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# THE COMPARISON THEOREM OF HILBERT-SPACE-VALUED TANGENT SEQUENCES

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Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability space, $X$ be a Hilbelt space. When $X$ has a Schauder basis $(e_i)_{i\geq 1}$, we consider $\varphi\colon X \to \mathbb{R}^\infty = \{(a_i)_{i\geq 1} \mid a_i \in \mathbb{R}\}$, $\varphi(\sum_{i\geq 1} a_i e_i) = (a_i)_{i\geq 1}$; Let $f$ be an $X$-valued random variable, then $\varphi(f)$ is a serie of random functions, there exists RCPD (regular conditional probability distribution) $P_{\varphi(f)}$ of $\varphi(f)$ w.r.t. $\mathcal{B}$, where $\mathcal{B}$ is a subalgebra of $\mathcal{F}$. Let $\mathcal{B}^\infty$ be the Borel algebra of $\mathbb{R}^\infty$, $\mathcal{B}_X$ be the Borel algebra of $X$, $\varphi(\mathcal{B}_X) = \{\varphi(B) \mid B \in \mathcal{B}_X\}$. Let $\chi_A$ be the characteristic function of $A \in \mathcal{F}$.

In this article, integrability means Bochner integrability.

**Lemma 1.** *Let $X$ be a Banach space, $f$ be an $X$-valued random variable with almost separable values, $\mathcal{B}$ be a subalgebra of $\mathcal{F}$, then there exists regular conditional probability distribution of $f$ w.r.t. $\mathcal{B}$ denoted by $P_f$.*

*Proof.* see [1].

**Theorem 2.** *Let $X$ be a Hilbert space, $f$ be an $X$-valued integrable random variable, then $E(f|\mathcal{B})(t) = \int_X x P_f(t, dx)$. a.e.*

*Proof.* Since an $X$-valued integrable random variable is strong measurable, it is almost separably-valued by the Pettis theorem. We need only consider the case where $X$ is a separable Hilbert space. Let $(e_n)_{n\geq 1}$ be an orthonormal basis of $X$, $p_n$ respectively $q_n$ be the projections of $X$ respectively $\mathbb{R}^\infty$ to the $n$'th coordinate, then for all $x \in X$, $p_n(x) = q_n(\varphi(x))$;

$$
\begin{aligned}
E(f|\mathcal{B})(t) &= E\Big(\sum_{n=1}^{\infty}(p_n f)e_n\Big|\mathcal{B}\Big)(t) = \sum_{n=1}^{\infty} E(p_n f|\mathcal{B})(t)e_n \\
&\quad \Big(\text{Since } \Big\|\sum_{n=1}^{k} p_n f(t)e_n\Big\| \le \|f(t)\|\Big) \\
&= \sum_{n=1}^{\infty} E\Big(q_n(\varphi(f))\Big|\mathcal{B}\Big)(t)e_n = \sum_{n=1}^{\infty}\int_{\mathbb{R}^\infty} q_n(y)P_{\varphi\circ f}(t,dy)e_n \\
&= \sum_{n=1}^{\infty}\int_X q_n(\varphi(x))P_f(t,dx)e_n = \sum_{n=1}^{\infty}\int_X p_n(x)e_n P_f(t,dx) \\
&= \int_X \sum_{n=1}^{\infty} p_n(x)e_n P_f(t,dx) = \int_X xP_f(t,dx).
\end{aligned}
$$

**Theorem 3.** *Let $X$, $Y$ be Hilbert spaces, $f$ be an $X$-valued random variable with almost separable values, $h\colon X \to Y$ be Borel measurable, $h\circ f$ be integrable, Then $E(h\circ f|\mathcal{B})(t) = \int_X h(x)P_f(t,dx)$. a.e.*

*Proof.* Because $h\colon X \to Y$ is measurable, we can define

$$P_{h\circ f}(t,B) = P_f\big(t, h^{-1}(B)\big), \quad \forall t\in\Omega,\ B\in\mathcal{B}_Y$$

then $\forall t\in\Omega$, $P_{h\circ f}(t,*)$ is a probability measure on $\mathcal{B}_Y$. $\forall B\in\mathcal{B}_Y$,

$$
\begin{aligned}
P_{h\circ f}(t,B) = P_f\big(t,h^{-1}(B)\big) &= E\Big(f^{-1}\big(h^{-1}(B)\big)\Big|\mathcal{B}\Big)(t) \\
&= E\big((h\circ f)^{-1}(B)\big|\mathcal{B}\big)(t) \text{ a.e.}
\end{aligned}
$$

So $P_{h\circ f}$ is a regular distribution of $h\circ f$ w.r.t. $\mathcal{B}$. Choosing regular distribution pair such as these and using Theorem 2, we have

$$
\begin{aligned}
E(h\circ f|\mathcal{B})(t) &= \int_Y yP_{h\circ f}(t,dy) \\
&= \int_X h(x)P_f(t,dx).
\end{aligned}
$$

**Definition 4.** Let $(\mathcal{F}_n)_{n\ge0}$ be an increasing sub-$\sigma$-algebra sequence of $\mathcal{F}$, $(d_n)_{n\ge1}$, $(e_n)_{n\ge1}$ are $X$-valued random variables w.r.t $(F_n)_{n\ge1}$. We call $(d_n)_{n\ge1}$ and $(e_n)_{n\ge1}$ tangent, if $\forall A\in\mathcal{B}_X$, $\forall n\ge1$, $P\big(d_n^{-1}(A)\big|\mathcal{F}_{n-1}\big) =$

$P\big(e_n^{-1}(A)\big|\mathcal{F}_{n-1}\big)$ a.e. We call $(d_n)_{n\geq 1}$ conditionally symmetric, if $(-d_n)_{n\geq 1}$ and $(d_n)_{n\geq 1}$ are tangent.

Let $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function satisfying the condition $\Delta_2$, it means: $\exists C > 0$ such that $\forall x \geq 0$, $\Phi(2x) \leq C\Phi(x)$, and $\Phi(0) = 0$. Easily, we have: $\forall x,\ y \geq 0$, $\Phi(x+y) \leq C\Phi(x) + C\Phi(y)$. Let $(d_n)_{n\geq 1}$ be a random variable sequence, we define

$$d_0^* = 0, \quad d_n^* = \sup_{1\leq k\leq n} \|d_k\|, \quad d^* = \sup_{n\geq 1} \|d_n\|.$$

**Lemma 5.** *Let $X$ be a Hilblert space with orthonormal basis $(e_i)_{i\geq 1}$, $\varphi(\sum_{i\geq 1} a_i e_i) = (a_i)_{i\geq 1}$, then*

$$\varphi(\mathcal{B}_X) = \mathcal{B}^\infty \bigcap \varphi(X) = \{A \bigcap \varphi(X) \mid A \in \mathcal{B}^\infty\}.$$

*Proof.* Let $p_n$ be the projection of $\mathbb{R}^\infty$ to the first $n$ coordinates, $\mathcal{B}^n$ be the Borel algebra of $\mathbb{R}^n$. Then $\mathcal{B}^\infty = \sigma(T)$, $T = \bigcup_{n\geq 1} p_n^{-1}(\mathcal{B}^n)$, where $\sigma(T)$ is the $\sigma$ algebra generated by $T$. So $\sigma\big(T \bigcap \varphi(X)\big) = \mathcal{B}^\infty \bigcap \varphi(X) \subseteq \varphi(\mathcal{B}_X)$, $\mathcal{B}^\infty \bigcap \varphi(X) = \varphi(\mathcal{B}_X)$.

**Lemma 6.** *Let Hilbert spaces $X$, $Y$ have orthonormal bases $(e_{2k-1})_{k\geq 1}$, $(e_{2k})_{k\geq 1}$ respectively, we take product topology on $X \times Y$, then $\mathcal{B}_{X\times Y} = \mathcal{B}_X \times \mathcal{B}_Y$.*

*Proof.* We define $\varphi\colon Z = X \times Y \to \mathbb{R}^\infty$, $\varphi(\sum_{i\geq 1} a_i e_i) = (a_i)_{i\geq 1}$,

$$\mathbb{R}_1^\infty = \{(a_n)_{n\geq 1} \mid a_{2k} = 0, \forall k \geq 1\}$$
$$\mathbb{R}_2^\infty = \{(a_n)_{n\geq 1} \mid a_{2k-1} = 0, \forall k \geq 1\}$$

$\mathcal{B}_i^\infty$ is the Borel algebra of $\mathbb{R}_i^\infty (i = 1, 2)$. Then

$$\begin{aligned}\varphi(\mathcal{B}_Z) &= \mathcal{B}^\infty \bigcap \varphi(Z) = (\mathcal{B}_1^\infty \times \mathcal{B}_2^\infty) \bigcap (\varphi(X) \times \varphi(Y)) \\ &= (\mathcal{B}_1^\infty \bigcap \varphi(X)) \times (\mathcal{B}_2^\infty \bigcap \varphi(Y)) \\ &= \varphi(\mathcal{B}_X) \times \varphi(\mathcal{B}_Y) = \varphi(\mathcal{B}_X \times \mathcal{B}_Y).\end{aligned}$$

So $\mathcal{B}_Z = \mathcal{B}_X \times \mathcal{B}_Y$.

**Lemma 7.** *Let $X$ be a Hilbert space with basis, $(d_n)_{n\geq 1}$ be an $X$-valued conditionally symmetric sequence. We define*

$$\lambda_n = (d_1, \dots, d_n) \colon \Omega \to X^n = Y,$$
$$\xi_n = (d_1, \dots, d_{n-1}, -d_n) \colon \Omega \to Y,$$

*then both $\lambda_n$, $\xi_n$ are measurable w.r.t. $\mathcal{B}_Y$, we can take their RCPDs $P_{\lambda_n}$, $P_{\xi_n}$ w.r.t. $\mathcal{F}_{n-1}$ and $E \in \mathcal{F}_{n-1}$ such that $\mu(E) = 0$, and $\forall t \in \Omega \setminus E$, $\forall A \in \mathcal{B}_Y$, $P_{\lambda_n}(t, A) = P_{\xi_n}(t, A)$.*

*Proof.* By Lemma 6, $\mathcal{B}_Y = \mathcal{B}_X^n$, so $\forall A_i \in \mathcal{F}$, $\lambda_n$, $\xi_n$ are measurable w.r.t. $\mathcal{B}_Y$. Since $X$ is separable, $X$ is secondly denumbrable, we can take a countable set $\mathcal{A} = \{A_i \mid i \in N\}$ consisted of open sets of $X$ such that $\mathcal{A}$ generates the topology $\mathcal{T}_X$ of $X$. The $\sigma$ algebra generated by $\mathcal{T}_X$ is $\sigma(\mathcal{T}_X) = \mathcal{B}_X$, so $\sigma(\mathcal{A}) = \mathcal{B}_X$, $\sigma(\mathcal{A}^n) = \mathcal{B}_X^n = \mathcal{B}_Y$.

$$\begin{aligned}
\forall B_k \in \mathcal{B}_X, \quad P_{\lambda_n}(t, \prod_{k=1}^n B_k) &= P\big(\lambda_n^{-1}(\prod_{k=1}^n B_k)\big|\mathcal{F}_{n-1}\big)(t) \\
&= P\big(\bigcap_{k=1}^n d_k^{-1}(B_k)\big|\mathcal{F}_{n-1}\big)(t) \\
&= \prod_{k=1}^{n-1} \chi_{D_k} E(\chi_{D_k}|\mathcal{F}_{n-1})(t) \\
&\quad \text{(where } D_k = d_k^{-1}(B_k)) \\
&= \prod_{k=1}^{n-1} \chi_{D_k} E(\chi_{E_n}|\mathcal{F}_{n-1})(t) \\
&\quad \text{(where } E_n = (-d_n)^{-1}(B_n)) \\
&= E(\prod_{k=1}^{n-1} \chi_{D_k} \circ \chi_{E_n}|\mathcal{F}_{n-1})(t) \\
&= P\big(\xi_n^{-1}(\prod_{k=1}^n B_k)\big|\mathcal{F}_{n-1}\big)(t) \\
&= P_{\xi_n}(t, \prod_{k=1}^n B_k). \text{ a.e.}
\end{aligned}$$

For $k_i \in N$, we take $E(k_1, \dots, k_n) \in \mathcal{F}_{n-1}$, such that $\mu\big(E(k_1, \dots, k_n)\big) = 0$, $\forall t \in \Omega \setminus E(k_1, \dots, k_n)$, $P_{\lambda_n}(t, \prod_{i=1}^n A_{k_i}) = P_{\xi_n}(t, \prod_{i=1}^n A_{k_i})$. (1)

Let $E = \bigcup\{E(k_1, \dots, k_n) \mid k_i \in N, 1 \le i \le n\}$, then $\mu(E) = 0$, $\forall t \in \Omega \setminus E$, $\forall k_i \in N$, (1) holds. Since $P_{\lambda_n}(t, *)$ and $P_{\xi_n}(t, *)$ are probability measures on $\mathcal{B}_Y$, by (1), they are equal on $\mathcal{A}$ that generates $\mathcal{B}_Y$, so $\forall t \in \Omega \setminus E$, $\forall A \in \mathcal{B}_Y$, $P_{\lambda_n}(t, A) = P_{\xi_n}(t, A)$. For $t \in E$, we take

$$P_{\lambda_n}(t, A) = \mu(\lambda_n^{-1}(A)), \quad P_{\xi_n}(t, A) = \mu(\xi_n^{-1}(A)).$$

**Lemma 8.** *Let $X$ be a Hilbert space, $(d_n)_{n\ge1}$ be a conditionally symmetric sequence, $d_n \in L_1(\mu, X)$. We denote $f_n = \sum_{k=1}^n d_k$, then $f = (f_n)_{n\ge1}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\ge1}$, and we have decomposition $f_n = g_n + h_n$, such that $g = (g_n)_{n\ge1}$ and $h = (h_n)_{n\ge1}$ are martingales w.r.t. $(\mathcal{F}_n)_{n\ge1}$, where*

$$g_n = \sum_{k=1}^n a_k = \sum_{k=1}^n d_k \chi_{A_k}, \quad h_n = \sum_{k=1}^n b_k = \sum_{k=1}^n d_k \chi_{B_k}$$
$$A_k = \{\|d_k\| \le 2d_{k-1}^*\}, \quad B_k = \{\|d_k\| > 2d_{k-1}^*\}$$

*Proof.* Similar to the proof of Theorem 2, we need only consider the case where $X$ is a separable Hilbert space. We define $\mathcal{B}\colon X^n \to X$ by

$$\beta(x_1, \dots, x_n) = x_n \chi_{\{\|x_n\| \le 2y_{n-1}\}}$$
$$y_{n-1} = \max\{\|x_1\|, \dots, \|x_{n-1}\|\}, \quad y_0 = 0$$

then

$$\begin{aligned} E(a_n|\mathcal{F}_{n-1})(t) &= E(\beta(d_1, \dots, d_n)|\mathcal{F}_{n-1})(t) = E(\beta \circ \lambda_n|\mathcal{F}_{n-1})(t) \\ &= \int_{X^n} \beta(x) P_{\lambda_n}(t, dx) = \int_{X^n} \beta(x) P_{\xi_n}(t, dx) \\ &= E(\beta \circ \xi_n|\mathcal{F}_{n-1})(t) = E(-a_n|\mathcal{F}_{n-1})(t) \quad \text{a.e.} \end{aligned}$$

So $E(a_n|\mathcal{F}_{n-1}) = 0$ a.e., $g$ is a martingale. Similarly, $h$ is a martingale. We denote the RCPD of $(d_n)_{n\ge1}$ and $(-d_n)_{n\ge1}$ w.r.t. $\mathcal{F}_{n-1}$ by $P_+$, $P_-$ respectively, then $P_+ = P_-$ a.e., similarly to Lemma 7, using separability of $X$. By this result

$$\begin{aligned} E(d_n|\mathcal{F}_{n-1})(t) &= \int_X x P_+(t, dx) = \int_X x P_-(t, dx) \\ &= E(-d_n|\mathcal{F}_{n-1})(t) \Rightarrow E(d_n|\mathcal{F}_{n-1}) = 0 \quad \text{a.e.}, \end{aligned}$$

so $f$ is a martingale.

**Lemma 9.** *Let $X$ be a Hilbert space, then there exists a constant $C > 0$ dependent only on $\Phi$, such that for all $L_1(\mu, X)$ bounded martingale $f = (f_n)_{n\geq 1}$ satisfying $\|d_n\| \leq w_{n-1}$, where $d_n = f_n - f_{n-1}$, $w_n$ is $\mathcal{F}_n$ measurable, we have*

$$E\Phi(f^*) \leq CE\Phi(S(f)) + CE\Phi(w^*) \tag{1}$$

$$E\Phi(S(f)) \leq CE\Phi(f^*) + CE\Phi(w^*) \tag{2}$$

*Proof.*

$$\begin{aligned} S_n(f) &= \Big(\sum_{k=1}^{n-1} \|d_k\|^2 + \|d_n\|^2\Big)^{1/2} \leq \big(S_{n-1}(f)^2 + w_{n-1}^2\big)^{1/2} \\ &\leq S_{n-1}(f) + w_{n-1} = \varrho_{n-1} \end{aligned}$$

For $\beta > 0$, $\lambda > 0$, we define a stopping time

$$S = \inf\{n \mid \varrho_n > \beta\lambda\}.$$

We consider martingale $f^{(S)} = (f_{n\wedge S})_{n\geq 0}$ and define a stopping time

$$T = \inf\{n \mid \|f_n^{(S)}\| > \lambda\}.$$

Then $\forall\alpha > 1$, denoting $|A|$ the measure of $A$, we have

$$\begin{aligned} \big|\{f^* > \alpha\lambda\}\big| &\leq \big|\{f^* > \alpha\lambda,\ S = \infty\}\big| + \big|\{S < \infty\}\big| \\ &\leq \big|\{f^{(S)^*} > \alpha\}\big| + \big|\{S < \infty\}\big| \\ &\leq \big|\{f^{(S)^*} - f_{T-1}^{(S)^{\ *}} > (\alpha - 1)\lambda\}\big| + \big|\{S < \infty\}\big| \end{aligned} \tag{3}$$

Now we consider a new $\sigma$ algebra sequence $(\mathcal{F}'_n)_{n\geq 0}$, where $\mathcal{F}'_n = \mathcal{F}_{n+T}$. We define $g' = (g'_n)_{n\geq 0}$, $g'_n = f_{n+T}^{(S)} - f_{T-1}^{(S)}$, then $g'$ is a martingale, because

$$\begin{aligned} E(g'_{n+1}|\mathcal{F}'_n) &= E(f_{n+T+1}^{(S)} - f_{T-1}^{(S)}|\mathcal{F}_{n+T}) \\ &= E(f_{(n+T+1)\wedge S} - f_{(T-1)\wedge S}|\mathcal{F}_{n+T}) \\ &= E(f_{(n+T+1)\wedge S}|\mathcal{F}_{n+T}) - f_{(T-1)\wedge S} \end{aligned}$$

and

$$\begin{aligned} &E(f_{(n+T+1)\wedge S}|\mathcal{F}_{n+T}) \\ &\quad = E(f_S\chi_{\{S\leq n+T\}}|\mathcal{F}_{n+T}) + E(f_{n+T+1}\chi_{\{S\geq n+T+1\}}|\mathcal{F}_{n+T}) \\ &\quad = f_S\chi_{\{S\leq n+T\}} + \chi_{\{S\geq n+T+1\}}E(f_{n+T+1}|\mathcal{F}_{n+T}) \\ &\quad = f_S\chi_{\{S\leq n+T\}} + f_{n+T}\chi_{\{S\geq n+T+1\}} = f_{(n+T)\wedge S} \end{aligned}$$

So $E(g'_{n+1}|\mathcal{F}'_n) = g'_n$.
Because ${f_{T-1}^{(S)}}^* = \sup_{n\geq 0} \|f^{(S)}_{n\wedge(T-1)}\|$, if $\exists m \leq T-1$, such that $f^{(S)^*} = \|f_m^{(S)}\|$, then $f^{(S)^*} = \|f_m^{(S)}\|$, $f^{(S)^*} - {f_{T-1}^{(S)}}^* = 0$. If $m > T-1$, $f^{(S)^*} > f_m^{(S)}$, we have

$$f^{(S)^*} - {f_{T-1}^{(S)}}^* \leq \sup_{m\geq T} \|f_m^{(S)}\| - \|f_{T-1}^{(S)}\| \leq \sup_{m\geq T} \|f_m^{(S)}\| - \|f_{n+T}^{(S)}\|.$$

$$\begin{aligned} S(g') &= \Big(\sum_{n\geq 0} \|f^{(S)}_{n+T+1} - f^{(S)}_{n+T}\|^2\Big)^{1/2} \\ &= \Big(\sum_{n\geq 0} \|f^{(S)}_{n+T+1} - f^{(S)}_{n+T}\|^2\Big)^{1/2} \chi_{\{T<\infty\}} \\ &\leq S(f^{(S)})\chi_{\{T<\infty\}} \leq \varrho_{T-1}\chi_{\{T<\infty\}} \leq \beta\lambda\chi_{\{S<\infty\}}. \end{aligned}$$

$$\begin{aligned} \big|\{f^{(S)^*} > \alpha\lambda\}\big| &\leq \Big|\{(g')^* > (\alpha-1)\lambda\}\Big| \\ &\leq E(g')^*/(\alpha-1)\lambda \leq CES(g')/(\alpha-1)\lambda \\ &\quad \text{(using [5, p.414, Theorem 7])} \\ &\leq C\beta\big|\{T<\infty\}\big|/(\alpha-1) \leq C\beta\big|\{f^{(S)^*} > \lambda\}\big|/(\alpha-1) \\ &\leq C\beta\big|\{f^* > \lambda\}\big|/(\alpha-1). \end{aligned}$$

So $\big|\{f^* > \alpha\lambda\}\big| \leq \Big(C\beta\big|\{f^* > \lambda\}\big|/(\alpha-1)\Big) + \big|\{S(f) + w^* > \beta\lambda\}\big|$ it means that $\big(f^*, S(f) + w^*\big)$ satisfies "good $\lambda$ inequality", so we have (1).
The proof of (2) is similar. With $\|f_n\| \leq f^*_{n-1} + w_{n-1} = \varrho_{n-1}$, we define a stopping time

$$S = \inf\{n \mid \varrho_n > \beta\lambda\}, \quad \forall\beta > 0,\ \lambda > 0$$

We consider martingale $f^{(S)} = (f_{n\wedge S})_{n\geq 0}$, and define a stopping time

$$T = \inf\big\{n \mid S_n(f^{(S)}) > \lambda\big\}$$

then for $\alpha > 1$, we have

$$\begin{aligned} \Big|\big\{S(f) > \alpha\lambda\big\}\Big| &\leq \Big|\big\{S(f^{(S)}) > \alpha\lambda\big\}\Big| + \big|\{S < \infty\}\big| \\ &\leq \Big|\big\{S(f^{(S)}) - S_{T-1}(f^{(S)}) > (\alpha-1)\lambda\big\}\Big| + \big|\{S < \infty\}\big| \end{aligned}$$

Because

$$\begin{aligned} S(f^{(S)}) - S_{T-1}(f^{(S)}) &\le \Big(\sum_{n\ge T} \|f_n^{(S)} - f_{n-1}^{(S)}\|^2\Big)^{1/2} \\ &= \big(S(f^{(S)})^2 - S_{T-1}(f^{(S)})^2\big)^{1/2}, \\ \sup_{m\ge T} \|f_m^{(S)} - f_{T-1}^{(S)}\| &\le 2f^{(S)*}\chi_{\{T<\infty\}} \\ &\le 2\beta\lambda\chi_{\{T<\infty\}}. \end{aligned}$$

So

$$\begin{aligned} \Big|\{S(f^{(S)}) > \alpha\lambda\}\Big| &\le \Big|\{S(f^{(S)}) - S_{T-1}(f^{(S)}) > (\alpha-1)\lambda\}\Big| \\ &\le \Big|\Big\{\big(S(f^{(S)})^2 - S_{T-1}(f^{(S)})^2\big)^{1/2} > (\alpha-1)\lambda\Big\}\Big| \\ &\le E\big(S(f^{(S)})^2 - S_{T-1}(f^{(S)})^2\big)^{1/2}/(\alpha-1)\lambda \\ &= E\Big(\sum_{n\ge T} \|f_n^{(S)} - f_{n-1}^{(S)}\|^2\Big)^{1/2}/(\alpha-1)\lambda \\ &\le CE\big(\sup_{m\ge T} \|f_m^{(S)} - f_{T-1}^{(S)}\|\big)/(\alpha-1)\lambda \\ &\quad \text{(using [5, p.411, Theorem 4])} \\ &\le 2\beta C\big|\{T<\infty\}\big|/(\alpha-1) \\ &= 2\beta C\Big|\{S(f^{(S)}) > \lambda\}\Big|/(\alpha-1) \\ &\le 2\beta C\Big|\{S(f) > \lambda\}\Big|/(\alpha-1). \end{aligned}$$

So

$$\Big|\{S(f) > \alpha\lambda\}\Big| \le 2\beta C\Big|\{S(f) > \lambda\}\Big|/(\alpha-1) + \big|\{f^* + w^* > \beta\lambda\}\big|,$$

it means $\big(S(f), f^* + w^*\big)$ satisfies "good $\lambda$ inequality", (2) holds.

**Theorem 10.** *Let $X$ be a Hilbert space, then there exists a constant $C$ dependent only on $\Phi$, such that for all integrable conditionally symmetric $X$-valued sequence $(d_n)_{n\ge 1}$ with respect to $(\mathcal{F}_n)_{n\ge 0}$, denoting $f_n = \sum_{1\le k\le n} d_k$, we have*

$$C^{-1}E\Phi\big(S(f)\big) \le E\Phi(f^*) \le CE\Phi\big(S(f)\big).$$

*Proof.* Let

$$g_n = \sum_{1\le k\le n} a_k = \sum_{1\le k\le n} d_k\chi_{A_k}, \quad A_k = \{\|d_k\| \le 2d^*_{k-1}\}$$
$$h_n = \sum_{1\le k\le n} b_k = \sum_{1\le k\le n} d_k\chi_{B_k}, \quad B_k = \{\|d_k\| > 2d^*_{k-1}\}$$

then

$$f^* \le g^* + h^* \le g^* + \sum_{n\ge1} \|b_n\|$$
$$S(g) \le S(f) + S(h) \le S(f) + \sum_{n\ge1} \|b_n\|$$
$$E\Phi(f^*) \le CE\Phi(g^*) + CE\Phi\Big(\sum_{n\ge1} \|b_n\|\Big)$$

By Lemma 9, we have

$$\begin{aligned} E\Phi(g^*) &\le CE\Phi S(g) + CE\Phi(2d^*) \\ &\le CE\Phi\big(S(f)\big) + CE\Phi\Big(\sum_{n\ge1}\|b_n\|\Big) + CE\Phi(d^*). \end{aligned}$$

$$d^* \le S(f) \to \sum_{n\ge1} \|b_n\| \le 2d^* \le 2S(f)$$

So $E\Phi(f^*) \le CE\Phi\big(S(f)\big)$. The proof of the other inequality is similar, using $S(f) \le S(g) + \sum_{n\ge1} \|b_n\|$, $g^* \le f^* + \sum_{n\ge1} \|b_n\|$.

**Remark 11.** Let $\Phi(x) = x^p$, $0 < p < +\infty$, we can obtain (11.2) in [2] and (1.6), (1,7) of Thorem 1 in [3] with different constants by inequality $\|f\|_p \le \|f^*\|_p$, but the constants in [3] are the best possible.

**Lemma 12.** *There exists a constant $C$ depending only on $\Phi$ such that for all nonnegative $\mathbb{R}$-valued tangent sequences $(d_n)_{n\ge1}$, $(e_n)_{n\ge1}$, we have*

$$E\Phi\Big(\sum_{n\ge1} d_n\Big) \le CE\Phi\Big(\sum_{n\ge1} e_n\Big)$$

*Proof.* see [4, Theorem 2].

**Corollary 13.** *Let $X$ be a Hilbert space, then there exists a constant $C$ depending only on $\Phi$ such that for all $X$-valued conditionally symmetric sequences $(d_n)_{n\geq 1}$, $(e_n)_{n\geq 1}$, $d_n$, $e_n \in L_1(\mu, X)$, when $\left(\|d_n\|\right)_{n\geq 1}$ and $\left(\|e_n\|\right)_{n\geq 1}$ are tangent, we have*

$$E\Phi(f^*) \leq CE\Phi(g^*)$$

*where $f_n = \sum_{1\leq k\leq n} d_k$, $g_n = \sum_{1\leq k\leq n} e_k$.*

*Proof.* From Lemma 12,

$$E\Phi(f^*) \leq CE\Phi\big(S(f)\big),$$
$$E\Phi\big(S(g)\big) \leq CE\Phi(g^*)$$

and we take $\Phi(t^{1/2})$ in stead of $\Phi(t)$ and take tangent sequences $\left(\|d_n\|^2\right)_{n\geq 1}$, $\left(\|e_n\|^2\right)_{n\geq 1}$, and get

$$E\Phi\big(S(f)\big) \leq E\Phi\big(S(g)\big)$$

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