1.3}$$

Moreover, if $\delta = 0$, i.e., $\mathrm{Ric}_M(\nabla f, \nabla f) \geq 0$, then f is an affine function and $\Phi\colon X \to M$ is an isometry, where $Z := f^{-1}(0)$ is endowed with the totally geodesic induced Riemannian metric and $X = \mathbb{R} \times Z$ means the Riemannian direct product.

Next suppose $\delta = 1$ and $|\Delta f| \equiv m - 1$. Then $\Phi\colon X \to M$ is an isometry, where X is endowed with a warped product metric $\mathbb{R} \times_\psi Z$ with $\psi(t) = \exp(\pm t)$ and the induced metric on Z. Moreover, f is a Busemann function defined by asymptotic rays $t \mapsto (t, p)$(or, $t \mapsto (-t, p)$).

Namely, standard warped product spaces appear as the extremal cases of the inequality (1.3). In fact, in [7] we have assumed that $\mathrm{Ric}_M \geq$

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$-(m-1)\delta$ in stead of (1.2). However, the proof presented there works under the assumption (1.2).

Now in [8], we considered the perturbed version of the above extremal cases, and asked what happens for the case where $\mathrm{Ric}_M \geq -(m-1)\kappa$ and $\mathrm{Ric}_M(\nabla f, \nabla f) \geq -(m-1)\delta$, or the case where $\mathrm{Ric}_M \geq -(m-1)\kappa$, $\mathrm{Ric}_M(\nabla f, \nabla f) \geq -(m-1)-\delta$ and $||\Delta f| - (m-1)| < \delta$, where δ, κ are positive constants and δ is sufficiently small. We set $\Psi = \Phi^{-1}$, i.e., $\Psi(p) = (f(p), \pi(p))$ with $\varphi_{f(p)}(\pi(p)) = p$, where $\pi(p) = \varphi_{-f(p)}(p)$ is the foot of the perpendicular of p on Z along a trajectory of ∇f. Then applying Cheeger-Colding's ideas ([5]) to our rather restricted situation, we showed that for any $R > 0$, $\Psi := \Psi\,|_{B_R(p;M)} \colon B_R(p; M) \to X$, restriction of Ψ to a distance R-ball, satisfies

$$|d_M(x, x') - d_{X,\chi}(\Psi(x), \Psi(x'))| < \epsilon, \tag{1.4}$$

namely, for the Gromov-Hausdorff distance

$$d_{GH}(B_R(p; M), B_R(p; d_{X,\chi})) < \epsilon \tag{1.5}$$

holds if we take $\chi > 0$ and $\delta = \delta(\epsilon, m, \kappa, \chi, R) > 0$ sufficiently small. Here, d_χ denotes the distance on Z which is close to the distance obtained from the induced Riemannian metric depending on the parameter χ, and $d_{X,\chi}$ denotes the warped product distance on $X = \mathbb{R} \times_\psi (Z; d_\chi)$ with warping function $\psi(t) = 1$ or $\psi(t) = \exp(\pm t)$ corresponding the above two cases, respectively. In fact, we have (1.4), (1.5) under somewhat weaker conditions (see [8] for details).

Now in the present paper, we consider a general warped product space $X = \mathbb{R} \times_\psi Z$ as a model space. Namely, let Z be a complete connected Riemannian manifold of dimension $m-1$, and $\psi \colon \mathbb{R} \to \mathbb{R}^+$ a positive smooth function. Then the warped product metric with the warping function ψ on $X = \mathbb{R} \times_\psi Z$ is given by

$$ds_X^2 = ds_{\mathbb{R}}^2 + \psi^2 ds_Z^2.$$

Note that the projection $\underline{f} \colon X \to \mathbb{R}$ onto the first factor gives an example of functions satisfying (1.1). We may assume that $\psi(0) = 1$ in the following without loss of generality. In [5], Cheeger-Colding showed that if for a Riemannian manifold M^m the volume or diameter is almost maximal compared with warped product manifolds $(a, b) \times_\psi N^{m-1}$ relative to the behaivor of the Ricci curvature, then it is close to $(a, b) \times_\psi Z$ in the Gromov-Hausdorff topology.

In this note, first we generalize results in [7]. For a given ψ as above we set $k(t) := \frac{\psi'(t)}{\psi(t)}$. We also consider $\psi^* \colon \mathbb{R} \to \mathbb{R}$ defined by $\psi^*(t) := \psi(-t)$.

Theorem 1.1. *Let M be a complete connected Riemannian m-manifold admitting a smooth function f with* (1.1). *Suppose*

$$\mathrm{Ric}_M(\nabla f(x),\nabla f(x)) \geq -(m-1)\frac{\psi''(f(x))}{\psi(f(x))} \quad (\mathit{resp.}, \geq -(m-1)\frac{(\psi^*)''(f(x))}{\psi^*(f(x))}) \tag{1.6}$$

for any $x \in M$. Then we have the following:

(1) If $\Delta f \equiv -(m-1)k \circ f$ (resp., $\Delta f \equiv (m-1)k \circ (-f)$) holds, then $\Phi\colon X = \mathbb{R} \times_\psi Z \to M$ (resp., $\mathbb{R} \times_{\psi^*} Z \to M$) is an isometry.

(2) If $\int_0^\infty \frac{ds}{\psi^2(s)} = +\infty$, then we get

$$\Delta f(x) \leq -(m-1)k(f(x)) \quad (\mathit{resp.}, \Delta f(x) \geq (m-1)k(-f(x))) \tag{1.7}$$

for any $x \in M$.

(3) If $\int_{-\infty}^0 \frac{ds}{\psi^2(s)} = +\infty$, then we get

$$\Delta f(x) \geq -(m-1)k(f(x)) \quad (\mathit{resp.}, \Delta f(x) \leq (m-1)k(-f(x))) \tag{1.8}$$

for any $x \in M$.

Note that taking $\psi(t) \equiv 1$ or $\psi(t) = \exp(\pm t)$, we get results given in [7]. As another corollary we have the following rigidity result which generalizes the nonnegative Ricci curvature case ([7]).

Corollary 1.2. *Let M be a complete connected Riemannian m-manifold admitting a function f with* (1.1). *Suppose*

$$\mathrm{Ric}_M(\nabla f(x),\nabla f(x)) \geq -(m-1)\frac{\psi''(f(x))}{\psi(f(x))},$$

where $\psi\colon \mathbb{R} \to \mathbb{R}^+$ is a positive smooth function satisfying $\int_0^\infty \frac{ds}{\psi^2(s)} = +\infty$ and $\int_{-\infty}^0 \frac{ds}{\psi^2(s)} = +\infty$. Then $\Phi\colon X = \mathbb{R} \times_\psi Z \to M$ is an isometry.

Corollary 1.2 follows directly from (2), (3) and (1) of Theorem 1.

Next we consider the perturbed version of the above rigidity result and get

Theorem 1.3. *Let M be a complete connected Riemannian manifold of dimension m which admits a smooth function f with $\|\nabla f\| = 1$. Let $\psi\colon \mathbb{R} \to \mathbb{R}^+$ be a positive smooth convex function, in the sense that $\psi''(s) \geq 0$. Suppose that the Ricci curvature of M satisfies*

$$(1.9) \qquad \mathrm{Ric}_M \geq -(m-1)\kappa, \quad \mathrm{Ric}_M(\nabla f, \nabla f) \geq -(m-1)\frac{\psi''\circ f}{\psi\circ f} - \delta$$

on $B_{\tilde{R}}(p; M)$, where $\kappa, \delta(< 1)$ are positive constants, and that

$$(1.10) \qquad \frac{1}{\mathrm{vol}\, B_{\tilde{R}}(p;M)} \int_{B_{\tilde{R}}(p;M)} |\Delta f(x) + (m-1)k \circ f(x)|^2 d\nu_g < \delta^2.$$

holds for the Laplacian of f. Then for any $(1 >)\chi > 0, \epsilon > 0$ and $R > 0$, there exists $\tau = \tau(m, \epsilon, \kappa, \chi, R) > 0$ such that if (1.9), (1.10) hold for $(0 <)\delta < \tau$ with sufficiently large $\tilde{R}$ compared with given $R > 0$, e.g.,$\tilde{R} = 30R$, then we have the following :

There exists a distance d_χ on $Z := f^{-1}(0)$ defined by (4.21) which is close to the distance on Z obtained from the induced metric. Let $d_{X,\chi}$ be the distance on X which is the warped product distance $\mathbb{R} \times_\psi (Z, d_\chi)$ defined by (4.22) with warping function ψ. Then $\Psi := \Psi\,|_{B_R(p;M)}$ satisfies

$$(1.11) \qquad |d_M(x, x') - d_{X,\chi}(\Psi(x), \Psi(x'))| < \varphi(\epsilon \mid \chi \mid \psi, R)$$

and for any $y \in B_R(p; (X, d_{X,\chi}))$ there exists $x \in B_R(p; M)$ such that

$$(1.12) \qquad d_{X,\chi}(y, \Psi(x)) < \varphi(\epsilon \mid \chi \mid \psi, R).$$

In particular, for the Gromov-Hausdorff distance we get

$$(1.13) \qquad d_{GH}(B_R(p; M), B_R(p; (X, d_{X,\chi}))) < \varphi(\epsilon \mid \chi \mid \psi, R).$$

In the above $\varphi(\epsilon \mid \chi \mid \psi, R)$ means that for fixed ψ and $R > 0$ we have $\varphi(\epsilon \mid \chi \mid \psi, R) \downarrow 0$ as $\chi \downarrow 0$ and taking $\epsilon = \epsilon(\chi) \downarrow 0$.

In §2 we are concerned with some geometry of warped product space $X = \mathbb{R} \times_\psi Z$, and in §3, 4 we give proofs of Theorem 1.1 and Theorem 1.3, respectively. We mainly use the Bochner formula and standard comparison theorem for the proof of Theorem 1.1. Our proof of Theorem 1.2 essentially follows the same strategy as in the previous paper [8], and much owes to recent important Cheeger-Colding's ideas ([5]).

2. Preliminaries from the geometry of warped product space. In this section we review some geometry of a warped product space $X = \mathbb{R} \times_\psi Z$. Recall that the projection $r = \underline{f}\colon X \to \mathbb{R}$ onto the first factor is a smooth function satisfying $\|\nabla \underline{f}\| \equiv 1$.

2.1. First we recall a result which characterizes the warped product spaces among Riemannian manifolds admitting a smooth function f with $\|\nabla f\| \equiv 1$. Note that the Hessian $D^2 f$ of f defines the second fundamental form when restricted to each level of f, and its mean curvature with respect to the unit normal ∇f is given by $-\frac{\Delta f}{m-1}$.

Lemma 2.1 ([8, Lemma 3.1]). *Let (M, g) be a complete connected Riemannian manifold admitting a function f with $\|\nabla f\| \equiv 1$. Let $Z := f^{-1}(0)$ be endowed with the induced Riemannian metric. Then M is isometric to a warped product manifold $\mathbb{R} \times_\psi Z$ with $\psi(0) = 1$ so that f corresponds to the canonical projection onto the first factor if and only if there exists a smooth function $k\colon \mathbb{R} \to \mathbb{R}$ such that*

$$D^2 f = (k \circ f)\{g - df \otimes df\}, \tag{2.1}$$

where k, ψ are related by $k(s) = \frac{\psi'(s)}{\psi(s)}$ or $\psi(s) = \exp(\int_0^t k(t)dt)$.

Note that in this case we have

$$k \circ \underline{f} = -\frac{\Delta \underline{f}}{m-1}. \tag{2.2}$$

Therefore, each level of $\underline{f}$ is a totally umbilical hypersurface of X with principal curvature $k \circ \underline{f}$.

2.2. Next we are concerned with Jacobi fields along radial geodesics $s \to (s, p), p \in Z$ in X. Since $r = \underline{f}\colon X \to \mathbb{R}$ is a Riemannian submersion, for any vector field V on Z, we have

$$\nabla_{\frac{\partial}{\partial r}} V = \frac{\psi' \circ r}{\psi \circ r} V, \tag{2.3}$$

where we set $\frac{\partial}{\partial r} := \nabla r = \nabla \underline{f}$ (see e.g., [1]). Namely, $E := \frac{V}{\psi \circ r}$ is a parallel vector field along a radial geodesic. Conversely, for a parallel vector field E with $E(0) \in T_p Z$ along a radial geodesic $\underline{\gamma}$, $\underline{Y}(s) := \psi(s)E(s)$ is a Jacobi field satisfying the initial conditions $\underline{Y}(0) \in T_p Z, \nabla \underline{Y}(0) = k(0)\underline{Y}(0)$, where $k(0)$ is the principal curvature of the totally umbilical hypersurface Z. Such a Jacobi field is called a Z-Jacobi field and may be characterized as the variation vector field of a variation consisting of radial geodesics. Then any Z-Jacobi field $\underline{Y}$ along $\underline{\gamma}$ may be written in the form $\underline{Y}(s) = \psi(s)E(s)$ with a parallel E, and from the Jacobi equation we get

$$R(E(s), \frac{\partial}{\partial r})\frac{\partial}{\partial r} = -\frac{\psi''(s)}{\psi(s)}E(s). \tag{2.4}$$

In particular, we have $\mathrm{Ric}_M(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = -(m-1)\frac{\psi'' \circ r}{\psi \circ r}$. Since any radial geodesic $\underline{\gamma}$ is free of Z-focal points, such Z-Jacobi fields span the tangent space $T_{\underline{\gamma}(s)}\underline{f}^{-1}(s)$ to any level of $\underline{f}$. Note also that $\nabla\underline{Y}(s) = k(s)\underline{Y}(s)$.

Next we consider a Jacobi field Y along $\underline{\gamma}$ in the form

$$Y(s) = y(s)\underline{Y}(s),$$

where $\underline{Y}$ is a Z-Jacobi field. Then from the Jacobi equation $\nabla\nabla Y(s) + R(Y(s), \frac{\partial}{\partial r})\frac{\partial}{\partial r} = 0$, we obtain

$$\psi(s)y''(s) + 2\psi'(s)y'(s) = 0. \tag{2.5}$$

Solving this differential equation we get

$$y(s) = y(0) + y'(0)\int_0^s \frac{ds}{\psi^2(s)}. \tag{2.6}$$

Note that $Y(s) = 0$ holds for $s > 0$ if and only if we have

$$y'(0) = -\frac{y(0)}{\int_0^s \frac{ds}{\psi^2(s)}}.$$

Now let $\tilde{Z}$ be a hypersuface perpendicular to a radial geodesic $\underline{\gamma}$, which is totally umbilical at $p := \underline{\gamma}(0) \in Z$. Let $\tilde{\lambda}$ be the principal curvature of $\tilde{Z}$ at p with respect to the unit normal $\dot{\underline{\gamma}}(0) = \nabla\underline{f}(p)$.

Lemma 2.2. *Suppose $\int_0^\infty \frac{ds}{\psi^2(s)} = +\infty$. Then there appears a focal point $\underline{\gamma}(s)$ of $\tilde{Z}$ at $s > 0$ if and only if $\tilde{\lambda} < k(0)$.*

Proof. Suppose $\tilde{\lambda} < k(0)$. Then consider a Jacobi field along $\underline{\gamma}$ given by $\tilde{Y}(s) := y(s)\underline{Y}(s)$, where $\underline{Y}(s)$ is a nonzero Z-Jacobi field and $y(0) = 1$. Then $y(s)$ satisfies (2.6) and $\nabla\tilde{Y}(0) = (y'(0) + k(0))\underline{Y}(0)$. Hence $\tilde{Y}(s)$ is a $\tilde{Z}$-Jacobi field if and only if

$$y(s) = 1 + (\tilde{\lambda} - k(0))\int_0^s \frac{ds}{\psi^2(s)}$$

by (2.6). Then we have $y(s) = 0$ for $s > 0$ satisfying the equation $\frac{1}{k(0)-\tilde{\lambda}} = \int_0^s \frac{ds}{\psi^2(s)}$, namely a focal point $\underline{\gamma}(s)$ of $\tilde{Z}$ along $\underline{\gamma}$ appears at $s > 0$. Conversely, suppose $\tilde{Y}(s) = 0, s > 0$ holds for a nonzero $\tilde{Z}$-Jacobi field $\tilde{Y}$ along $\underline{\gamma}$. Let $\underline{Y}_i$ $(i = 1, \dots, m-1)$ be a basis for the space of Z-Jacobi fields along $\underline{\gamma}$. Then we may write $\tilde{Y}(s) = \sum_i y_i(s)\underline{Y}_i(s)$ with

$$y_i(s) = y_i(0)\{1 + (\tilde{\lambda} - k(0)) \int_0^s \frac{ds}{\psi^2(s)}\} \quad (i = 1, \dots, m-1).$$

Therefore, $\tilde{Y}(s) = 0$ for $s > 0$ implies that $\tilde{\lambda} < k(0)$.

Remark 2.3. Suppose $\tilde{Z}$ is a hypersurface through $p := \underline{\gamma}(s_0) \in f^{-1}(s_0)$ perpendicular to a radial geodesic $\underline{\gamma}$, which is totally umbilical at p with the principal curvature $\tilde{\lambda}$ with respect to the unit normal $\dot{\underline{\gamma}}(s_0) = \nabla \underline{f}(p)$. Suppose $\int_0^\infty \frac{ds}{\psi^2(s)} = +\infty$. Then by the same argument as above we see that there appears a focal point $\underline{\gamma}(s)$ of $\tilde{Z}$ for $s > s_0$ if and only if $\tilde{\lambda} < k(s_0)$ holds. Similarly, under the above situation, considering a radial geodesic $s \to \underline{\gamma}(-s)$ reversing the orientation of $\underline{\gamma}$, we have the following: Suppose $\int_{-\infty}^0 \frac{ds}{\psi^2(s)} = +\infty$. Then there appears a focal point $\underline{\gamma}(s)$ of $\tilde{Z}$ for $s < s_0$ along the reversed radial geodesic if and only if $\tilde{\lambda} > k(s_0)$ holds.

2.3. Next we are concerned with geodesics and distance function of the warped product space $X = \mathbb{R} \times_\psi Z$. Let $\underline{\gamma}$ be a geodesic parametrized by arclength in X. We set

$$\underline{\mathcal{U}}(s) := \underline{f}(\underline{\gamma}(s)). \tag{2.7}$$

Then we have

$$\underline{\mathcal{U}}'(s) = \langle \nabla \underline{f}, \dot{\underline{\gamma}}(s) \rangle,$$
$$\underline{\mathcal{U}}''(s) = D^2\underline{f}(\dot{\underline{\gamma}}(s), \dot{\underline{\gamma}}(s)) = (k \circ \underline{f})\{g - d\underline{f} \otimes d\underline{f}\}(\dot{\underline{\gamma}}(s), \dot{\underline{\gamma}}(s)).$$

Therefore, $\underline{\mathcal{U}}$ satisfies the differential equation

$$\underline{\mathcal{U}}''(s) + k(\underline{\mathcal{U}}(s))(\underline{\mathcal{U}}'(s)^2 - 1) = 0. \tag{2.8}$$

Now set $F(t) := \int_0^t \psi(s)ds$, which is strictly monotone increasing, and we consider the inverse function $G := F^{-1}$. Setting $\underline{y}(s) := F(\underline{\mathcal{U}}(s))$ we obtain

$$\underline{y}'' = H(\underline{y}) \quad \text{with} \quad H := \psi' \circ G. \tag{2.9}$$

This means that $\underline{y}(s)$ and consequently $\underline{\mathcal{U}}(s)$ admit first integrals.

We assume that $\psi''(s) \geq 0$ in the following. Now the next result is given in [5].

Lemma 2.4. *Suppose $\psi''(s) \geq 0$. Then the boundary value problem for (2.8) has a unique solution. Namely, for fixed $\underline{\mathcal{U}}(0)$ and l, the map $\underline{\mathcal{U}}'(0) \mapsto \underline{\mathcal{U}}(l)$ defines a smooth strictly monotone increasing map $\beta\colon (-1,1) \to (-l+r_0, l+r_0)$, and therefore β^{-1} is smooth and stricly monotone increasing.*

Proof. Since (2.8) depends only on $k = \frac{\psi'}{\psi}$, we may consider $\underline{\mathcal{U}}(s)$ in a particular warped product space $X_0 := \mathbb{R} \times_\psi \mathbb{R}$, which is an Hadamard 2-manifold by the assumption $\psi''(s) \geq 0$. Take $x_0 \in X_0$ with $\underline{f}(x_0) = \underline{\mathcal{U}}(0) =: r_0$, and a geodesic $\underline{\gamma}_{\alpha_0}$ emanating from x_0 in X_0 parametrized by arclength with $\angle(\dot{\underline{\gamma}}_{\alpha_0}(0), \nabla \underline{f}(x_0)) = \alpha_0 \quad (0 < \alpha_0 < \pi)$. Then we consider a family of geodesics $\underline{\gamma}_\alpha$, $0 < \alpha < \pi$ containing $\underline{\gamma}_{\alpha_0}$, which emanate from x_0 and are parametrized by arclength with $\angle(\dot{\underline{\gamma}}_\alpha(0), \nabla \underline{f}(x_0)) = \alpha$. Then the map

$$\beta : u = \cos\alpha := \underline{\mathcal{U}}'(0) \in (-1,1) \mapsto \underline{\mathcal{U}}(l) = \underline{f}(\underline{\gamma}_\alpha(l)) \in (-l+r_0, l+r_0)$$

is a smooth map with derivative

$$\beta'(u) = -\langle \nabla \underline{f}(\underline{\gamma}_\alpha(l)), \frac{\partial \underline{\gamma}_\alpha}{\partial \alpha}(l)\rangle \frac{1}{\sin\alpha}.$$

Now if $\beta'(u) = 0$, then the Jacobi field $s \mapsto \frac{\partial \underline{\gamma}_\alpha}{\partial \alpha}(s)$ along $\underline{\gamma}_\alpha$ vanishes at $s = l$, which is a contradiction. Hence $\beta'(u) \neq 0$ everywhere and the assertion of the lemma follows. Note that $\beta'(\pm 1) = -\frac{d}{d\alpha}\,|_{\alpha=0,\pi}\, \langle \nabla \underline{f}(\underline{\gamma}_\alpha(l)), \frac{\partial \underline{\gamma}_\alpha}{\partial \alpha}(l)\rangle$.

Remark 2.5. (1) If $\underline{\mathcal{U}}'(0) = \pm 1$, then the corresponding geodesic in X_0 is a radial (or reversed radial) geodesic, and we have $\underline{\mathcal{U}}(s) = \pm s + r_0$.

(2) From the above lemma, for given $l > 0$ and $r_0, r_l \in \mathbb{R}$ with $|r_l - r_0| \leq l$ we have a unique $\underline{\mathcal{U}}(s)$ satisfying (2.8) and the boundary conditions $\underline{\mathcal{U}}(0) = r_0, \underline{\mathcal{U}}(l) = r_l$, which will be denoted by

$$\underline{\mathcal{U}}(s) =: \underline{\mathcal{U}}(s; r_0, r_l, l). \tag{2.10}$$

Note that $\underline{\mathcal{U}}$ depends smoothly on parameters. We may also write

$$\underline{\mathcal{U}}'(s) =: \cos\underline{\theta}(s) \quad \textit{with} \quad \underline{\theta}(s) =: \underline{\theta}(s; r_0, r_l, l), \tag{2.11}$$

where $\theta(s)$ equals the angle $\angle(\dot{\underline{\gamma}}(s), \nabla \underline{f}(\underline{\gamma}(s)))$ between ∇f and the corresponding geodesic $\underline{\gamma}$ in X.

Then we get the following by arguing as in the last part of §2 of [5]: Let d_X (resp. d_Z) denotes the distance on X (resp. Z) defined from the Riemannian metric g (resp. induced metric on Z). First suppose $\underline{x}_i, \underline{y}_i \in X (i = 1, 2)$ satisfy $\underline{\pi}(\underline{x}_1) = \underline{\pi}(\underline{y}_1)$, $\underline{\pi}(\underline{x}_2) = \underline{\pi}(\underline{y}_2)$. Then we may write the distance function d_X in the form

$$d_X(\underline{y}_1, \underline{y}_2) = Q(\underline{r}(\underline{x}_1), \underline{r}(\underline{y}_1), \underline{r}(\underline{x}_2), \underline{r}(\underline{y}_2), d_X(\underline{x}_1, \underline{x}_2)). \tag{2.12}$$

Next considering the case where $x_1, x_2 \in Z$ and letting $x_2 \to x_1$ we get a formula representing d_X in terms of d_Z. Namely, we have a function $\rho\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by which we may write

$$d_X((r_1, z_1), (r_2, z_2)) = \rho(r_1, r_2, d_Z(z_1, z_2)). \tag{2.13}$$

For instance, if $z_1, z_2 \in Z$ we have

$$d_Z(z_1, z_2) = \int_0^{d_X(z_1,z_2)} \exp(-\underline{\mathcal{U}}(s))\sqrt{1 - \underline{\mathcal{U}}'(s)^2}ds,$$

where we set $\underline{\mathcal{U}}(s) := \underline{\mathcal{U}}(s; 0, 0, d_X(z_1, z_2))$. Then the above formulas determine the warped product distance on X with warping function ψ, once a distance is given on Z. Note that for fixed $r_0, r_N, \{v_i\}_{i=1}^N$ there exist $r_1, \ldots, r_{N-1}$ such that

$$\sum_{i=0}^{N-1} \rho(r_i, r_{i+1}, v_{i+1}) = \rho(r_0, r_N, \sum_{i=1}^{N} v_i).$$

We also remark that setting $w(v) := \rho(0, 0, v)$, which is strictly monotone increasing, we have $w(v) \le v$, $w(0) = 0$ and $w'(0) = 1$, and that by the first variation formula $\frac{\partial Q}{\partial v}(0, r, 0, r', v) \ge 0$.

3. Proof of Theorem 1.1. First suppose we have $\Delta f \equiv -(m-1)k \circ f$ under the assumption (1.6) on the Ricci curvature. Then applying the Bochner formula

$$\frac{1}{2}\Delta\|\nabla u\|^2 = \langle \nabla u, \nabla \Delta u\rangle - \mathrm{Ric}_M(\nabla u,\ \nabla u) - \|D^2 u\|^2, \tag{3.1}$$

to our function f, and noting that $\|\nabla f\| = 1$ and $\nabla\Delta f = -(m-1)(k'\circ f)\nabla f$ with $k(s) = \frac{\psi'(s)}{\psi(s)}$, we obtain

$$\begin{aligned}\|D^2 f\|^2 &= -\mathrm{Ric}_M(\nabla f, \nabla f) + \langle \nabla\Delta f, \nabla f\rangle \\ &\le (m-1)\frac{\psi'' \circ f}{\psi \circ f} - (m-1)k' \circ f,\end{aligned}$$

and therefore

$$\begin{aligned}
&\|D^2 f - (k \circ f)\{g - df \otimes df\}\|^2 \\
&= \|D^2 f\|^2 + 2(k \circ f)\Delta f + (m-1)k^2 \circ f \\
&\leq (m-1)\frac{\psi'' \circ f}{\psi \circ f} - (m-1)k' \circ f - (m-1)k^2 \circ f \\
&= (m-1)\left\{\frac{\psi'' \circ f}{\psi \circ f} - \frac{\psi'' \circ f}{\psi \circ f} + k^2 \circ f - k^2 \circ f\right\} \\
&= 0.
\end{aligned}$$

It follows that

$$D^2 f - (k \circ f)\{g - df \otimes df\} = 0 \tag{3.2}$$

and (1) of Theorem 1.1 follows from Lemma 2.1.

Now we turn to the proof of (2). Suppose $\Delta f(x) > -(m-1)k(f(x))$ at some $x \in M$. Set $s_0 := f(x)$ and recall that the second fundamental form of the level $Z_{s_0} := f^{-1}(s_0)$ with respect to the unit normal $\nabla f(x)$ is given by $D^2 f(x)$. Let γ be a radial geodesic through x with $\gamma(0) = x, f(\gamma(s)) = s + s_0$, which is a trajectory of ∇f, and we consider Z_{s_0}-Jacobi fields along γ. Next, in the model space $X := \mathbb{R} \times_{\psi_1} Z_{s_0}$, where $\psi_1(s) := \psi(s + s_0)$ and Z_{s_0} is endowed with the induced metric, take a hypersurface $\tilde{Z}$ through $\Phi_1(x) := (0, x) \in X$, which is perpendicular to the radial geodesic $\underline{\gamma}$: $s \mapsto (s, x)$ in X emanating from $\Phi_1(x)$ and is totally umbilical at $\Phi(x)$ with principal curvature $-\frac{\Delta f(x)}{m-1}$ with respect to $\dot{\underline{\gamma}}(0)$. Recall that $-\frac{\Delta f(x)}{m-1}$ is the mean curvature of Z_{s_0} at x in M. On the other hand, the mean curvature of Z_{s_0} in X at $\Phi_1(x)$ with respect to the unit normal $\dot{\underline{\gamma}}(0)$ is given by $k_1(0) = k(s_0)$. By applying Lemma 2.2 with $\tilde{\lambda} = -\frac{\Delta f(x)}{m-1}$, it follows that along the radial geodesic $\underline{\gamma}$ in X there appears a focal point $\underline{\gamma}(s_1)$ of $\tilde{Z}$ at $s_1 > 0$ with $\frac{1}{k(s_0) - \tilde{\lambda}} = \int_0^{s_1} \frac{ds}{\psi_1^2(s)}$.

Now we apply the Heintze-Karcher comparison theorem (see e.g., [9], p.147-148) to Z_{s_0}-Jacobi fields along γ in M and $\tilde{Z}$-Jacobi fields along $\underline{\gamma}$ in X. Recall that we have

$$\begin{aligned}
&\mathrm{Ric}_M(\dot{\gamma}(s), \dot{\gamma}(s)) = \mathrm{Ric}_M(\nabla f, \nabla f)(\gamma(s)) \\
&\geq -(m-1)\frac{\psi''(s+s_0)}{\psi(s+s_0)} = -(m-1)\frac{\psi_1''(s)}{\psi_1(s)} = \mathrm{Ric}_X(\dot{\underline{\gamma}}(s), \dot{\underline{\gamma}}(s)).
\end{aligned}$$

Since $-\frac{\Delta f(x)}{m-1}$ is the mean curvature of Z_{s_0} in M, and $\tilde{Z}$ is totally umbilical at $\underline{\gamma}(s_0)$ with the same mean curvature as $\tilde{Z}$ in X, the above comparison theorem implies that there also appears a focal point $\gamma(s)$ of Z_{s_0} at some $s_1 \geq s > 0$. In fact, the comparison theorem given in the reference [9] is stated in the case where $\mathrm{Ric}_X(\dot{\underline{\gamma}}(s), \dot{\underline{\gamma}}(s))$ is bounded below by a constant. However, the same proof works for the present case, and we have a contradiction. Hence $\Delta f(x) \leq -(m-1)k(f(x))$ for any $x \in M$. The assertion (3) may be proved by the same manner considering the reversed radial geodesics and noting Remark 2.3. Then the alternate assertions in (2), (3) follow considering ψ^* in stead of ψ.

4. Proof of Theorem 1.3. Let (M, g) be a complete connected Riemannian manifold of dimension m admitting a function $f\colon M \to \mathbb{R}$ with $\|\nabla f\| \equiv 1$. Here we are concerned with the perturbed version of Theorem 1.1 (1). Under the assumption on the Ricci curvature and the Laplacian of f compared with those of the warped product space $X = \mathbb{R} \times_\psi Z$ with warping function ψ, where $Z = f^{-1}(0)$ is endowed with some appropriate distance, we want to show that, when restricted to distance balls, M is close in the Gromov-Hausdorff topology to $X = \mathbb{R} \times_\psi Z$. We begin with the following lemma.

Suppose that the Ricci curvature of M satisfies

$$\mathrm{Ric}_M \geq -(m-1)\kappa \quad \mathrm{Ric}_M(\nabla f, \nabla f) \geq -(m-1)\frac{\psi'' \circ f}{\psi \circ f} - \delta \tag{4.1}$$

on $B_{2R}(p; M)$, where $\kappa, \delta (< 1)$ are positive constants, and that

$$\frac{1}{\mathrm{vol}\, B_{2R}(p; M)} \int_{B_{2R}(p;M)} |\Delta f(x) + (m-1)k \circ f(x)|^2 d\nu_g < \delta^2. \tag{4.2}$$

Lemma 4.1. *There exsists $C = C(m, \psi, \kappa, R) > 0$ such that for $p \in M$ if we assume that* (4.1) *and* (4.2) *hold on $B_{2R}(p; M)$ then we have*

$$\frac{1}{\mathrm{vol}\, B_R(p)} \int_{B_R(p)} \|D^2 f - (k \circ f)\{g - df \otimes df\}\|^2 d\nu_g < C\delta. \tag{4.3}$$

Proof. Take a function $\tilde{f}$ such that $\Delta \tilde{f} = -(m-1)k \circ f$ on $B_{2R}(p)$. Then we get

$$\Delta(f - \tilde{f}) = \Delta f + (m-1)k \circ f$$

and

$$\langle \nabla f, \nabla \Delta f \rangle = \langle \nabla f, \nabla(\Delta f - \Delta \tilde{f}) \rangle - (m-1)k' \circ f.$$

Now from the Bochner formula we obtain

$$\begin{aligned}\|D^2 f\|^2 &= -\mathrm{Ric}_M(\nabla f, \nabla f) + \langle \nabla \Delta f, \nabla f\rangle \\ &\le (m-1)\frac{\psi'' \circ f}{\psi \circ f} + \langle \nabla \Delta f, \nabla f\rangle + \delta.\end{aligned}$$

It follows that

$$\begin{aligned}&\|D^2 f + \frac{\Delta f}{m-1}(g - df \otimes df)\|^2 = \|D^2 f\|^2 - \frac{(\Delta f)^2}{m-1} \\ &= -\mathrm{Ric}_M(\nabla f, \nabla f) + \langle \nabla \Delta f, \nabla f\rangle - \frac{(\Delta f)^2}{m-1} \\ &\le (m-1)\frac{\psi'' \circ f}{\psi \circ f} - \frac{(\Delta f)^2}{m-1} + \langle \nabla \Delta f, \nabla f\rangle + \delta \\ &= (m-1)(\frac{\psi'' \circ f}{\psi \circ f} - k' \circ f) - \frac{(\Delta f)^2}{m-1} + \langle \nabla f, \nabla(\Delta f - \Delta \tilde{f})\rangle + \delta \\ &= \frac{1}{m-1}\{(m-1)^2 k^2 \circ f - (\Delta f)^2\} + \langle \nabla f, \nabla(\Delta f - \Delta \tilde{f})\rangle + \delta.\end{aligned}$$

On the other hand we get from the assumption

$$\begin{aligned}&\int_{B_{2R}(p;M)} |(m-1)k \circ f - \Delta f|^2 d\nu_g, \quad \int_{B_{2R}(p;M)} |\Delta f|^2 d\nu_g \\ &\le \{2\delta^2 + c(m, R, \psi)\} \mathrm{vol}\, B_{2R}(p; M).\end{aligned}$$

Now take a cut off function ϕ: $M \to [0,1]$ such that

$$\phi \mid B_R(p) \equiv 1,\ \mathrm{supp}\, \phi \subset B_{2R}(p),\ \|\nabla \phi\|, |\Delta \phi| \le c(m, \kappa, R) \tag{4.4}$$

(see [5], Theorem 6.33). Then we have

$$\begin{aligned}&\phi\langle \nabla f, \nabla(\Delta f - \Delta \tilde{f})\rangle \\ &= \mathrm{div}(\phi \Delta(f - \tilde{f}) \nabla f) - \Delta(f - \tilde{f})\langle \nabla \phi, \nabla f\rangle + \phi \Delta f \Delta(f - \tilde{f}).\end{aligned}$$

It follows from the above using Green theorem and Cauchy-Schwarz inequality that

$$\begin{aligned}
&\int_{B_R(p)} \|D^2 f + \frac{\Delta f}{m-1}(g - df \otimes df)\|^2 d\nu_g \\
&\leq \int_{B_{2R}(p)} \phi \|D^2 f + \frac{\Delta f}{m-1}(g - df \otimes df)\|^2 d\nu_g \\
&\leq \delta c_1(m, \psi, R) \operatorname{vol} B_{2R}(p) + \left| \int_{B_{2R}(p)} \phi \langle \nabla f, \nabla(\Delta f - \Delta \tilde{f}) \rangle d\nu_g \right| \\
&\leq \delta c_2(m, \kappa, \psi, R) \operatorname{vol} B_{2R}(p) \leq \delta c_2(m, \kappa, \psi, R) \frac{\operatorname{vol} B_{2R}(p)}{\operatorname{vol} B_R(p)} \operatorname{vol} B_R(p) \\
&\leq \delta c_2(m, \kappa, \psi, R) \frac{v^m_{-\kappa}(2R)}{v^m_{-\kappa}(R)} \operatorname{vol} B_R(p) \\
&\leq \delta C(m, \kappa, \psi, R) \operatorname{vol} B_R(p),
\end{aligned}$$

where $v^m_{-\kappa}(R)$ denotes the volume of a distance R-ball in the simply connected space form of constant curvature κ, and we have used the Bishop-Gromov volume comparison theorem. On the other hand, note that

$$\begin{aligned}
&\|D^2 f - (k \circ f)\{g - df \otimes df\}\|^2 \\
&= \|D^2 f + \frac{\Delta f}{m-1}(g - df \otimes df)\|^2 + (m-1)(\frac{\Delta f}{m-1} + k \circ f)^2.
\end{aligned}$$

Then integrating the above as before the lemma easily follows.

Remark 4.2. Assume the following rather strong condition for f

$$(4.5) \qquad |\Delta f(x) + (m-1) k \circ f(x)| < \delta, \quad \nabla f(\Delta f) + (m-1) k' \circ f < \delta.$$

Then under the Ricci condition

$$(4.6) \qquad \operatorname{Ric}_M(\nabla f, \nabla f) \geq -(m-1) \frac{\psi'' \circ f}{\psi \circ f} - \delta,$$

we have the following by the same argument as in Propositions 2.1, 3.2 of [8]: Suppose M and f with $\|\nabla f\| \equiv 1$ satisfy (4.5) and (4.6). Then for any $\epsilon > 0$ and any $R > 0$ there exists $\tau = \tau(\epsilon, m, \psi, R) > 0$ such that if $0 < \delta < \tau$, then for Φ restricted to $B_R(p; X) \subset X$, $p \in Z$ we have

$$(4.7) \qquad (1-\epsilon) d_X(x, y) \leq d_M(\Phi(x), \Phi(y)) \leq (1+\epsilon) d_X(x, y),$$

where $X = \mathbb{R} \times_\psi Z$ is a warped product metric with warping finction ψ and the induced metric on Z. Furthermore, levels of f are almost totally umbilical in the sense that we have for a positive constant $C = C(m, \psi, R)$

$$\|D^2 f - (k \circ f)\{g - df \otimes df\}\|^2 < C\delta. \tag{4.8}$$

In fact, it suffices to assume that (4.5) and (4.6) hold on a larger concentric distance ball e.g, $B_{2R}(p; X)$.

Now for $y_1, y_2 \in B_{2R_1}(p; M)$ let $\gamma = \gamma_{y_1 y_2}$ be a minimal geodesic joining y_1 to y_2 in M parametrized by arclength. As in (2.7) we set $\mathcal{U}(s) = \mathcal{U}(s; y_1, y_2) := f(\gamma(s))$, and get

$$\mathcal{U}'(s) = \langle \nabla f, \dot\gamma(s)\rangle, \quad \mathcal{U}''(s) = D^2 f(\dot\gamma(s), \dot\gamma(s)).$$

We want to compare $\mathcal{U}(s)$ with the corresponding $\underline{\mathcal{U}}(s)$ satisfying the same boundary condition in the model space $X = \mathbb{R} \times_\psi Z$, where $Z = f^{-1}(0)$ is endowed with a Riemannian metric. Namely, we compare $\mathcal{U}(s) = \mathcal{U}(s; y_1, y_2)$ with $\underline{\mathcal{U}}(s) = \underline{\mathcal{U}}(s; r_0, r_l, l)$ setting $r_0 = f(y_1) = \mathcal{U}(0), r_l = f(y_2) = \mathcal{U}(l), l = d(y_1, y_2)$.

First we assume that

$$\int_0^l |\mathcal{U}''(s) + k(\mathcal{U}(s))((\mathcal{U}'(s))^2 - 1)| ds < \epsilon_2. \tag{4.9}$$

Setting $b(s) := \mathcal{U}''(s) + k(\mathcal{U}(s))((\mathcal{U}'(s))^2 - 1)$ and $y(s) := F(\mathcal{U}(s)), \underline{y}(s) := F(\underline{\mathcal{U}}(s))$ (see (2.9)) we have

$$\begin{aligned} &y''(s) = H(y(s)) + \psi(G(y(s)))b(s), \quad \underline{y}''(s) = H(\underline{y}(s)), \\ &y(0) = \underline{y}(0), \quad y(l) = \underline{y}(l). \end{aligned}$$

Note that $G(y(s)) = \mathcal{U}(s)$. Since $|\mathcal{U}'(s)|, |\underline{\mathcal{U}}'(s)| \le 1$ we have a uniform bound $|\mathcal{U}(s)|, |\underline{\mathcal{U}}(s)| \le C(l, r_0); 0 \le s \le l$. Now we set $z = y - \underline{y}$ and get

$$\begin{aligned} z''(s) &= H(y(s)) - H(\underline{y}(s)) + a(s) \\ &= H'(\underline{y}(s) + \theta(y(s) - \underline{y}(s)))z(s) + a(s), \end{aligned}$$

where we set $a(s) = \psi(\mathcal{U}(s))b(s)$. Note that $z(0) = z(l) = 0$ and $H' = \frac{\psi''}{\psi} \circ G \ge 0$. It follows from (4.9) that

$$\begin{aligned} 0 &= \int_0^l \{z'(s)z(s)\}' ds = \int_0^l \{z'(s)^2 + z''(s)z(s)\} ds \\ &\ge \int_0^l z'(s)^2 ds + \int_0^l a(s)z(s) ds. \end{aligned}$$

Hence

$$(4.10)\qquad \int_0^l z'(s)^2 ds \le \left|\int_0^l a(s)z(s)ds\right| < C(R_1,\psi)\epsilon_2.$$

From the above we easily see that $|z(s)|$ is small enough and $|z'(s)-z'(0)| < \varphi(\epsilon_2 \mid \psi, R_1)$ for $0 \le s \le l$. Considering the boundary condition $z(0) = z(l) = 0$, it follows that $|z'(s)|$ is also small, namely,

$$(4.11)\qquad |\mathcal{U}'(s) - \underline{\mathcal{U}}'(s)|,\quad |\mathcal{U}(s) - \underline{\mathcal{U}}(s)| < \varphi(\epsilon_2 \mid \psi, R_1) \quad (0 \le s \le l),$$

where $\varphi(\epsilon_2 \mid \psi, R_1)$ means that for given ψ, R_1 we have $\varphi(\epsilon_2 \mid \psi, R_1) \downarrow 0$ as $\epsilon_2 \downarrow 0$.

Now to compare $\mathcal{U}(s)$ with $\underline{\mathcal{U}}(s)$ for general $\gamma_{y_1y_2}$ we appeal to [5], Theorem 2.11, which is first given in [2] and has many important applications (see [3, 4]), and get

Lemma 4.3. *Suppose that* $\mathrm{Ric}_M \ge -(m-1)\kappa$ *on* $B_{4R_1}(p)$ *and*

$$(4.12)\qquad \frac{1}{\mathrm{vol}\, B_{4R_1}(p)} \int_{B_{4R_1}(p)} \|D^2 f - (k\circ f)\{g - df\otimes df\}\| d\nu_g < \epsilon_1.$$

Then there exists $C = C(m,\kappa,R_1) > 0$ *such that*

(4.13)

$$\frac{1}{(\mathrm{vol}\, B_{2R_1}(p))^2} \int_{B_{2R_1}(p)\times B_{2R_1}(p)} d\nu_{g\oplus g} \int_0^l |\mathcal{U}''(s) + k(\mathcal{U}(s))(\mathcal{U}'(s)^2 - 1)| ds < C\epsilon_1$$

holds, where we set $\mathcal{U}(s) = \mathcal{U}(s; y_1, y_2)$ *and* $l = d_M(y_1, y_2)$.

Note that from Lemma 4.2 and Cauchy-Schwarz inequality, (4.12) holds for any $4R_1 > 0$ taking $\delta = \delta(m,\kappa,\psi,R_1)$ in (4.1), (4.2) with $2R = 8R_1$ sufficiently small. This lemma roughly means that for any $\epsilon_2 > 0$ we have

$$\int_0^l |\mathcal{U}''(s) + k(\mathcal{U}(s))((\mathcal{U}'(s))^2 - 1)| ds < \epsilon_2$$

for almost all (namely, except for a set of very small volume) $y_1 \in B_{2R_1}(p)$ and $y_2 \in B_{2R_1}(p)$, where we set $\mathcal{U}(s) := \mathcal{U}(s; y_1, y_2)$. To be more precise, we denote by $D_{\epsilon_2}(y_1)$, $y_1 \in B_{2R_1}(p)$, the set of points $y_2 \in B_{2R_1}(p)$ such that there exists a unique minimal geodesic $\gamma_{y_1y_2}$ joining y_1 to y_2 parametrized by arclength, and that $\int_0^l |\mathcal{U}''(s) + k(\mathcal{U}(s))((\mathcal{U}'(s))^2 - 1)|ds < \epsilon_2$ holds for $\mathcal{U}(s) := \mathcal{U}(s; y_1, y_2), l := d(y_1, y_2)$. Note that $\gamma_{y_1 y_2} \subset B_{4R_1}(p)$. Next we set

$$Q_{\epsilon_2} := \{y_1 \in B_{2R_1}(p) \mid \mathrm{vol}\, D_{\epsilon_2}(y_1) \geq (1 - \epsilon_2)\mathrm{vol}\, B_{2R_1}(p)\}.$$

Then for any $\epsilon_2 > 0$ there exists $\tau = \tau(\epsilon_2, m, \kappa, R_1) > 0$ such that if $0 < \epsilon_1 < \tau$ and

$$\frac{1}{\mathrm{vol}\, B_{4R_1}(p)} \int_{B_{4R_1}(p)} \|D^2 f - k \circ f(g - df \otimes df)\| d\nu_g < \epsilon_1$$

holds, we have

$$\mathrm{vol}\, Q_{\epsilon_2} \geq (1 - \epsilon_2)\mathrm{vol}\, B_{2R_1}(p). \tag{4.14}$$

Note that for $y_1 \in Q_{\epsilon_2}$, $y_2 \in D_{\epsilon_2}(y_1)$ we have

$$|\mathcal{U}(s) - \underline{\mathcal{U}}(s)|, \quad |\mathcal{U}'(s) - \underline{\mathcal{U}}'(s)| < \varphi(\epsilon_2 \mid \psi, R_1)$$

for $\mathcal{U}(s) := \mathcal{U}(s; y_1, y_2)$ by (4.11). Therefore, for any fixed $y \in Q_{\epsilon_2}$,

$$|\mathcal{U}'(d(y,z); y, z) - \underline{\mathcal{U}}'(d(y,z); f(y), f(z), d(y,z))| < \varphi(\epsilon_2 \mid \psi, R_1)$$

holds for almost all $z \in B_{2R_1}(p)$. Namely, we get

$$\begin{aligned} &\frac{1}{\mathrm{vol}\, B_{2R_1}(p)} \int_{z \in B_{2R_1}(p)} |\mathcal{U}'(l; y, z) - \underline{\mathcal{U}}'(l; f(y), f(z), l)| d\nu_g \\ &\qquad < \varphi(\epsilon_2 \mid \psi, R_1), \end{aligned} \tag{4.15}$$

where we set $l = d(y, z)$.

Proof of the next lemma is essentially the same as in Proposition 2.80 of [5], and Lemmas 2.5, 3.5 of [8]. See §§2.3 for the definition of Q.

Lemma 4.4. *For any $\epsilon > 0$ there exists $\zeta = \zeta(\epsilon, m, \kappa, \psi, R_1) > 0$ such that if*

$$\frac{1}{\mathrm{vol}\, B_{4R_1}(p)} \int_{B_{4R_1}(p)} \|D^2 f - (k \circ f)\{g - df \otimes df\}\| d\nu_g < \epsilon_1 \quad ((4.12))$$

holds for $0 < \epsilon_1 < \zeta$, then we have

$$|d_M(y_1, y_2) - Q(f(x_1), f(y_1), f(x_2), f(y_2), d_M(x_1, x_2))| < \epsilon \tag{4.16}$$

for any $x_i, y_i \in B_{R_1}(p)$ with $f(y_i) - f(x_i) = d(x_i, y_i)$ $(i = 1, 2)$.

We give a sketch of proof. Indeed, considering a minimal geodesic $\gamma_{x_1y_2}$, which is contained in $B_{2R_1}(p;M)$, it suffices to show our claim in the case of $x_1 = y_1 = x$. We may assume that $d(x_2, y_2) > \eta$ for a fixed sufficiently small $\eta > 0$. Taking $\epsilon_2 > 0$ sufficiently small we may choose $y \in B_{\eta^3}(x) \cap Q_{\eta^3}, q \in B_{\eta^3}(x_2), w \in B_{\eta^3}(y_2) \cap D_{\eta^3}(q)$ by virtue of (4.14) and the Bishop-Gromov volume comparison theorem. Set

$$\lambda = \gamma_{qw}, \quad l(s) = d(y, \lambda(s)), \quad d = d(q, w).$$

Now again applying Theorem 2.11 of [5] to (4.15) as in Lemma 4.3 we may assume in the above that

$$\int_0^d |\mathcal{U}'(l(s); y, \lambda(s)) - \underline{\mathcal{U}}'(l(s); f(y), f(\lambda(s)), l(s))| ds < \eta^2, \tag{4.17}$$

where

$$\underline{\mathcal{U}}'(l(s); f(y), f(\lambda(s)), l(s)) = \cos \underline{\theta}(l(s); f(y), f(\lambda(s)), l(s))$$

(see (2.11)). On the other hand, since $w \in D_{\eta^3}(q)$, noting that $d(x_2, y_2) > \eta$ and (4.11), we have

$$\begin{cases} |f(\lambda(s)) - (f(x_2) + s)| < \varphi(\epsilon_2, \eta \mid \psi, R_1) \\ |1 - \langle \dot{\lambda}(s), \nabla f(\lambda(s)) \rangle| < \varphi(\epsilon_2, \eta \mid \psi, R_1). \end{cases} \tag{4.18}$$

Now we set $\alpha(s) := \angle(\dot{\gamma}_{y\lambda(s)}(l(s)), \dot{\lambda}(s))$. Then we get $l'(s) = \cos \alpha(s)$ for almost all s by the first variation formula. Note that $|\cos \alpha(s) - \mathcal{U}'(l(s); y, \lambda(s))| < \varphi(\epsilon_2, \eta \mid \psi, R_1)$ holds by (4.18). Then it follows from (4.17), (4.18) that

$$\begin{aligned} \int_0^d l(s) | \cos \alpha(s) - \cos \underline{\theta}(l(s); f(y), f(x_2) + s, l(s)) | ds \\ < \varphi(\epsilon_2, \eta \mid \psi, R_1). \end{aligned} \tag{4.19}$$

Now consider $\underline{y}, \underline{q}, \underline{w}$ in X_0 satisfying the same conditions as above, and define $\underline{\lambda}, \underline{l}(s), \underline{\alpha}$ etc. with $\underline{l}(0) = l(0)$ in the same manner. Then we get

$$\begin{aligned} \int_0^d \underline{l}(s) | \cos \underline{\alpha}(s) - \cos \underline{\theta}(\underline{l}(s); f(y), f(x_2) + s, \underline{l}(s)) | ds \\ < \varphi(\epsilon_2, \eta \mid \psi, R_1). \end{aligned} \tag{4.20}$$

It follows that for $0 \le s \le d$

$$\begin{aligned}\frac{1}{2}|l^2(s) - \underline{l}^2(s)| &\le \int_0^s |l(s)l'(s) - \underline{l}(s)\underline{l}'(s)|ds \\ &= \int_0^s |l(s)\cos\alpha(s) - \underline{l}(s)\cos\underline{\alpha}(s)|ds \\ &\le \int_0^s |l(s)\cos\underline{\theta}(l(s); f(y), f(x_2)+s, l(s)) \\ &\quad - \underline{l}(s)\cos\underline{\theta}(\underline{l}(s); f(y), f(x_2)+s, \underline{l}(s))|ds + \varphi(\epsilon_2, \eta \mid \psi, R_1) \\ &\le \frac{C}{2}\int_0^s |l(s) - \underline{l}(s)|ds + \frac{1}{2}\varphi(\epsilon_2, \eta \mid \psi, R_1),\end{aligned}$$

where $C = C(\psi, R_1)$ is a positive constant. Now set $\varphi_1 := \frac{-2CR_1}{\log\varphi(\epsilon_2,\eta|\psi,R_1)}$ for the last $\varphi(\epsilon_2, \eta \mid \psi, R_1)$. We may assume that $l(d) + \underline{l}(d) > \varphi_1$. First suppose that $l(s) + \underline{l}(s) > \varphi_1$ for $0 \le s \le d$. Then we get

$$|l^2(s) - \underline{l}^2(s)| \le \frac{2C}{\varphi_1}\int_0^s |l^2(s) - \underline{l}^2(s)|ds + \varphi(\epsilon_2, \eta \mid \psi, R_1).$$

Now we note the following fact: If $|k(s)| \le C\int_0^s |k(s)|ds + \epsilon$ holds, then we have $|k(s)| \le \epsilon\exp(Cs)$.

Applying the above to our case it follows that

$$|l^2(d) - \underline{l}^2(d)| < \varphi^{\frac{1}{2}}(\epsilon_2, \eta \mid \psi, R_1).$$

In general, we set $s_0 := \sup\{s \in [0, d) \mid l(s) + \underline{l}(s) \le \varphi_1\}$ and apply the above argument to the interval $[s_0, d]$. Summing up we get $|l^2(d) - \underline{l}^2(d)|$ is small, namely

$$|l(d) - \underline{l}(d)| < \varphi(\epsilon_2, \eta \mid \psi, R_1),$$

where $\varphi(\epsilon_2, \eta \mid \psi, R_1)$ merely means that for fixed ψ and $R > 0$ we have $\varphi(\epsilon \mid \chi \mid \psi, R) \downarrow 0$ as $\chi \downarrow 0$ and taking $\epsilon = \epsilon(\chi) \downarrow 0$. Then recalling that $l(d) = d_M(y, w), \underline{l}(d) = d_{X_0}(\underline{y}, \underline{w})$ and $|d(y, w) - d(x, y_2)|, |d(\underline{y}, \underline{w}) - Q(f(x), f(x), f(x_2), f(y_2), d_M(x, x_2))| < 2\eta^2$, Lemma 4.4 follows. Note also that the assertion of Lemma 4.4 holds under the assumption that $|f(y_i) - f(x_i)| = d(x_i, y_i)$ $(i = 1, 2)$.

Now for small $(1 >)\chi > 0$ we define a distance d_χ on Z by

$$d_\chi(z, z') := \inf\{\sum_{i=0}^{N_1-1} \tilde{d}_Z(z_i, z_{i+1}) \mid z_i \in Z (i = 0, \ldots, N_1 - 1), \quad (4.21)$$

$$z_0 = z, z_{N_1} = z', \tilde{d}_Z(z_i, z_{i+1}) < \chi\},$$

where $\tilde{d}_Z$ is the induced distance on Z, i.e., $\tilde{d}_Z(z,z') := d_M(z,z')$. Next noting (2.13), we define a distance $d_{X,\chi}$ on X by

$$d_{X,\chi}((r,z),(r',z')) := \rho(r,r',d_\chi(z,z')), \tag{4.22}$$

which is in fact a warped product distance with warping function ψ. Here note that $Q(0,r,0,r',\tilde{d}_Z(z,z')) = \rho(r,r',w^{-1}(\tilde{d}_Z(z,z')))$ and therefore

$$\begin{aligned} \rho(r,r',\tilde{d}_Z(z,z')) &\leq Q(0,r,0,r',\tilde{d}_Z(z,z')) \\ &\leq \rho(r,r',\tilde{d}_Z(z,z')) + C(r,r',\psi)\tilde{d}_Z(z,z')^2 \end{aligned} \tag{4.23}$$

holds if $\tilde{d}_Z(z,z') < 1$ with some positive constant $C(r,r',\psi)$ (see §§2.3).

Now our aim is to show that

$$|d_M(x,x') - d_{X,\chi}(\Psi(x),\Psi(x'))| < \varphi(\epsilon \mid \chi \mid \psi, R) \tag{4.24}$$

for $x,x' \in B_R(p;M)$, where $\epsilon > 0$ in (4.16) may be arbitrary small, if we choose sufficiently small δ in (4.1), (4.2) for large concentric distance ball, e.g., of radius $\tilde{R} = 30R$. For that purpose we take a minimal geodesic $\gamma = \gamma_{xx'}$ of M parametrized by arclength and a subdivision $\{\gamma(s_i); s_i = \frac{il}{N}, i = 0,\dots,N\}$ of γ, where we set $l = d(x,x')$. Note that $\gamma(s_i) \in B_{2R}(p;M)$, $\pi(\gamma(s_i)) \in B_{4R}(p;M)$. Then applying Lemma 4.4 with $R_1 = 4R$, we obtain

$$\begin{aligned} |d_M(\pi(\gamma(s_i)),&\pi(\gamma(s_{i+1}))) \\ &- Q(f(\gamma(s_i)),0,f(\gamma(s_{i+1})),0,d_M(\gamma(s_i),\gamma(s_{i+1})))| < \epsilon. \end{aligned}$$

Taking a sufficiently large positive integer $N := N(\chi,\psi,R)$ it follows that

$$d_M(\pi(\gamma(s_i)),\pi(\gamma(s_{i+1}))) < \chi. \tag{4.25}$$

Then from Lemma 4.4 and (4.25), we obtain as in [8]

$$d_M(x,x') = \sum_{i=0}^{N-1} d_M(\gamma(s_i),\gamma(s_{i+1})) \geq d_{X,\chi}(\Psi(x),\Psi(x')) - \epsilon N,$$

where ϵN may be arbitrarily small if we take $\epsilon > 0$ sufficiently small.

On the other hand, for $x,x' \in B_R(p;M)$ we may assume that there exist points $(r_i,z_i) \in \mathbb{R} \times Z\,(i = 0,\dots,N_1)$ such that

$$d_{X,\chi}(\Psi(x),\Psi(x')) = \sum_{i=0}^{N_1-1} d_{X,\chi}(\Psi(x_i),\Psi(x_{i+1}))$$

with

$$\sum_{i=0}^{N_1-1} d_\chi(z_i, z_{i+1}) = d_\chi(\pi(x), \pi(x')), \quad d_M(z_i, z_{i+1}) < \chi,$$

where we set $x_i = \Phi(r_i, z_i)$, namely, $(r_i, z_i) = \Psi(x_i)$. Note that $|r_i| < 2R + \frac{\chi}{2}, z_i \in B_{4R}(p; M), x_i \in B_{7\hat{R}}(p; M)$. Then Lemma 4.4 with (4.23) implies that

$$d_M(x_i, x_{i+1}) - d_{X,\chi}((r_i, z_i), (r_{i+1}, z_{i+1})) < \epsilon + C\chi^2,$$

where $C = C(\psi, R)$ is a positive constant. Since

$$d_\chi(z_i, z_{i+1}) + d_\chi(z_{i+1}, z_{i+2}) \geq \chi$$

we see that $N_1 \leq \frac{C_1(R,\chi)}{\chi}$. It follows that

$$\begin{aligned} d_M(x, x') &\leq \sum_{i=0}^{N_1-1} d_M(x_i, x_{i+1}) \leq d_{X,\chi}(\Psi(x), \Psi(x')) + N_1\epsilon + \chi C C_1 \\ &\leq d_{X,\chi}(\Psi(x), \Psi(x')) + \varphi(\epsilon \mid \chi \mid \psi, R), \end{aligned}$$

where for fixed $R > 0$, $\varphi(\epsilon \mid \chi \mid \psi, R)$ becomes arbitrary small if we take first $\chi > 0$ small and then choose $\epsilon = \epsilon(\chi)$ further small. We may proceed with the remaining argument as in [8], proof of Theorem 1.1, Theorem 1.2.

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