Math. J. Okayama Univ. **39** (1997), 135–145

ALGEBRAIC LOOPS ON FIBREWISE POINTED SPACES

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1. Introduction. In ordinary homotopy theory, the set $\pi(K, X)$ of homotopy classes of based maps from K to X is a group under certain conditions on either K or X. For example, if X is a group-like space, then $\pi(K, X)$ is a group. On the other hand, if X is an H-space, then $\pi(K, X)$ is a group under some conditions on K, as the following shows.

Recall that an *algebraic loop* is a set M together with a binary operation, written multiplicatively, having the following two properties:

(i) there is a two-sided identity element e,

(ii) for every two elements $a, b \in M$ the equations

$$a \cdot x = b, \quad y \cdot a = b,$$

admit unique solutions x, y in M.

The following theorem was proved by James. (See [5, Theorem1.1].)

Theorem 1.1. *Let K be a CW-space and let X be a path-connected H-space. Then $\pi(K, X)$ is an algebraic loop.*

Recall that the *Lusternik-Schnirelmann category*, cat$X$, of X is the least number of open subsets, each contractible in X, required to cover X. For example, cat$X = 1$ if and only if X is contractible.

The following theorem was proved by O'Neill. (See [7, Theorem3.1].)

Theorem 1.2. *Let K be a connected CW-space and let X be an H-space. If cat$K \leq 3$, then $\pi(K, X)$ is a group.*

Recall that a group Γ is said to be *nilpotent* if for some $n \geq 1$ there exists a sequence

$$\Gamma = \Gamma_1 \supset \ldots \supset \Gamma_n = \{e\}$$

of subgroups such that the commutator of Γ and Γ_i is contained in Γ_{i+1} for $i = 1, \ldots, n-1$. Such a sequence is called a *central chain* of length $n-1$ for the group Γ, and the minimum length of such chains is called the *nilpotency class* of Γ.

The following theorem was proved by G. W. Whitehead.(See [10].)

Theorem 1.3. *Let K have finite Lusternik-Schnirelmann category and let X be a path-connected group-like space. Then the group $\pi(K, X)$ is nilpotent of class less than $catK$.*

The paper is organised as follows. The above three theorems are generalized to the fibrewise pointed theory. In particular, in Section 2 we introduce the basic concepts of fibrewise pointed theory. In Section 3 we obtain the generalization of Theorem 1.1. In Section 4 we describe the fibrewise pointed category, to which the Lusternik-Schnirelmann category is generalized. In Section 5 we obtain the generalizations of Theorem 1.2 and Theorem 1.3.

2. Fibrewise pointed spaces. We work over a base space B. By a *fibrewise pointed space* over B, we mean a space X together with maps

$$B \xrightarrow{s} X \xrightarrow{p} B$$

such that $p \circ s = 1_B$. We refer to s as the *section* and p the *projection.* When p is a fibration we describe X as *fibrant.* We regard $A \times B$, for any pointed space A, as a fibrewise pointed space with section given by the basepoint of A.

If X is a fibrewise pointed space over B, the section embeds B as a subspace and one may refer to the pair (X, B).

If $X_i, i = 1, 2$, is a fibrewise pointed space over B with section s_i and projection p_i, a *fibrewise pointed map* ϕ: $X_1 \to X_2$ is a map such that $\phi \circ s_1 = s_2$ and $p_2 \circ \phi = p_1$. The notion *fibrewise pointed homotopy* is defined similarly and denoted by $\simeq^B_B$. A fibrewise pointed homotopy into the fibrewise pointed constant map $s_2 \circ p_1$ is called a *fibrewise pointed nulhomotopy.*

The *fibrewise product* $X_1 \times_B X_2$, defined in the usual way as the pull-back of p_1 and p_2, is considered as a fibrewise pointed space with section given by (s_1, s_2). The subspace

$$X_1 \times_B s_2(B) \cup s_1(B) \times_B X_2 \subset X_1 \times_B X_2$$

is denoted by $X_1 \vee_B X_2$ and called the *fibrewise wedge.*

The set of fibrewise pointed homotopy classes of fibrewise pointed maps of X into Y is denoted by $\pi^B_B(X, Y)$. The operation of composition for fibrewise pointed maps induces a map

$$\pi^B_B(Y, Z) \times \pi^B_B(X, Y) \to \pi^B_B(X, Z)$$

for fibrewise pointed spaces X, Y, Z over B. There is a natural equivalence

$$\pi_B^B(X, Y_1 \times_B Y_2) \equiv \pi_B^B(X, Y_1) \times \pi_B^B(X, Y_2)$$

for all fibrewise pointed spaces X, Y_1, Y_2 over B.

Given a fibrewise pointed space X over B, a fibrewise pointed map $m\colon X \times_B X \to X$ is called a *fibrewise multiplication.* If m is fibrewise pointed homotopic to $m \circ t$, where $t\colon X \times_B X \to X \times_B X$ switches factors, we say that m is *fibrewise homotopy-commutative.* If the two maps

$$m \circ (m \times 1_X),\ m \circ (1_X \times m)\colon X \times_B X \times_B X \to X$$

are fibrewise pointed homotopic, we say that m is *fibrewise homotopy-associative.* If the two maps

$$m \circ (1_X \times c) \circ \Delta,\ m \circ (c \times 1_X) \circ \Delta\colon X \to X$$

are fibrewise pointed homotopic to 1_X, where c is a fibrewise pointed constant map, we say that m is a *fibrewise Hopf structure* on X and that X, with this structure, is a fibrewise H-space. A *fibrewise homotopy right inverse* for a fibrewise multiplication m on X is a fibrewise pointed map $\sigma\colon X \to X$ such that $m \circ (1_X \times \sigma) \circ \Delta\colon X \to X$ is fibrewise pointed nul-homotopic. *Fibrewise homotopy left inverses* are defined similarly. When m is fibrewise homotopy-associative, a fibrewise homotopy right inverse is also a fibrewise homotopy left inverse, and the term *fibrewise homotopy inverse* may be used.

A fibrewise homotopy-associative fibrewise H-space for which the fibrewise multiplication admits a fibrewise homotopy inverse is called a *fibrewise group-like space.*

A fibrewise multiplication on the fibrewise pointed space Y over B determines a multiplication on the pointed set $\pi_B^B(X, Y)$ for all fibrewise pointed spaces X over B. If the former is a fibrewise Hopf structure on Y, then the latter has the following property:

$$[f] \cdot [c] = [f] = [c] \cdot [f].$$

If the former is fibrewise homotopy-commutative, then the latter is commutative, and similarly for the other conditions mentioned above. Thus $\pi_B^B(X, Y)$ is a group if Y is fibrewise group-like.

3. Generalization of Theorem 1.1. In this section, we prove the following theorem which is a generalization of Theorem 1.1.

Theorem 3.1. *Let B be a CW-space, K a fibrant fibrewise pointed space over B with CW-fibres, and X a fibrant fibrewise H-space over B such that X, as a space, is path-connected. Then $\pi_B^B(K,X)$ is an algebraic loop.*

Note that a fibrant fibrewise pointed space over a CW-space is also a CW-space if the fibre is a CW-space.(See [3].)

The following proposition is a somewhat stronger version of Proposition 9.1 of [3].

Proposition 3.2. *Let B be a CW-space, (K,L) a fibrant fibrewise pointed pair over B with CW-fibres, and (X,Y) a fibrant fibrewise pointed pair over B such that both X and Y, as spaces, are path-connected. If $\pi_1(X,Y)=0$ and for all $n>1$, (X,Y) is n-simple and*

$$H^n(K,L;\pi_n(X,Y))=0,$$

then every fibrewise pointed map $f\colon (K,L)\to(X,Y)$ is compressed into Y by a fibrewise pointed homotopy.

The following proposition is also a stronger version of Theorem 9.2 of [3].

Proposition 3.3. *Let B be a CW-space, K a fibrant fibrewise pointed space over B with CW-fibres, and X,Y fibrant fibrewise H-spaces over B such that both X and Y, as spaces, are path-connected. Let $f\colon X\to Y$ be an n-connected fibrewise pointed map. Then the induced map*

$$f_*\colon \pi_B^B(K,X)\to\pi_B^B(K,Y)$$

is injective when $\dim K<n$, *and surjective when* $\dim K\leq n$.

In particular, we have

Corollary 3.4. *Let B be a CW-space, K a fibrant fibrewise pointed space over B with CW-fibres, and X,Y fibrant fibrewise H-spaces over B such that both X and Y, as spaces, are path-connected. Let $f\colon X\to Y$ be a weak homotopy equivalence fibrewise pointed map. Then the induced map*

$$f_*\colon \pi_B^B(K,X)\to\pi_B^B(K,Y)$$

is bijective.

We use these results to prove Theorem 3.1.

Proof of Theorem 3.1. The class of fibrewise pointed constant maps is a two-sided identity of $\pi_B^B(K, X)$. We now show that for every two elements $\alpha, \beta \in \pi_B^B(K, X)$, the equations

$$\eta \cdot \alpha = \beta, \quad \alpha \cdot \xi = \beta$$

admit unique solutions η, ξ in $\pi_B^B(K, X)$. Let $m \colon X \times_B X \to X$ be a fibrewise Hopf structure on X. Then the fibrewise pointed map

$$p \colon X \times_B X \to X \times_B X,$$

defined by $p(x, y) = (m(x, y), y)$, for $x, y \in X$, induces a map

$$p_* \colon \pi_B^B(K, X \times_B X) \to \pi_B^B(K, X \times_B X),$$

given by $p_*(\alpha, \beta) = (\alpha \cdot \beta, \beta)$ for $\alpha, \beta \in \pi_B^B(K, X)$, since we can identify $\pi_B^B(K, X \times_B X)$ with $\pi_B^B(K, X) \times \pi_B^B(K, X)$. Since X is fibrant, $X \times_B X$ is also fibrant.

We now consider the homotopy exact sequences of fibrations:

$$\begin{array}{ccccccccc}
\cdots \to \pi_{i+1}(B) & \to & \pi_i(X_b \times X_b) & \to & \pi_i(X \times_B X) & \to & \pi_i(B) & \to & \pi_{i-1}(X_b \times X_b) \to \cdots \\
\| & & \downarrow (p|_{X_b \times X_b})_* & & \downarrow p_* & & \| & & \downarrow (p|_{X_b \times X_b})_* \\
\cdots \to \pi_{i+1}(B) & \to & \pi_i(X_b \times X_b) & \to & \pi_i(X \times_B X) & \to & \pi_i(B) & \to & \pi_{i-1}(X_b \times X_b) \to \cdots
\end{array}$$

The induced map

$$(p\,|_{X_b \times X_b})_* \colon \pi_i(X_b \times X_b) \to \pi_i(X_b \times X_b), \text{ where } b \in B,$$

is bijective if $i \geq 1$, and injective if $i = 0$. Thus

$$p_* \colon \pi_i(X \times_B X) \to \pi_i(X \times_B X)$$

is an isomorphism for all $i \geq 0$, that is, p is a weak homotopy equivalence. By Corollary 3.4, we see that

$$p_* \colon \pi_B^B(K, X \times_B X) \to \pi_B^B(K, X \times_B X)$$

is a bijection. Hence for the element $(\beta, \alpha) \in \pi_B^B(K, X \times_B X)$, there exists a unique element $\gamma = (\eta, \omega) \in \pi_B^B(K, X \times_B X)$ such that $p_*(\gamma) = (\beta, \alpha)$, then $\eta \cdot \omega = \beta$ and $\omega = \alpha$. Hence there exists a unique solution η of $\eta \cdot \alpha = \beta$. A similarly argument shows that there exists a unique solution ξ of $\alpha \cdot \xi = \beta$.

4. Fibrewise pointed category. In this section we recall some results from [6].

We describe a subset U of a fibrewise pointed space X over B as *fibrewise pointed categorical* if U contains the section and the inclusion $U \to X$ is fibrewise pointed nulhomotopic.

The *fibrewise pointed category*, $\mathrm{cat}_B^B X$, of a fibrewise pointed space X over B is defined to be the least number of fibrewise pointed categorical open sets required to cover X. For example, $\mathrm{cat}_B^B X = 1$ if and only if X is fibrewise pointed contractible.

Definition 4.1. The fibrewise pointed space X over B is a fibrewise absolute neighbourhood retract (fibrewise ANR for short) if X can be fibrewise pointed embedded as an open subspace of $A \times B$ for some absolute retract A.

Of course, an open subspace of a fibrewise ANR is again a fibrewise ANR. We shall be using the following property of fibrewise ANRs. Let E be a fibrewise pointed space such that $E \times I^r$ is normal, as a space, for $r = 0, 1, \ldots$. Let us call such a fibrewise pointed space *fully normal.* Let F be a closed subspace of E. If $\phi\colon E \to X$ is a fibrewise pointed map and $\psi_t\colon F \to X$ is a fibrewise pointed homotopy such that $\psi_0 = \phi \mid F$, then ψ_t can be extended to a fibrewise pointed homotopy $\phi_t\colon E \to X$ such that $\phi_0 = \phi$.

If X is a fibrewise pointed space such that X is normal, as a space, then any open covering of X by fibrewise pointed categorical subsets can be shrunk to a closed covering. If, moreover, X is a fibrewise ANR, then any closed covering of X by fibrewise pointed categorical subsets can be expanded to an open covering of X by fibrewise pointed categorical subsets, and so we may replace the word "open" by the word "closed" in the definition of fibrewise pointed category.

We now show that under certain conditions, fibrewise pointed categories can be characterised in terms of the compressibility of the diagonal. Specifically, consider the fibrewise topological products $\Pi_B^n X, n = 1, 2, \ldots$, of the fibrewise pointed space X with itself. Let $\Pi_B^n(X, B) \subset \Pi_B^n X$ denote the subspace for which the fibre over each point $b \in B$ is the fat wedge

$$\Pi^n(p^{-1}(b), s(b)).$$

Then $\Pi_B^n X$ contains the diagonal ΔX of X, while $\Pi_B^n(X, B)$ contains the diagonal ΔB of B. In other words, the pair $(\Pi_B^n X, \Pi_B^n(X, B))$ contains the diagonal $(\Delta X, \Delta B)$. Note that

$$\Pi_B^{n+1} X = p^* \Pi_B^n X,$$
$$\Pi_B^{n+1}(X, B) = p^* \Pi_B^n(X, B) \cup \Pi_B^n X \times_B B.$$

Proposition 4.2 ([6, Proposition 6.1]). *Let X be a fibrewise pointed space over B. Suppose that X admits a fibrewise pointed categorical neighbourhood of B. If, for some $n \geq 1$, the diagonal*

$$\Delta \colon X \to \Pi_B^n X$$

can be compressed into $\Pi_B^n(X, B)$ by a fibrewise pointed homotopy, then $cat_B^B X \leq n$.

Proposition 4.3 ([6, Proposition 6.2]). *Let X be a fibrewise pointed space over B with a closed section such that X, as a space, is fully normal. If $cat_B^B X \leq n$ for some $n \geq 1$, then the diagonal*

$$\Delta \colon X \to \Pi_B^n X$$

can be compressed into $\Pi_B^n(X, B)$ by a fibrewise pointed homotopy.

5. Generalizations of Theorem 1.2 and Theorem 1.3. We first prove the following theorem which is a generalization of Theorem 1.2.

Theorem 5.1. *Let B a CW-space, K a fibrant fibrewise pointed space over B with CW-fibres and a closed section such that K, as a space, is fully normal, and X a fibrant fibrewise H-space over B such that X, as a space, is path-connected. If $cat_B^B K \leq 3$, then $\pi_B^B(K, X)$ is a group.*

For example, $\pi_B^B(K, X)$ is commutative whenever $\mathrm{cat}_B^B K \leq 2$. (See [6, Section 5].)

Proof. By Theorem 3.1, the set $\pi_B^B(K, X)$ is an algebraic loop and hence it suffices to show that $\pi_B^B(K, X)$ is associative. Let $i \colon \Pi_B^3(K, B) \to \Pi_B^3 K$ and $j \colon \Pi_B^3(X, B) \to \Pi_B^3 X$ be the inclusion maps, and let $\Delta \colon K \to \Pi_B^3 K$ be the diagonal map. By Proposition 4.3, there exists a fibrewise pointed map $k \colon K \to \Pi_B^3(K, B)$ such that $\Delta \simeq i \circ k$. By the assumption, X is a fibrewise H-space over B. Let $m \colon X \times_B X \to X$ be its fibrewise multiplication. Then we have

$$m \circ (1_X \times m) \circ j \simeq m \circ (m \times 1_X) \circ j.$$

Let f, g, $h\colon K \to X$ be any three fibrewise pointed maps and denote by $\Pi_B^3(f,g,h)\colon \Pi_B^3(K,B) \to \Pi_B^3(X,B)$ the fibrewise pointed map induced by $f \times g \times h\colon \Pi_B^3 K \to \Pi_B^3 X$. Consider the element $[f]\cdot([g]\cdot[h]) \in \pi_B^B(K,X)$. We have

$$\begin{aligned}[f]\cdot([g]\cdot[h]) &= [m \circ (1_X \times m) \circ (f \times g \times h) \circ \Delta] \\ &= [m \circ (1_X \times m) \circ (f \times g \times h) \circ i \circ k] \\ &= [m \circ (1_X \times m) \circ j \circ \Pi_B^3(f,g,h) \circ k] \\ &= [m \circ (m \times 1_X) \circ j \circ \Pi_B^3(f,g,h) \circ k] \\ &= ([f]\cdot[g])\cdot[h].\end{aligned}$$

Therefore, $\pi_B^B(K,X)$ is associative, and this proves the theorem.

The following proposition is a generalization of Theorem 1.3.

Proposition 5.2 ([6, Proposition 5.1]). *Let X be a fibrewise group-like space over B. Let K be a fibrewise pointed space over B with a closed section. Suppose that K is a fibrewise ANR and, as a space, is fully normal. Then the group $\pi_B^B(K,X)$ is nilpotent of class less than $cat_B^B K$.*

We now consider the nilpotency class of $\pi_B^B(K,X)$ which is an algebraic loop.

Definition 5.3. Let Γ be an algebraic loop.

(i) If $x, y \in \Gamma$, their *commutator* is the element

$$[x,y] = (x\cdot y)\cdot(y\cdot x)^{-1},$$

where $(y\cdot x)^{-1}$ is the right inverse of $(y\cdot x)$, that is, $(y\cdot x)\cdot(y\cdot x)^{-1} = e$.

(ii) Γ is said to be *nilpotent* if for some $n \geq 1$ there exists a sequence

$$\Gamma = \Gamma_1 \supset \ldots \supset \Gamma_{n+1} = \{e\}$$

of subloops such that the commutator of Γ and Γ_i is contained in Γ_{i+1} for $i = 1, \ldots, n$. The least such number n is called the *nilpotency class* of Γ.

Let B be a CW-space and X a fibrant fibrewise H-space over B with CW-fibres such that X, as a space, is path-connected. By Theorem 3.1, the set $\pi_B^B(X,X)$ is an algebraic loop. Let $[\sigma] \in \pi_B^B(X,X)$ be a right inverse of $[1_X]$. Then σ is a fibrewise homotopy right inverse for m and we denote $x^{-1} = \sigma(x)$. Let $m\colon X \times_B X \to X$ be a fibrewise Hopf structure on X. We define the commutator map

$$C_m\colon X \times_B X \to X$$

by $C_m(x,y) = (xy)(yx)^{-1} = m(m(x,y), \sigma(m(y,x)))$.

Then we have

Lemma 5.4. *Let B be a CW-space and X a fibrant fibrewise H-space over B with CW-fibres and a closed section such that X, as a space, is path-connected. Then the restriction $C_m\mid_{X\vee_B X}$ is fibrewise pointed nulhomotopic.*

Definition 5.5. Let K and X be fibrewise pointed spaces over B. If $f, g\colon K \to X$ are fibrewise pointed maps, their *commutator* is the element

$$[f,g] = C_m \circ (f \times g) \circ \Delta\colon K \to X.$$

Remark 5.6. If σ is a fibrewise pointed right inverse, for every element $[f] \in \pi_B^B(K,X)$, where K and X are fibrewise pointed spaces over B, the element $[\sigma \circ f]$ is a right inverse of $[f]$. Thus the commutator $[[f],[g]]$ of the homotopy classes $[f]$ and $[g]$ in the algebraic loop $\pi_B^B(K,X)$ as defined in Definition 5.3 is the same as the fibrewise pointed homotopy class of the commutator $[f,g]$ of the fibrewise pointed maps f and g as defined in Definition 5.5.

Theorem 5.7. *Let B be a CW-space, K a fibrant fibrewise pointed space over B with CW-fibres and a closed section, and X a fibrant fibrewise H-space over B with CW-fibres and a closed section such that X, as a space, is path-connected. Suppose that K is a fibrewise ANR and, as a space, is fully normal. Then the algebraic loop $\pi_B^B(K,X)$ is nilpotent of class less than $cat_B^B K$.*

Proof. Since K is normal, we can define the fibrewise pointed category in terms of closed, rather than open, fibrewise pointed categorical coverings. Let $\{A_1, ..., A_n\}$ be such a covering, where $n=\text{cat}_B^B K$ and let

$\Gamma_i = Ker\{Res\colon \pi_B^B(K,X) \to \pi_B^B(A_1 \cup ... \cup A_i, X)\}$ for $i = 1, ..., n$, where Res is the map induced by the inclusion $A_1 \cup ... \cup A_i \to K$. The fibrewise Hopf suructure m determines a binary operation on $\pi_B^B(A_1 \cup ... \cup A_i, X)$ and the class of fibrewise pointed constant maps is a two-sided identity. The map Res is a homomorphism, that is, $Res(\alpha, \beta) = Res(\alpha) \cdot Res(\beta)$, $Res(e) = e$ so that Γ_i is a subloop of $\Gamma = \pi_B^B(K, X)$. Since $A_1 \cup ... \cup A_n = X$, we have $\Gamma_n = \{e\}$. Let $[f]$ be any element of Γ, p a projection of K and s, t sections of K, X, respectively. Since A_1 is a fibrewise pointed categorical subset, we have

$$i \simeq_B^B s \circ p\,|_{A_1}$$

where $i\colon A_1 \to K$ is the inclusion map. Therefore

$$f\,|_{A_1} = f \circ i \simeq_B^B f \circ s \circ p\,|_{A_1} = t \circ p\,|_{A_1},$$

that is, $f\,|_{A_1}$ is fibrewise pointed nulhomotopic, which shows that $[f] \in \Gamma_1$ and so $\Gamma = \Gamma_1$. An analogous argument proves that

$$\Gamma = Ker\big\{Res\colon \pi_B^B(K, X) \to \pi_B^B(A_i, X)\big\}$$

for any $i = 2, ..., n$.

For $i < n$, let $[f]$ and $[g]$ be elements of Γ and Γ_i, respectively. By the assumption, there exists an element $f' \in \Gamma$ with $f' = s \circ p$ on A_{i+1} such that $f \simeq_B^B f'$ and an element $g' \in \Gamma_i$ with $g' = s \circ p$ on $A_1 \cup ... \cup A_i$ such that $g \simeq_B^B g'$. Then the commutator $[f, g]$ is fibrewise pointed homotopic to $[f', g']$, that is, to the following composition:

$$K \xrightarrow{\ \Delta\ } K \times_B K \xrightarrow{f' \times g'} X \times_B X \xrightarrow{\ C_m\ } X$$

Since the composition $(f' \times g') \circ \Delta$ maps $A_1 \cup ... \cup A_{i+1}$ into $X \vee_B X$, by Lemma 5.4, we have

$$[f', g'] \simeq_B^B s \circ p \ \text{ on } A_1 \cup ... A_{i+1}.$$

Therefore

$$[f, g] \simeq_B^B s \circ p \text{ on } A_1 \cup ... A_{i+1}.$$

Hence the commutator $[[f], [g]]$ lies in Γ_{i+1}. This proves the theorem.

Acknowledgement. The present paper is the English version of the author's Master thesis written at the Osaka City University under the supervision of Professor H. Oshima, to whom the author is deeply grateful.

REFERENCES

[1] I. M. JAMES: Fibrewise Topology, Cambridge Univ. Press, Cambridge, 1990.
[2] I. M. JAMES: General Topology and Homotopy Theory, Springer-Verlag, New York Berlin Heidelberg Tokyo, 1984.
[3] I. M. JAMES: Introduction to Fibrewise Homotopy Theory, Chapter 4 of Handbook of Algebraic Topology, North-Holland, 1995.
[4] I. M. JAMES: On category, in the sense of Lusternik-Schnirelmann, Topology **17** (1978), 331–348.
[5] I. M. JAMES: On H-spaces and their homotopy groups, Quart. J. Math. Oxford Ser.(2) **11** (1960), 161–179.
[6] I. M. JAMES AND J. R. MORRIS: Fibrewise category, Proc. Roy. Soc. Edinburgh **199A** (1991), 177–190.
[7] R. C. O'NEILL: On H-spaces that are CW-complexes I, Illinois J. Math. **8** (1964), 280–290.
[8] E. H. SPANIER: Algebraic Topology, McGraw-Hill, 1966.
[9] M. MIMURA-H. TODA: Homotopy Theory, Kinokuniya-shoten, 1975, (in Japanese).
[10] G. W. WHITEHEAD: Elements of Homotopy Theory, Springer-Verlag, New York Heidelberg Berlin, 1978.

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(*Received June 23, 1998*)