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THE TWISTOR SPACE OF DISTRIBUTIONS

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1. Introduction. In this paper a naturally defined kind of distributions on the twistor spase of distributions over some manifold N is established. It is defined by an ordered pair (D,ω) of a distribution D on the base manifold N and a linear connection ω on it. Concerning integrability of this distribution (denoted by D^{ω}) it is shown that cases $\dim D^{\omega}=2$ and $\dim D^{\omega} = \dim N$ are distinguished. Nevertheless D^{ω} is integrable very rare in any case (see theorems 2.2 and 2.4). Moreover consequences of induced action of a gauge transformation on this distribution are also considered. One works in the C^{∞} category.

2. Constructions. Let $\pi_L\colon L(N)\to N$ be the principal $\mathrm{GL}(n;\mathbb{R})$ bundle of linear frames over an $n=p+q$-dimensional manifold N. The twistor space of p-dimensional distributions over N is constructed as follows: for the subgroup

$$(2.1)\qquad G=\left\{\begin{pmatrix} A & B\\ O & C\end{pmatrix}\in \mathrm{GL}(n;\mathbb{R})\,\middle|\, A\in \mathrm{GL}(p;\mathbb{R}),\ C\in \mathrm{GL}(q;\mathbb{R})\right\}$$

any reduced G-subbundle $Q\subset L(N)$ corresponds to some p-dimensional distribution $D\subset TN$ on the manifold N, [3]. There exists a 1:1 correspondence between such subbundles and the set of sections of a fiber bundle associated with $L(N)$ whose standard fiber is the space $\mathrm{GL}(n;\mathbb{R})/G$. This fiber bundle can be naturally identified with the quotient $L(N)/G$; and this quotient represents the twistor space Z of p-dimensional distributions over manifold N. This generalized framework is considered in [1] and [4] in another setting.

Fibrations are exposed by

$$(2.2)\qquad L(N)\xrightarrow{\pi_Z} Z=L(N)/G\xrightarrow{\pi} N.$$

Thus any p-dimensional distribution $D\subset TN$ on N uniquely determines a global section (also denoted by) $D\colon N\to Z$ which further defines a reduced G-subbundle $Q:=\pi_Z^{-1}(D(N))$. By definition some linear connection ω

on $L(N)$ is reducible to Q iff its horizontal distribution $H^\omega \subset TL(N)$ is tangent to Q. Equivalently, ω is reducible to Q iff

$$(\pi_Z)_*(H^\omega|_Q) = D_*(TN) \tag{2.3}$$

holds. In the same manner a horizontal distribution on the space Z is defined by

$$(\pi_Z)_*(H^\omega) \subset TZ. \tag{2.4}$$

Here ω need not be reducible to Q. Consequently if $D^H \subset H^\omega$ is the horizontal lift of a distribution $D \subset TN$ defined by the connection ω on $L(N)$, then a distribution $D^\omega \subset TZ$ is defined on the space Z as

$$D^\omega := (\pi_Z)_*(D^H). \tag{2.5}$$

Before considering the question of integrability of this distribution first will be proved the following very useful

Lemma 2.1. *Let B_i be the standard horizontal vector fields on bundle $L(N)$ defined by a connection ω and basis vectors $u_i \in \mathbb{R}^n$ $(i = 1, \dots, n)$, [2]. Denote by Ω and T the curvature and the torsion of ω respectively (viewed on $L(N)$). Then the following commutator formula holds.*

$$[B_i, B_j] = -T^k(B_i, B_j)B_k - \Omega_l^k(B_i, B_j)E_k^l. \tag{2.6}$$

Here (E_k^l) is the standard basis in $\mathrm{gl}(n;\mathbb{R})$, and one identifies (i.e., uses the same letter) elements from $\mathrm{gl}(n;\mathbb{R})$ and corresponding fundamental fields on $L(N)$ (a convention is $v_1 \wedge v_2 := v_1 \otimes v_2 - v_2 \otimes v_1$).

Proof. Let M be a manifold and (e_α) and (ε^α) some frame with corresponding coframe on M respectively. Then holds the following relations ([5, p. 26])

$$\begin{aligned} [e_\alpha, e_\beta] &= C^\gamma_{\alpha\beta} e_\gamma, \\ d\varepsilon^\gamma &= -\frac{1}{2} C^\gamma_{\alpha\beta} \varepsilon^\alpha \wedge \varepsilon^\beta. \end{aligned} \tag{2.7}$$

Here C-symbols are defined by the choice of frame (e_α). Let

$$\begin{aligned} d\theta^i &= -\omega^i_j \wedge \theta^j + \frac{1}{2} T^i_{jk} \theta^j \wedge \theta^k \\ d\omega^i_j &= -\omega^i_k \wedge \omega^k_j + \frac{1}{2} \Omega^i_{jkl} \theta^k \wedge \theta^l \end{aligned} \tag{2.8}$$

be the structure equations of the connection ω. Here $\Omega^i_{jkl} := \Omega^i_j(B_k, B_l)$, $T^i_{jk} := T^i(B_j, B_k)$, and (θ^i) are components of the canonical 1-form on $L(N)$. Because $(\theta^i; \omega^i_j)$ is a (global) coframe on $L(N)$ which corresponds to the frame $(B_i; E^i_j)$, by comparing (2.7) and (2.8) one obtains (2.6).

Let X^ω, Y^ω and $[X,Y]^\omega$ be horizontal lifts to $L(N)$ of vector fields X, Y and $[X,Y]$ on N respectiovely which are defined by the connection ω on $L(N)$. Then by using (2.6) one finds the following relation

$$[X^\omega, Y^\omega] = [X,Y]^\omega - T^k(X^\omega, Y^\omega)B_k - \Omega(X^\omega, Y^\omega). \tag{2.9}$$

By this formula one is able to prove the following

Theorem 2.2. *Let $D \subset TN$ be a p-dimensional foliation on a manifold N endowed with a torsionfree connection ω by which the distribution $D^\omega \subset TZ$ introduced in* (2.5) *is defined. Suppose that vector fields Y_1 and Y_2 take value in D while Y_3 is any vector field on N. Then the distribution D^ω will be integrable if*

$$R(Y_1, Y_2)Y_3 = fY_3 \tag{2.10}$$

holds. Here R is the curvature tensor of ω while f is a function on N which does not depend on Y_3.

Proof. From (2.9) it follows that D^ω will be integrable iff $\Omega(X^\omega, Y^\omega)$ takes value in $\Im$, the Lie algebra of G, when X and Y take value in D. But fields X^ω and Y^ω are right $\mathrm{GL}(n;\mathbb{R})$-invariant, consequently

$$\Omega_{pg}(X^\omega, Y^\omega) = \mathrm{Ad}(g^{-1})\Omega_p(X^\omega, Y^\omega) \tag{2.11}$$

holds for all $p \in L(N)$, $g \in \mathrm{GL}(n;\mathbb{R})$. Thus in order that D^ω is integrable $\Omega(X^\omega, Y^\omega)$ must take value in an ideal of $\mathfrak{gl}(n;\mathbb{R})$ which is contained in $\Im$, e.g.,

$$\Omega(X^\omega, Y^\omega) = \hat{f} I \tag{2.12}$$

can hold. where $\hat{f} \in C^\infty\big(L(N)\big)$. This condition is expressed on the level of N in a form given in (2.10).

Because it is supposed that ω is a torsionfree connection, the first Bianchi identity

$$R(Y_1, Y_2)Y_3 + R(Y_2, Y_3)Y_1 + R(Y_3, Y_1)Y_2 = 0 \tag{2.13}$$

holds. Hence if in (2.10) also $Y_3 \in D$ and moreover Y_1, Y_2, Y_3 are linearly independent, then from (2.13) it follows that $f \equiv 0$. Thus in the setting of Theorem 2.2 we have

Corollary 2.3. *If* $\dim D > 2$*, then* D^ω *is integrable if* ω *is flat.*

On the other hand, from (2.13) and (2.10) it does not follows that there are no nontrivial examples (i.e., ω is not flat) of integrable D^ω in case of $\dim D = 2$. Important special case is $\dim D = \dim N$, i.e., $D^\omega = (\pi_Z)_* H^\omega$ from (2.4): here there is no need to impose condition that a connection ω is torsionfree because the term $T^k(X^\omega, Y^\omega)B_k$ in (2.9) is not troublesome at all now. Consequently the first Bianchi identity does not make restrictions so that the following holds.

Theorem 2.4. *The horizontal distribution* D^ω *on the twistor space* Z *which is introduced in* (2.4) *is integrable if*

$$R(Y_1, Y_2)Y_3 = fY_3 \tag{2.14}$$

holds for any vector fields Y_i *on* N*. Here the function* f *on* N *does not depend on* Y_3*.*

One can notice that if ω is a torsionfree connection on $L(N)$ which is reducible to a G-subbundle $Q \subset L(N)$ defined by a foliation $D \subset TN$, then the distribution $D^\omega|_{D(N)}$ on the submanifold $D(N)$ is integrable.

3. On gauge-action upon distributions. Let $f\colon L(N) \to L(N)$ be any gauge-transformation in the bundle $L(N)$. Because its fiber-preserving action commutes with the right action of the group $\mathrm{GL}(n;\mathbb{R})$ on $L(N)$, it follows that f induces in a natural manner a transformation (also denoted by f) on the twistor space Z. Consequently f maps any distribution $D\colon N \to Z$ into another fD. Then $fD = D$ holds iff f preserves the G-structure $Q \subset L(N)$ which corresponds to D. On the other hand f acts naturally upon $TL(N)$ and further upon TZ. In this setting we have

Theorem 3.1. *If the subbundle* $Q \subset L(N)$ *which corresponds to a distribution* $D \subset TN$ *is left invariant by a gauge-transformation* f*, and some connection* ω *on* $L(N)$ *is reducible to* Q*, then the distribution* $D^\omega|_{D(N)} \subset TD(N)$ *is preserved under the action of* f *upon* TZ*.*

Proof. A sufficient condition that f preserves the distribution $D^H \subset TL(N)$ is that it preserves the connection ω, i.e., its horizontal distribution (if $D^H \equiv H^\omega$, i.e., $D \equiv TN$, this condition is also necessary). But if $fD = D$ and ω is reducible to Q then $f^*\omega$ is also reducible to Q. Consequently by (2.3)

$$(\pi_Z)_*(H^\omega|_Q) = (\pi_Z)_*(H^{f^*\omega}|_Q)$$

where from follows the statement of the theorem.

From this Theorem it follows that if $fQ = Q$ and ω is reducible to Q, then

$$D^{f^*\omega}|_{D(N)} = D^{\omega}|_{D(N)}.$$

From this does not follows that $D^{f^*\omega} = D^{\omega}$ on the whole space Z. A sufficient condition for it is, say, that $f^*\omega - \omega$ take value in the center $\{cI \mid c \in \mathbb{R}\}$ of $\mathfrak{gl}(n;\mathbb{R})$.

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