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A GENERALIZATION OF THE WHITEHEAD PRODUCT

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1. Introduction. Barratt [4] and Arkowitz [1] defined the *generalized Whitehead product*

$$[\alpha, \beta] \in [\Sigma(X \wedge Y), Z]$$

for any elements α of $[\Sigma X, Z]$ and β of $[\Sigma Y, Z]$, where ΣA is the suspension space of a space A with a base point and $[A, Z]$ is the set of base point preserving homotopy classes of base point preserving continuous maps from A to Z.

Let Γ be a co-Hopf space. For any space X, we write $\Gamma X = \Gamma \wedge X$, the smash product of Γ and X. We call ΓX the Γ-*suspension space* of a space X. If $\Gamma = \Sigma = S^1$ (1-sphere), then $\Gamma X = \Sigma X = S^1 \wedge X$ is the usual suspension space of X. In general, if Γ is a co-grouplike space, that is, an associative co-Hopf space with an inverse, then ΓX has a co-grouplike structure induced by that of Γ. The co-grouplike structure of ΓX enables us to define a generalization of the product of Barratt [4] and Arkowitz [1]; we shall define an element

$$[\alpha, \beta]_\Gamma \in [\Gamma(X \wedge Y), Z]$$

for any elements α of $[\Gamma X, Z]$ and β of $[\Gamma Y, Z]$. We refer to the element $[\alpha, \beta]_\Gamma$ as the Γ-*Whitehead product* of α and β. The definition of the Γ-Whitehead product $[\alpha, \beta]_\Gamma$ is obtained by making use of the commutator. This generalizes the Whitehead product studied by Barratt and Arkowitz. We show that the Γ-Whitehead product enjoys many properties of the usual Whitehead product. We also study the dual results.

In the first section we define the Γ-Whitehead product $[\alpha, \beta]_\Gamma$. If $\alpha \in [\Gamma X, Z]$, $\beta \in [\Gamma Y, Z]$ and $\gamma \in [\Gamma X, Z]$ (or $\gamma \in [\Gamma Y, Z]$), then γ acts on the Γ-Whitehead product $[\alpha, \beta]_\Gamma$, which we denote by $[\alpha, \beta]^\gamma_\Gamma$. The Γ-Whitehead product with this action has many of the properties of the usual Whitehead product. It is natural with respect to induced maps between homotopy sets. It satisfies the relation $\Gamma_2([\alpha, \beta]^\gamma_{\Gamma_1}) = 0$ for any co-grouplike space Γ_1 and any co-Hopf space Γ_2, which is a generalization of the well-known fact that the suspension of any Whitehead product vanishes. Moreover, we prove that the Γ-Whitehead product $[\alpha, \beta]_\Gamma$ is biadditive and satisfies the

Jacobi identity under some conditions with the action of γ (Theorems 2.8, 2.9 and 2.11). The author did not find in the literature the explict formulas with the action of γ and the explicit proof of the Jacobi identity even in the case $\Gamma = S^1$. In the second section we define Γ^*-*Whitehead product* $[\alpha,\beta]_{\Gamma^*}$, which is a dual concept of the Γ-Whitehead product $[\alpha,\beta]_\Gamma$. This generalizes the *dual product* of Arkowitz in §6 of [1]. We will see that some of the results in the preceding section can be dualized. But the dual results are not complete ones. In the third section we study the relation between the Γ-Whitehead product and the Samelson product and also the relation between the Γ^*-Whitehead product and the dual Samelson product.

We work in the category of topological spaces with a *nondegenerate* base point $*$ (cf. [9]). For any spaces X and Y, the wedge sum (one point union) $X \vee Y$ is a subspace of the product space $X \times Y$ with the inclusion map $j\colon X \vee Y \to X \times Y$. The smash product $X \wedge Y$ is defined by $X \wedge Y = (X \times Y)/(X \vee Y)$. We use the same notation for a map and its homotopy class.

2. Γ-Whitehead product.

We write $\Gamma \wedge X = \Gamma X$ (the Γ-*suspension space* of X) for any co-Hopf space Γ and any space X. A map $f\colon X \to Y$ induces a Γ-*suspension map* $\Gamma f\colon \Gamma X \to \Gamma Y$. We see $\Gamma g \circ \Gamma f = \Gamma(g \circ f)$ for any maps $f\colon X \to Y$ and $g\colon Y \to Z$.

Let A be a co-Hopf space with a co-multiplication $\theta\colon A \to A \vee A$. For any maps α, $\beta\colon A \to Z$, we define $\alpha \dot{+} \beta\colon A \to Z$ by

$$\alpha \dot{+} \beta = \nabla_Z \circ (\alpha \vee \beta) \circ \theta,$$

where $\nabla_Z\colon Z \vee Z \to Z$ is the folding map. This defines a homotopy class $\alpha \dot{+} \beta \in [A, Z]$ for any elements α, $\beta \in [A, Z]$.

A space A is called a *co-grouplike space* (cf. Whitehead [9]) if it is a homotopy associative co-Hopf space with a homotopy inverse (inversion) $\nu\colon A \to A$, namely, $1_A \dot{+} \nu \simeq * \simeq \nu \dot{+} 1_A$, where $1_A\colon A \to A$ is the identity map. Let Γ be a **co-grouplike space** with a co-multiplication $\theta\colon \Gamma \to \Gamma \vee \Gamma$. We define

$$\theta_X = \theta \wedge 1_X\colon \Gamma X = \Gamma \wedge X \to (\Gamma \vee \Gamma) \wedge X \cong (\Gamma \wedge X) \vee (\Gamma \wedge X) = \Gamma X \vee \Gamma X.$$

Then ΓX is a co-grouplike space with a co-multiplication θ_X for any space X (cf. 6.3.14(a) of Maunder [7]).

Consider a cofibration sequence

$$\Gamma(X \vee Y) \xrightarrow{\Gamma j} \Gamma(X \times Y) \xrightarrow{\Gamma q} \Gamma(X \wedge Y) \xrightarrow{\Gamma \delta} \Sigma\Gamma(X \vee Y) \xrightarrow{\Sigma\Gamma j} \Sigma\Gamma(X \times Y).$$

Let $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ be the projections and $j_1 \colon X \to X \vee Y$ and $j_2 \colon Y \to X \vee Y$ the inclusion maps. We define a map $\rho \colon \Gamma(X \times Y) \to \Gamma(X \vee Y)$ by $\rho = \Gamma(j_1 \circ p_1) \dotplus \Gamma(j_2 \circ p_2)$. Then the map ρ satisfies the relation $\rho \circ \Gamma j \simeq 1_{\Gamma(X \vee Y)}$. Then by a long homotopy exact sequence, we have a short exact sequence of groups

$$(2.1)\quad 0 \longrightarrow [\Gamma(X \wedge Y), Z] \xrightarrow{(\Gamma q)^*} [\Gamma(X \times Y), Z] \xrightarrow{(\Gamma j)^*} [\Gamma(X \vee Y), Z] \longrightarrow 0$$

for any space Z. We define

$$\overline{\alpha} = \alpha \circ \Gamma p_1 \colon \Gamma(X \times Y) \to Z \quad \text{and} \quad \overline{\beta} = \beta \circ \Gamma p_2 \colon \Gamma(X \times Y) \to Z$$

for any maps $\alpha \colon \Gamma X \to Z$ and $\beta \colon \Gamma Y \to Z$. Let

$$(2.2)\qquad [\overline{\alpha}, \overline{\beta}] = \overline{\alpha} \dotplus \overline{\beta} \dot{-} \overline{\alpha} \dot{-} \overline{\beta} \in [\Gamma(X \times Y), Z]$$

be a commutator defined for any elements $\alpha \in [\Gamma X, Z]$ and $\beta \in [\Gamma Y, Z]$ by making use of the co-grouplike structure of $\Gamma(X \times Y)$. Since $(\Gamma j)^*([\overline{\alpha}, \overline{\beta}]) = 0$ in $[\Gamma(X \vee Y), Z]$, we can find a unique element

$$[\alpha, \beta]_\Gamma \in [\Gamma(X \wedge Y), Z]$$

with $(\Gamma q)^*([\alpha, \beta]_\Gamma) = [\overline{\alpha}, \overline{\beta}]$ for the identification map $q \colon X \times Y \to X \wedge Y$. We refer to the element $[\alpha, \beta]_\Gamma \in [\Gamma(X \wedge Y), Z]$ as the Γ-*Whitehead product* of $\alpha \in [\Gamma X, Z]$ and $\beta \in [\Gamma Y, Z]$. If $\Gamma = S^1$ (1-sphere), then $[\alpha, \beta]_\Gamma$ coincides with the *generalized Whitehead product* $[\alpha, \beta]$ of Arkowitz [1].

Remark 2.1. Arkowitz [1] uses the commutator $(x, y) = x^{-1} y^{-1} x y$ in the defintion of the generalized Whitehead product $[\alpha, \beta]$. Then we have $[\alpha, \beta]_{S^1} = [-\alpha, -\beta]$.

In a group G, an "action" of a on x is defined by conjugation $x^a = a x a^{-1}$ for any $x, a \in G$ and relations between commutators and actions are known as in Lemma 2.7. We now define actions of elements of homotopy groups on Γ-Whitehead products.

If $\alpha \in [\Gamma X, Z]$, $\beta \in [\Gamma Y, Z]$ and $\gamma \in [\Gamma X, Z]$ (or $\gamma \in [\Gamma Y, Z]$), then γ acts on the Γ-Whitehead product $[\alpha, \beta]_\Gamma$, denoted by $[\alpha, \beta]_\Gamma^\gamma$, by the formula

$$(2.3)\qquad (\Gamma q)^*([\alpha, \beta]_\Gamma^\gamma) = \overline{\gamma} \dotplus [\overline{\alpha}, \overline{\beta}] \dot{-} \overline{\gamma} = [\overline{\alpha}, \overline{\beta}]^{\overline{\gamma}}$$

for the monomorphism $(\Gamma q)^* \colon [\Gamma(X \wedge Y), Z] \to [\Gamma(X \times Y), Z]$ in (2.1) (cf. Marcum [6]).

We remark that any element $\gamma \in [\Gamma(X \times Y), Z]$ acts on any element $\alpha \in [\Gamma(X \wedge Y), Z]$ by

$$(\Gamma q)^*(\alpha^\gamma) = \gamma \dotplus \alpha \circ \Gamma q \dot{-} \gamma = (\alpha \circ \Gamma q)^\gamma \in [\Gamma(X \times Y), Z].$$

The Γ-Whitehead product with action $[\alpha, \beta]_\Gamma^\gamma$ does not always have the form $[\alpha', \beta']_\Gamma$ for some $\alpha' \in [\Gamma X, Z]$ and $\beta' \in [\Gamma Y, Z]$. Bearing these in mind, we have the following formulas: If $[\alpha, \beta]_\Gamma = 0$, then $[\alpha, \beta]_\Gamma^\gamma = 0$ for any γ; $[\alpha, \beta]_\Gamma^0 = [\alpha, \beta]_\Gamma$; $([\alpha, \beta]_\Gamma^{\gamma_1})^{\gamma_2} = [\alpha, \beta]_\Gamma^{\gamma_2 \dotplus \gamma_1}$ for any γ_1 and γ_2.

The construction of Γ-Whitehead product is natural in the following sense.

Proposition 2.2. *Let $\alpha \in [\Gamma X, Z]$, $\beta \in [\Gamma Y, Z]$ and $\gamma \in [\Gamma X, Z]$ (or $\gamma \in [\Gamma Y, Z]$) be any elements. Then we have the following results.*

(i) $\omega \circ [\alpha, \beta]_\Gamma^\gamma = [\omega \circ \alpha, \omega \circ \beta]_\Gamma^{\omega \circ \gamma}$ *for any $\omega\colon Z \to Z'$.*

(ii) $[\alpha, \beta]_\Gamma^\gamma \circ \Gamma(\delta \wedge \varepsilon) = [\alpha \circ \Gamma\delta, \beta \circ \Gamma\varepsilon]_\Gamma^{\gamma'}$ *for any $\delta\colon X' \to X$ and $\varepsilon\colon Y' \to Y$, where $\gamma' = \gamma \circ \Gamma\delta$ if $\gamma \in [\Gamma X, Z]$ and $\gamma' = \gamma \circ \Gamma\varepsilon$ if $\gamma \in [\Gamma Y, Z]$.*

Proof. (i) Consider the following commutative diagram.

$$\begin{array}{ccccc}
0 & \longrightarrow & [\Gamma(X \wedge Y), Z] & \xrightarrow{(\Gamma q)^*} & [\Gamma(X \times Y), Z] \\
 & & \downarrow \omega_* & & \downarrow \omega_* \\
0 & \longrightarrow & [\Gamma(X \wedge Y), Z'] & \xrightarrow{(\Gamma q)^*} & [\Gamma(X \times Y), Z']
\end{array}$$

We see

$$\begin{aligned}
(\Gamma q)^*(\omega \circ [\alpha, \beta]_\Gamma^\gamma) &= (\Gamma q)^*\{\omega_*([\alpha, \beta]_\Gamma^\gamma)\} = \omega_*\{(\Gamma q)^*([\alpha, \beta]_\Gamma^\gamma)\} \\
&= \omega_*(\overline{\gamma} \dotplus \overline{\alpha} \dotplus \overline{\beta} \dot{-} \overline{\alpha} \dot{-} \overline{\beta} \dot{-} \overline{\gamma}) \\
&= (\omega \circ \overline{\gamma}) \dotplus (\omega \circ \overline{\alpha}) \dotplus (\omega \circ \overline{\beta}) \dot{-} (\omega \circ \overline{\alpha}) \dot{-} (\omega \circ \overline{\beta}) \dot{-} (\omega \circ \overline{\gamma}) \\
&= \overline{\omega \circ \gamma} \dotplus \overline{\omega \circ \alpha} \dotplus \overline{\omega \circ \beta} \dot{-} \overline{\omega \circ \alpha} \dot{-} \overline{\omega \circ \beta} \dot{-} \overline{\omega \circ \gamma} \\
&= [\overline{\omega \circ \alpha}, \overline{\omega \circ \beta}]^{\overline{\omega \circ \gamma}} = (\Gamma q)^*([\omega \circ \alpha, \omega \circ \beta]_\Gamma^{\omega \circ \gamma}).
\end{aligned}$$

Then we have $\omega \circ [\alpha, \beta]_\Gamma^\gamma = [\omega \circ \alpha, \omega \circ \beta]_\Gamma^{\omega \circ \gamma}$.

(ii) We prove the case $\gamma \in [\Gamma X, Z]$. The other case is proved similarly. Consider the following commutative diagram.

$$\begin{array}{ccccc}
0 & \longrightarrow & [\Gamma(X \wedge Y), Z] & \xrightarrow{(\Gamma q)^*} & [\Gamma(X \times Y), Z] \\
 & & \downarrow{\scriptstyle \{\Gamma(\delta\wedge\varepsilon)\}^*} & & \downarrow{\scriptstyle \{\Gamma(\delta\times\varepsilon)\}^*} \\
0 & \longrightarrow & [\Gamma(X' \wedge Y'), Z] & \xrightarrow{(\Gamma q')^*} & [\Gamma(X' \times Y'), Z]
\end{array}$$

Let us use the notation that $\overline{\zeta}' = \zeta \circ \Gamma p_1'$ and $\overline{\eta}' = \eta \circ \Gamma p_2'$ for any maps $\zeta\colon \Gamma X' \to Z$ and $\eta\colon \Gamma Y' \to Z$ and projections $p_1'\colon X' \times Y' \to X'$ and $p_2'\colon X' \times Y' \to Y'$. Then we see

$$\begin{aligned}
(\Gamma q')^*\{[\alpha,\beta]_\Gamma^\gamma \circ \Gamma(\delta \wedge \varepsilon)\} &= (\Gamma q')^*\{(\Gamma(\delta \wedge \varepsilon))^*([\alpha,\beta]_\Gamma^\gamma)\} \\
&= (\Gamma(\delta \times \varepsilon))^*\{(\Gamma q)^*([\alpha,\beta]_\Gamma^\gamma)\} \\
&= (\Gamma(\delta \times \varepsilon))^*(\overline{\gamma} \dotplus \overline{\alpha} \dotplus \overline{\beta} \dot{-} \overline{\alpha} \dot{-} \overline{\beta} \dot{-} \overline{\gamma}) \\
&= \{\overline{\gamma} \circ \Gamma(\delta \times \varepsilon)\} \dotplus \{\overline{\alpha} \circ \Gamma(\delta \times \varepsilon)\} \dotplus \{\overline{\beta} \circ \Gamma(\delta \times \varepsilon)\} \\
&\quad \dot{-} \{\overline{\alpha} \circ \Gamma(\delta \times \varepsilon)\} \dot{-} \{\overline{\beta} \circ \Gamma(\delta \times \varepsilon)\} \dot{-} \{\overline{\gamma} \circ \Gamma(\delta \times \varepsilon)\} \\
&= \overline{\gamma \circ \Gamma\delta}' \dotplus \overline{\alpha \circ \Gamma\delta}' \dotplus \overline{\beta \circ \Gamma\varepsilon}' \dot{-} \overline{\alpha \circ \Gamma\delta}' \dot{-} \overline{\beta \circ \Gamma\varepsilon}' \dot{-} \overline{\gamma \circ \Gamma\delta}' \\
&= (\Gamma q')^*([\alpha \circ \Gamma\delta, \beta \circ \Gamma\varepsilon]_\Gamma^{\gamma'}),
\end{aligned}$$

since we have the following commutative diagrams for α and β and a similar diagram for γ:

$$\begin{array}{ccccc}
\Gamma(X' \times Y') & \xrightarrow{\Gamma p_1'} & \Gamma X' & & \\
\downarrow{\scriptstyle \Gamma(\delta\times\varepsilon)} & & \downarrow{\scriptstyle \Gamma\delta} & & \\
\Gamma(X \times Y) & \xrightarrow{\Gamma p_1} & \Gamma X & \xrightarrow{\alpha} & Z
\end{array}$$

$$\begin{array}{ccccc}
\Gamma(X' \times Y') & \xrightarrow{\Gamma p_2'} & \Gamma Y' & & \\
\downarrow{\scriptstyle \Gamma(\delta\times\varepsilon)} & & \downarrow{\scriptstyle \Gamma\varepsilon} & & \\
\Gamma(X \times Y) & \xrightarrow{\Gamma p_2} & \Gamma Y & \xrightarrow{\beta} & Z
\end{array}$$

When Γ is a commutative co-grouplike space, e.g. an iterated suspension space $\Gamma = \Sigma^n W = S^n \wedge W$ for some space W and the n-sphere S^n for $n \geq 2$, we have the following result.

Proposition 2.3. *Let Γ be a commutative co-grouplike space. Then* $[\alpha,\beta]^{\gamma}_{\Gamma} = 0$ *for any* $\alpha \in [\Gamma X, Z]$, $\beta \in [\Gamma Y, Z]$ *and* $\gamma \in [\Gamma X, Z]$ *(or* $\gamma \in [\Gamma Y, Z]$*).*

Proof. Since $[\Gamma(X \times Y), Z]$ is an abelian group when Γ is a commutative co-grouplike space, we have

$$[\overline{\alpha},\overline{\beta}]^{\overline{\gamma}} = \overline{\gamma} \dotplus \overline{\alpha} \dotplus \overline{\beta} \dot{-} \overline{\alpha} \dot{-} \overline{\beta} \dot{-} \overline{\gamma} = 0$$

in (2.2) or (2.3). Hence we have the result.

Proposition 2.4. *If Z is a Hopf space, then* $[\alpha,\beta]^{\gamma}_{\Gamma} = 0$ *for any elements* $\alpha \in [\Gamma X, Z]$, $\beta \in [\Gamma Y, Z]$ *and* $\gamma \in [\Gamma X, Z]$ *(or* $\gamma \in [\Gamma Y, Z]$*).*

Proof. Since $[\Gamma(X \times Y), Z]$ is an abelian group when Z is a Hopf space, we see that $[\overline{\alpha},\overline{\beta}]^{\overline{\gamma}} = 0$ in (2.2) or (2.3). Then we have the result.

Proposition 2.5. *Let Γ_1 be a co-grouplike space and Γ_2 a co-Hopf space. Let*

$$\Gamma_2 \colon [\Gamma_1(X \wedge Y), Z] \to [\Gamma_2\Gamma_1(X \wedge Y), \Gamma_2 Z]$$

be the Γ_2-suspension homomorphism, then $\Gamma_2([\alpha,\beta]^{\gamma}_{\Gamma_1}) = 0$ *for any elements* $\alpha \in [\Gamma_1 X, Z]$, $\beta \in [\Gamma_1 Y, Z]$ *and* $\gamma \in [\Gamma_1 X, Z]$ *(or* $\gamma \in [\Gamma_1 Y, Z]$*).*

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc}
[\Gamma_2\Gamma_1(X \wedge Y), \Gamma_2 Z] & \xrightarrow{(\Gamma_2\Gamma_1 q)^*} & [\Gamma_2\Gamma_1(X \times Y), \Gamma_2 Z] \\
\tau \downarrow & & \downarrow \tau \\
[\Gamma_1(X \wedge Y), \Gamma_2^*\Gamma_2 Z] & \xrightarrow{(\Gamma_1 q)^*} & [\Gamma_1(X \times Y), \Gamma_2^*\Gamma_2 Z]
\end{array}$$

The map $\tau \colon [\Gamma_2 A, B] \to [A, \Gamma_2^* B]$ in the above diagram is the adjoint isomorphism.
It follows then that

$$(\Gamma_2\Gamma_1 q)^* \colon [\Gamma_2\Gamma_1(X \wedge Y), \Gamma_2 Z] \longrightarrow [\Gamma_2\Gamma_1(X \times Y), \Gamma_2 Z]$$

is a monomorphism for the identification map $q\colon X \times Y \to X \wedge Y$. We see $\Gamma_2([\overline{\alpha}, \overline{\beta}]^{\overline{\gamma}}) = 0$ in $[\Gamma_2\Gamma_1(X \times Y), \Gamma_2 Z]$, since the group is an abelian group when Γ_1 is a co-grouplike space and Γ_2 is a co-Hopf space. Then we have

$$\begin{aligned}(\Gamma_2\Gamma_1 q)^*\{\Gamma_2([\alpha,\beta]^{\gamma}_{\Gamma_1})\} &= \Gamma_2([\alpha,\beta]^{\gamma}_{\Gamma_1}) \circ \Gamma_2\Gamma_1 q \\ &= \Gamma_2([\alpha,\beta]^{\gamma}_{\Gamma_1} \circ \Gamma_1 q) \\ &= \Gamma_2([\overline{\alpha},\overline{\beta}]^{\overline{\gamma}}) = 0.\end{aligned}$$

Thus we have $\Gamma_2([\alpha,\beta]^{\gamma}_{\Gamma_1}) = 0$.

Theorem 2.6. *Let $\overline{T}\colon Y \wedge X \to X \wedge Y$ be a switching map defined by $\overline{T}(y \wedge x) = x \wedge y$ for any element $y \wedge x \in Y \wedge X$. Let $\alpha \in [\Gamma X, Z]$, $\beta \in [\Gamma Y, Z]$ and $\gamma \in [\Gamma X, Z]$ (or $\gamma \in [\Gamma Y, Z]$) be any elements. Then the Γ-Whitehead product $[\alpha,\beta]^{\gamma}_{\Gamma} \in [\Gamma(X \wedge Y), Z]$ satisfies*

$$(\Gamma\overline{T})^*([\alpha,\beta]^{\gamma}_{\Gamma}) = \dot{-}[\beta,\alpha]^{\gamma}_{\Gamma}.$$

Proof. We prove the case $\gamma \in [\Gamma X, Z]$. The other case is proved similarly. Consider the following commutative diagram.

$$\begin{array}{ccc} [\Gamma(X \wedge Y), Z] & \xrightarrow{(\Gamma q)^*} & [\Gamma(X \times Y), Z] \\ {\scriptstyle (\Gamma\overline{T})^*}\downarrow & & \downarrow{\scriptstyle (\Gamma T)^*} \\ [\Gamma(Y \wedge X), Z] & \xrightarrow{(\Gamma q')^*} & [\Gamma(Y \times X), Z] \end{array}$$

In the above diagram, the maps $q\colon X \times Y \to X \wedge Y$ and $q'\colon Y \times X \to Y \wedge X$ are the identification maps and $T\colon Y \times X \to X \times Y$ is a switching map. Then we have

$$\begin{aligned}(\Gamma q')^*\{(\Gamma\overline{T})^*([\alpha,\beta]^{\gamma}_{\Gamma})\} &= (\Gamma T)^*\{(\Gamma q)^*([\alpha,\beta]^{\gamma}_{\Gamma})\} \\ &= (\Gamma T)^*(\overline{\gamma} \dotplus \overline{\alpha} \dotplus \overline{\beta} \,\dot{-}\, \overline{\alpha} \,\dot{-}\, \overline{\beta} \,\dot{-}\, \overline{\gamma}) \\ &= \overline{\gamma}' \dotplus \overline{\alpha}' \dotplus \overline{\beta}' \,\dot{-}\, \overline{\alpha}' \,\dot{-}\, \overline{\beta}' \,\dot{-}\, \overline{\gamma}' \\ &= \dot{-}(\overline{\gamma}' \dotplus \overline{\beta}' \dotplus \overline{\alpha}' \,\dot{-}\, \overline{\beta}' \,\dot{-}\, \overline{\alpha}' \,\dot{-}\, \overline{\gamma}') \\ &= (\Gamma q')^*(\dot{-}[\beta,\alpha]^{\gamma}_{\Gamma}),\end{aligned}$$

where $\overline{\gamma}' = \gamma \circ \Gamma p_2'\colon \Gamma(Y \times X) \to \Gamma X \to Z$, $\overline{\alpha}' = \alpha \circ \Gamma p_2'\colon \Gamma(Y \times X) \to \Gamma X \to Z$ and $\overline{\beta}' = \beta \circ \Gamma p_1'\colon \Gamma(Y \times X) \to \Gamma Y \to Z$. Since $(\Gamma q')^*$ is a monomorphism, we have the result.

We use the following well-known results on commutators in a group. See, for example, 5.53 and 5.54 of Rotman [8].

Lemma 2.7. *Let G be a group. We define $x^a = axa^{-1}$ and $[x, y] = xyx^{-1}y^{-1}$ for any elements $x, y, a \in G$. Then, for any x, y, $z \in G$, we have*

(i) $[x, y]^{-1} = [y, x]$

(ii) $[x, yz] = [x, y][x, z]^y = [x, y][x, z][[z, x], y]$

(iii) $[xy, z] = [y, z]^x[x, z] = [y, z][[z, y], x][x, z]$

(iv) $[x, [y^{-1}, z]]^y[y, [z^{-1}, x]]^z[z, [x^{-1}, y]]^x = 1,$

$[[y^{-1}, x], z]^y[[z^{-1}, y], x]^z[[x^{-1}, z], y]^x = 1$

(v) $[[x, y], z]^{-1}[[y^{-1}, x], z]^y = [[x, y], [z^{-1}, y]]^z$

Theorem 2.8. (i) *Let $\alpha, \beta \in [\Gamma X, Z]$ and $\gamma \in [\Gamma Y, Z]$ be any elements. Then we have*

$$[\alpha \dot{+} \beta, \gamma]_\Gamma = [\beta, \gamma]_\Gamma^\alpha \dot{+} [\alpha, \gamma]_\Gamma.$$

(ii) *Let $\alpha \in [\Gamma X, Z]$ and $\beta, \gamma \in [\Gamma Y, Z]$ be any elements. Then we have*

$$[\alpha, \beta \dot{+} \gamma]_\Gamma = [\alpha, \beta]_\Gamma \dot{+} [\alpha, \gamma]_\Gamma^\beta.$$

Proof. These are obtained by Lemma 2.7 and the definition of the Γ-Whitehead product.

We use the notation $\mathrm{cat}(X)$ in the sense of Whitehead [9], Chapter X, so that $\mathrm{cat}(X) = 0$ if and only if X is contractible, and $\mathrm{cat}(X) \le 1$ if and only if X is a co-Hopf space. Then Theorem 9 of Fox [5] reads

$$\mathrm{cat}(X \times Y) \le \mathrm{cat}(X) + \mathrm{cat}(Y).$$

Theorem 2.9. *Let Γ be a co-grouplike space. If X and Y are co-Hopf spaces, then*

(i) $[\alpha \dot{+} \beta, \gamma]_\Gamma = [\alpha, \gamma]_\Gamma \dot{+} [\beta, \gamma]_\Gamma$ *for any* α, $\beta \in [\Gamma X, Z]$ *and* $\gamma \in [\Gamma Y, Z]$.

(ii) $[\alpha, \beta \dot{+} \gamma]_\Gamma = [\alpha, \beta]_\Gamma \dot{+} [\alpha, \gamma]_\Gamma$ *for any* $\alpha \in [\Gamma X, Z]$ *and* β, $\gamma \in [\Gamma Y, Z]$.

Proof. (i) Since X and Y are co-Hopf spaces, we have $\mathrm{cat}(X) \leq 1$ and $\mathrm{cat}(Y) \leq 1$. It follows then that $\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y) \leq 2$. Then any 3-fold commutators vanish in $[\Gamma(X \times Y), Z] \cong [X \times Y, \Gamma^* Z]$, namely $[[\overline{\gamma}, \overline{\beta}], \overline{\alpha}] = 0$. Consider the following exact sequence of (2.1).

$$0 \longrightarrow [\Gamma(X \wedge Y), Z] \xrightarrow{(\Gamma q)^*} [\Gamma(X \times Y), Z] \xrightarrow{(\Gamma j)^*} [\Gamma(X \vee Y), Z] \longrightarrow 0.$$

We remark that $[\Gamma(X \wedge Y), Z]$ is an abelian group, since X (or Y) is a co-Hopf space. Then we have

$$\begin{aligned}(\Gamma q)^*([\alpha \dot{+} \beta, \gamma]_\Gamma) &= [\overline{\alpha \dot{+} \beta}, \overline{\gamma}] = [\overline{\alpha} \dot{+} \overline{\beta}, \overline{\gamma}] \\ &= [\overline{\beta}, \overline{\gamma}] \dot{+} [[\overline{\gamma}, \overline{\beta}], \overline{\alpha}] \dot{+} [\overline{\alpha}, \overline{\gamma}] \\ &= [\overline{\beta}, \overline{\gamma}] \dot{+} [\overline{\alpha}, \overline{\gamma}] \\ &= [\overline{\alpha}, \overline{\gamma}] \dot{+} [\overline{\beta}, \overline{\gamma}] \\ &= (\Gamma q)^*([\alpha, \gamma]_\Gamma \dot{+} [\beta, \gamma]_\Gamma).\end{aligned}$$

Hence we have $[\alpha \dot{+} \beta, \gamma]_\Gamma = [\alpha, \gamma]_\Gamma \dot{+} [\beta, \gamma]_\Gamma$.
(ii) is proved similarly.

As a special case of Theorem 2.9, we have the following result.

Corollary 2.10. *Let $X = \Gamma_1 U$ and $Y = \Gamma_1 V$ for some spaces U and V and a co-Hopf space Γ_1. Let Γ be a co-grouplike space. Then we have*
(i) $[\alpha \dot{+} \beta, \gamma]_\Gamma = [\alpha, \gamma]_\Gamma \dot{+} [\beta, \gamma]_\Gamma$ *for any* $\alpha, \beta \in [\Gamma X, Z]$ *and* $\gamma \in [\Gamma Y, Z]$.
(ii) $[\alpha, \beta \dot{+} \gamma]_\Gamma = [\alpha, \beta]_\Gamma \dot{+} [\alpha, \gamma]_\Gamma$ *for any* $\alpha \in [\Gamma X, Z]$ *and* $\beta, \gamma \in [\Gamma Y, Z]$.

Now, we consider 3-fold Γ-Whitehead product of type $[[\alpha_1, \alpha_2]_\Gamma, \alpha_3]_\Gamma$ and an action of an element γ on it, namely, $[[\alpha_1, \alpha_2]_\Gamma, \alpha_3]_\Gamma^\gamma$.

Let X_1, X_2, X_3 and Z be any spaces. Let $\alpha_i \in [\Gamma X_i, Z]$ for $i = 1, 2, 3$. Since $[\alpha_1, \alpha_2]_\Gamma \in [\Gamma(X_1 \wedge X_2), Z]$, we have $[[\alpha_1, \alpha_2]_\Gamma, \alpha_3]_\Gamma \in [\Gamma(X_1 \wedge X_2 \wedge X_3), Z]$. We put

$$\beta_i = \alpha_i \circ \Gamma p_i \colon \Gamma(X_1 \times X_2 \times X_3) \longrightarrow \Gamma X_i \longrightarrow Z$$

which is the composite of α_i and the Γ-suspension of the projection $p_i \colon X_1 \times X_2 \times X_3 \to X_i$. We remark that the natural identification map $q \colon X_1 \times X_2 \times X_3 \to X_1 \wedge X_2 \wedge X_3$ implies a monomorphism

$$(\Gamma q)^* \colon [\Gamma(X_1 \wedge X_2 \wedge X_3), Z] \to [\Gamma(X_1 \times X_2 \times X_3), Z],$$

since it factors through following two monomorphisms

$$\begin{array}{c}[\Gamma(X_1 \wedge X_2 \wedge X_3), Z] \\ \downarrow (\Gamma q_{12} \wedge 1_{X3})^* \\ [\Gamma\{(X_1 \times X_2) \wedge X_3\}, Z] \\ \downarrow (\Gamma q_{12,3})^* \\ [\Gamma(X_1 \times X_2 \times X_3), Z].\end{array}$$

We remark that

$$(\Gamma q)^*([[\alpha_1, \alpha_2]_\Gamma, \alpha_3]_\Gamma) = [[\beta_1, \beta_2], \beta_3],$$

since, by the above factorization of $(\Gamma q)^*$ and Proposition 2.2 (ii), we have

$$\begin{aligned}(\Gamma q)^*([[\alpha_1, \alpha_2]_\Gamma, \alpha_3]_\Gamma) &= (\Gamma q_{12,3})^*(\Gamma q_{12} \wedge 1_{X3})^*([[\alpha_1, \alpha_2]_\Gamma, \alpha_3]_\Gamma) \\ &= (\Gamma q_{12,3})^*([[\alpha_1, \alpha_2]_\Gamma \circ \Gamma q_{12}, \alpha_3]_\Gamma) \\ &= ([[\overline{\alpha_1}, \overline{\alpha_2}], \alpha_3]_\Gamma) \circ \Gamma q_{12,3} \\ &= [[\beta_1, \beta_2], \beta_3],\end{aligned}$$

where $\overline{\alpha_i} = \alpha_i \circ \Gamma p_i \colon \Gamma(X_1 \times X_2) \to Z$ for $i = 1$, 2.

Let $\gamma \in [\Gamma X_k, Z]$ for some $k = 1, 2, 3$ and put

$$\delta = \gamma \circ \Gamma p_k \colon \Gamma(X_1 \times X_2 \times X_3) \longrightarrow \Gamma X_k \longrightarrow Z.$$

We define $[[\alpha_1, \alpha_2]_\Gamma, \alpha_3]_\Gamma^\gamma$ (the action of γ on $[[\alpha_1, \alpha_2]_\Gamma, \alpha_3]_\Gamma$) by

$$(\Gamma q)^*([\alpha_1, \alpha_2]_\Gamma, \alpha_3]_\Gamma^\gamma) = \delta \dot{+} [[\beta_1, \beta_2], \beta_3] \dot{-} \delta = [[\beta_1, \beta_2], \beta_3]^\delta.$$

We have to show that $[[\alpha_1, \alpha_2]_\Gamma, \alpha_3]_\Gamma^\gamma$ is an element of $[\Gamma(X_1 \wedge X_2 \wedge X_3), Z]$. To show this, we consider the following diagram.

$$\begin{array}{ccccc}\Gamma(X_1 \times X_2 \times X_3) & \xrightarrow{\Gamma q_{12,3}} & \Gamma((X_1 \times X_2) \wedge X_3) & \xrightarrow{\Gamma q_{12} \wedge 1_{X3}} & \Gamma(X_1 \wedge X_2 \wedge X_3) \\ \uparrow{\scriptstyle \Gamma j_{12,3}} & & \uparrow{\scriptstyle \Gamma(j_{12} \wedge 1_{X3})} & & \\ \Gamma((X_1 \times X_2) \vee X_3) & & \Gamma((X_1 \vee X_2) \wedge X_3) & & \end{array}$$

The row of the above diagram is the factorization of $\Gamma q \colon \Gamma(X_1 \times X_2 \times X_3) \to \Gamma(X_1 \wedge X_2 \wedge X_3)$. Since $[[\beta_1, \beta_2], \beta_3]^\delta \circ \Gamma j_{12,3} = 0$, there exists an element $[[\beta_1, \beta_2], \beta_3]^{\gamma}_{\Gamma} \in [\Gamma(X_1 \times X_2) \wedge X_3, Z]$ such that $[[\beta_1, \beta_2], \beta_3]^{\gamma}_{\Gamma} \circ \Gamma q_{12,3} = [[\beta_1, \beta_2], \beta_3]^{\delta}$. Moreover we see $[[\beta_1, \beta_2], \beta_3]^{\gamma}_{\Gamma} \circ \Gamma(j_{12} \wedge 1_{X3}) = [[\beta_1, \beta_2] \circ \Gamma j_{12}, \beta_3]^{\gamma'}_{\Gamma} = 0$ by Proposition 2.2 (ii), then there exists an element $[[\alpha_1, \alpha_2]_{\Gamma}, \alpha_3]^{\gamma}_{\Gamma} \in [\Gamma(X_1 \wedge X_2 \wedge X_3), Z]$ such that

$$(\Gamma q_{12} \wedge 1_{X3})^*([[\alpha_1, \alpha_2]_{\Gamma}, \alpha_3]^{\gamma}_{\Gamma}) = [[\beta_1, \beta_2], \beta_3]^{\gamma}_{\Gamma}$$

so that

$$\begin{aligned}(\Gamma q)^*([[\alpha_1, \alpha_2]_{\Gamma}, \alpha_3]^{\gamma}_{\Gamma}) &= [[\alpha_1, \alpha_2]_{\Gamma}, \alpha_3]^{\gamma}_{\Gamma} \circ \Gamma(q_{12} \wedge 1_{X3}) \circ \Gamma q_{12,3} \\ &= [[\beta_1, \beta_2], \beta_3]^{\delta}.\end{aligned}$$

This shows that the above definition is well-defined. We see that $[[\alpha_1, \alpha_2]_{\Gamma}, \alpha_3]^{0}_{\Gamma} = [[\alpha_1, \alpha_2]_{\Gamma}, \alpha_3]_{\Gamma}$ and

$$([[\alpha_1, \alpha_2]_{\Gamma}, \alpha_3]^{\gamma_1}_{\Gamma})^{\gamma_2} = [[\alpha_1, \alpha_2]_{\Gamma}, \alpha_3]^{\gamma_2 \dot{+} \gamma_1}_{\Gamma}.$$

Theorem 2.11. *Let Γ be a co-grouplike space. Let $\alpha_1 \in [\Gamma X_1, Z]$, $\alpha_2 \in [\Gamma X_2, Z]$ and $\alpha_3 \in [\Gamma X_3, Z]$ be any elements. Then we have*

$$(2.4) \quad (\Gamma \overline{T}_{213})^*([[\dot{-}\alpha_2, \alpha_1]_{\Gamma}, \alpha_3]^{\alpha_2}_{\Gamma}) \dot{+} (\Gamma \overline{T}_{321})^*([[\dot{-}\alpha_3, \alpha_2]_{\Gamma}, \alpha_1]^{\alpha_3}_{\Gamma}) \dot{+} (\Gamma \overline{T}_{132})^*([[\dot{-}\alpha_1, \alpha_3]_{\Gamma}, \alpha_2]^{\alpha_1}_{\Gamma}) = 0,$$

where $\overline{T}_{ijk}\colon X_1 \wedge X_2 \wedge X_3 \to X_i \wedge X_j \wedge X_k$ is the permutation of factors.

Moreover, if X_1, X_2 and X_3 are co-Hopf spaces, then we have

$$[[\alpha_1, \alpha_2]_{\Gamma}, \alpha_3]_{\Gamma} \dot{+} (\Gamma \overline{T}_{231})^*([[\alpha_2, \alpha_3]_{\Gamma}, \alpha_1]_{\Gamma}) \dot{+} (\Gamma \overline{T}_{312})^*([[\alpha_3, \alpha_1]_{\Gamma}, \alpha_2]_{\Gamma}) = 0.$$

Proof. Let $\beta_i = \alpha_i \circ \Gamma p_i\colon \Gamma(X_1 \times X_2 \times X_3) \longrightarrow \Gamma X_i \longrightarrow Z$ be the composite of α_i and the Γ-suspension of the projection p_i. Consider a monomorphism

$$(\Gamma q)^*\colon [\Gamma(X_1 \wedge X_2 \wedge X_3), Z] \to [\Gamma(X_1 \times X_2 \times X_3), Z]$$

induced by the natural identification map $q\colon X_1 \times X_2 \times X_3 \to X_1 \wedge X_2 \wedge X_3$ as remarked before.

Let $q_{ijk}\colon X_i \times X_j \times X_k \to X_i \wedge X_j \wedge X_k$ be the natural projections and $T_{ijk}\colon X_1 \times X_2 \times X_3 \to X_i \times X_j \times X_k$ be the permutation of factors so that we have the following commutative diagram, where $\gamma_{ijk} = [[\dot{-}\alpha_i, \alpha_j]_{\Gamma}, \alpha_k]^{\alpha_i}_{\Gamma}$.

$$\begin{array}{ccccc}
\Gamma(X_i \times X_j \times X_k) & \xrightarrow{\Gamma q_{ijk}} & \Gamma(X_i \wedge X_j \wedge X_k) & \xrightarrow{\gamma_{ijk}} & Z \\
\Gamma T_{ijk} \uparrow & & \uparrow \Gamma \overline{T}_{ijk} & & \\
\Gamma(X_1 \times X_2 \times X_3) & \xrightarrow{\Gamma q} & \Gamma(X_1 \wedge X_2 \wedge X_3) & &
\end{array}$$

Firstly we remark that

$$\begin{aligned}
(\Gamma q)^*\{(\Gamma\overline{T}_{ijk})^*([[\dot{-}\alpha_i, \alpha_j]_\Gamma, \alpha_k]_\Gamma^{\alpha_i})\} &= (\Gamma T_{ijk})^*\{(\Gamma q_{ijk})^*([[\dot{-}\alpha_i, \alpha_j]_\Gamma, \alpha_k]_\Gamma^{\alpha_i})\} \\
&= (\Gamma T_{ijk})^*([[\dot{-}\beta'_i, \beta'_j], \beta'_k]^{\beta'_i}) \\
&= [[\dot{-}\beta_i, \beta_j], \beta_k]^{\beta_i},
\end{aligned}$$

where $\beta'_s = \alpha_s \circ \Gamma p'_s \colon \Gamma(X_i \times X_j \times X_k) \to \Gamma X_s \to Z$, since $\beta'_s \circ \Gamma T_{ijk} = \alpha_s \circ \Gamma p'_s \circ \Gamma T_{ijk} = \alpha_s \circ \Gamma p_s = \beta_s$. Then we have a relation

$$\begin{aligned}
(\Gamma q)^*\{(\Gamma\overline{T}_{213})^*([[\dot{-}\alpha_2, \alpha_1]_\Gamma, \alpha_3]_\Gamma^{\alpha_2}) \dot{+} (\Gamma\overline{T}_{321})^*([[\dot{-}\alpha_3, \alpha_2]_\Gamma, \alpha_1]_\Gamma^{\alpha_3}) \\
\dot{+} (\Gamma\overline{T}_{132})^*([[\dot{-}\alpha_1, \alpha_3]_\Gamma, \alpha_2]_\Gamma^{\alpha_1})\} \\
= [[\dot{-}\beta_2, \beta_1], \beta_3]^{\beta_2} \dot{+} [[\dot{-}\beta_3, \beta_2], \beta_1]^{\beta_3} \dot{+} [[\dot{-}\beta_1, \beta_3], \beta_2]^{\beta_1} = 0
\end{aligned}$$

in $[\Gamma(X_1 \times X_2 \times X_3), Z]$ by Lemma 2.7(iv).
Then the first relation holds by the definition of Γ-Whitehead product.
Now suppose that X_i is a co-Hopf space and hence $\mathrm{cat}(X_i) \leq 1$ for $i = 1, 2$ and 3. Then we have

$$\mathrm{cat}(X_1 \times X_2 \times X_3) \leq \mathrm{cat}(X_1) + \mathrm{cat}(X_2) + \mathrm{cat}(X_3) \leq 3.$$

Then any four-fold commutators vanish in

$$[\Gamma(X_1 \times X_2 \times X_3), Z] \cong [X_1 \times X_2 \times X_3, \Gamma^* Z].$$

Thus we have

$$\dot{-}[[\beta_1, \beta_2], \beta_3] \dot{+} [[\dot{-}\beta_2, \beta_1], \beta_3]^{\beta_2} = [[\beta_1, \beta_2], [\dot{-}\beta_3, \beta_2]]^{\beta_3} = 0$$

by Lemma 2.7 (v). It follows that

$$[[\dot{-}\beta_2, \beta_1], \beta_3]^{\beta_2} = [[\beta_1, \beta_2], \beta_3].$$

Similarly we have

$$[[\dot{-}\beta_3, \beta_2], \beta_1]^{\beta_3} = [[\beta_2, \beta_3], \beta_1]$$

and

$$[[\dot{-}\beta_1, \beta_3], \beta_2]^{\beta_1} = [[\beta_3, \beta_1], \beta_2].$$

Then we have the result.

3. Γ^*-Whitehead product. Let $\mathrm{map}^*(A, Z)$ be the space of base point preserving maps from A to Z. We define $\Gamma^* X = \mathrm{map}^*(\Gamma, X)$ (the Γ-*loop space*) for any co-Hopf space Γ. A map $f\colon X \to Y$ induces a Γ-*loop map* $\Gamma^* f\colon \Gamma^* X \to \Gamma^* Y$. We see $\Gamma^* g \circ \Gamma^* f = \Gamma^*(g \circ f)$ for any maps $f\colon X \to Y$ and $g\colon Y \to Z$. We define

$$\theta_X^* = \mathrm{map}^*(\theta, X)\colon \mathrm{map}^*(\Gamma, X) \times \mathrm{map}^*(\Gamma, X) \cong \mathrm{map}^*(\Gamma \vee \Gamma, X) \to \mathrm{map}^*(\Gamma, X).$$

If Γ is a co-grouplike space, then $\Gamma^* X$ is a grouplike space with multiplication θ_X^* for any space X (cf. 6.3.14(b) of Maunder [7]). There is the following isomorphism as grouplike spaces;

$$\Gamma^*(X \times Y) \cong \Gamma^* X \times \Gamma^* Y.$$

If Z is a grouplike space with a grouplike structure $\mu\colon Z \times Z \to Z$, then we define $\alpha \dotplus \beta\colon A \to Z$ by

$$\alpha \dotplus \beta = \mu \circ (\alpha \times \beta) \circ \triangle_A$$

for any maps $\alpha, \beta\colon A \to Z$, where $\triangle_A\colon A \to A \times A$ is the diagonal map.

Let $X \flat Y$ be the homotopy fibre of the inclusion map $j\colon X \vee Y \to X \times Y$. We shall now define the Γ^*-*Whitehead product*

$$[\alpha, \beta]_{\Gamma^*} \in [W, \Gamma^*(X \flat Y)]$$

for any elements $\alpha \in [W, \Gamma^* X]$ and $\beta \in [W, \Gamma^* Y]$. If $\Gamma = S^1$, then the Γ^*-Whitehead product $[\alpha, \beta]_{\Gamma^*}$ coincides with the *dual product* in Definition 6.2 of Arkowitz [1]. (We have to remark that Arkowitz [1] uses the commutator $(x, y) = x^{-1} y^{-1} x y$ for the definition of the dual product $[\alpha, \beta]$. Then we have $[\alpha, \beta]_{S^{1*}} = [-\alpha, -\beta]$.)

Consider a fibration sequence

$$\cdots \longrightarrow \Omega\Gamma^*(X \vee Y) \xrightarrow{\Omega\Gamma^* j} \Omega\Gamma^*(X \times Y) \xrightarrow{\partial} \Gamma^*(X \flat Y) \xrightarrow{\Gamma^* i} \Gamma^*(X \vee Y) \xrightarrow{\Gamma^* j} \Gamma^*(X \times Y).$$

Let $p_1\colon X \times Y \to X$, $p_2\colon X \times Y \to Y$ be the projections and $j_1\colon X \to X \vee Y$, $j_2\colon Y \to X \vee Y$ the inclusion maps. We define $\sigma = \Gamma^*(j_1 \circ p_1) \dotplus$

$\Gamma^*(j_2 \circ p_2)$. Then we have $(\Gamma^* j) \circ \sigma \simeq 1_{\Gamma^*(X \times Y)}$. Therefore there is a short exact sequence of groups

(3.1)

$$0 \longrightarrow [W, \Gamma^*(X \flat Y)] \xrightarrow{(\Gamma^* i)_*} [W, \Gamma^*(X \vee Y)] \xrightarrow{(\Gamma^* j)_*} [W, \Gamma^*(X \times Y)] \longrightarrow 0$$

for any space W. We put

$$\underline{\alpha} = (\Gamma^* j_1) \circ \alpha \colon W \to \Gamma^*(X \vee Y) \text{ and } \underline{\beta} = (\Gamma^* j_2) \circ \beta \colon W \to \Gamma^*(X \vee Y)$$

for elements $\alpha \in [W, \Gamma^* X]$ and $\beta \in [W, \Gamma^* Y]$. We define

$$(3.2) \qquad [\underline{\alpha}, \underline{\beta}] = \underline{\alpha} \dotplus \underline{\beta} \dot{-} \underline{\alpha} \dot{-} \underline{\beta} \in [W, \Gamma^*(X \vee Y)].$$

We see that $(\Gamma^* j)_*([\underline{\alpha}, \underline{\beta}]) = 0 \in [W, \Gamma^*(X \times Y)]$, for $\Gamma^*(X \times Y) \cong \Gamma^* X \times \Gamma^* Y$. Since $(\Gamma^* i)_* \colon [W, \Gamma^*(X \flat Y)] \to [W, \Gamma^*(X \vee Y)]$ is a monomorphism, there exists a unique homotopy class $[\alpha, \beta]_{\Gamma^*} \in [W, \Gamma^*(X \flat Y)]$ such that

$$(\Gamma^* i)_*([\alpha, \beta]_{\Gamma^*}) = [\underline{\alpha}, \underline{\beta}].$$

We refer to the element $[\alpha, \beta]_{\Gamma^*} \in [W, \Gamma^*(X \flat Y)]$ as the Γ^*-*Whitehead product* of $\alpha \in [W, \Gamma^* X]$ and $\beta \in [W, \Gamma^* Y]$.

If $\alpha \in [W, \Gamma^* X]$, $\beta \in [W, \Gamma^* Y]$ and $\gamma \in [W, \Gamma^* X]$ (or $\gamma \in [W, \Gamma^* Y]$), then γ acts on the Γ^*-Whitehead product $[\alpha, \beta]_{\Gamma^*}$, denoted by $[\alpha, \beta]^{\gamma}_{\Gamma^*}$, by the formula

$$(3.3) \qquad (\Gamma^* i)_*([\alpha, \beta]^{\gamma}_{\Gamma^*}) = \underline{\gamma} \dotplus [\underline{\alpha}, \underline{\beta}] \dot{-} \underline{\gamma} = [\underline{\alpha}, \underline{\beta}]^{\underline{\gamma}}$$

for the monomorphism $(\Gamma^* i)_* \colon [W, \Gamma^*(X \flat Y)] \to [W, \Gamma^*(X \vee Y)]$ in (3.1).

The element $[\alpha, \beta]^{\gamma}_{\Gamma^*}$ does not always have the form $[\alpha', \beta']_{\Gamma^*}$ for some element $\alpha' \in [W, \Gamma^* X]$ and $\beta' \in [W, \Gamma^* Y]$. But, actually, any element $\gamma \in [W, \Gamma^*(X \vee Y)]$ acts on any element $\alpha \in [W, \Gamma^*(X \flat Y)]$ by the formula

$$(\Gamma^* i)_*(\alpha^{\gamma}) = \gamma \dotplus \Gamma^* i \circ \alpha \dot{-} \gamma = (\Gamma^* i \circ \alpha)^{\gamma}.$$

Then we have formulas: If $[\alpha, \beta]_{\Gamma^*} = 0$, then $[\alpha, \beta]^{\gamma}_{\Gamma^*} = 0$ for any γ; $[\alpha, \beta]^{0}_{\Gamma^*} = [\alpha, \beta]_{\Gamma^*}$; $([\alpha, \beta]^{\gamma_1}_{\Gamma^*})^{\gamma_2} = [\alpha, \beta]^{\gamma_2 \dotplus \gamma_1}_{\Gamma^*}$.

Proposition 3.1. *Let* $\alpha \in [W, \Gamma^* X]$, $\beta \in [W, \Gamma^* Y]$ *and* $\gamma \in [W, \Gamma^* X]$ *(or* $\gamma \in [W, \Gamma^* Y]$*) be any elements. Then we have the following results.*

(i) $[\alpha, \beta]^{\gamma}_{\Gamma^*} \circ \omega = [\alpha \circ \omega, \beta \circ \omega]^{\gamma \circ \omega}_{\Gamma^*}$ *for any* $\omega \colon W' \to W$.

(ii) $\Gamma^*(\delta \flat \varepsilon) \circ [\alpha, \beta]^{\gamma}_{\Gamma^*} = [(\Gamma^*\delta) \circ \alpha, (\Gamma^*\varepsilon) \circ \beta]^{\gamma'}_{\Gamma^*}$ *for any* $\delta \colon X \to X'$ *and* $\varepsilon \colon Y \to Y'$, *where* $\gamma' = (\Gamma^*\delta) \circ \gamma$ *if* $\gamma \in [W, \Gamma^* X]$ *and* $\gamma' = (\Gamma^*\varepsilon) \circ \gamma$ *if* $\gamma \in [W, \Gamma^* Y]$.

Proof. (i) Consider the following commutative diagram.

$$\begin{array}{ccccc} 0 & \longrightarrow & [W, \Gamma^*(X \flat Y)] & \xrightarrow{(\Gamma^* i)_*} & [W, \Gamma^*(X \vee Y)] \\ & & \Big\downarrow \omega^* & & \Big\downarrow \omega^* \\ 0 & \longrightarrow & [W', \Gamma^*(X \flat Y)] & \xrightarrow{(\Gamma^* i)_*} & [W', \Gamma^*(X \vee Y)] \end{array}$$

We have

$$\begin{aligned} (\Gamma^* i)_*([\alpha, \beta]^{\gamma}_{\Gamma^*} \circ \omega) &= (\Gamma^* i)_* \{\omega^*([\alpha, \beta]^{\gamma}_{\Gamma^*})\} = \omega^* \{(\Gamma^* i)_*([\alpha, \beta]^{\gamma}_{\Gamma^*})\} \\ &= \omega^*(\underline{\gamma} \dotplus \underline{\alpha} \dotplus \underline{\beta} \,\dot{-}\, \underline{\alpha} \,\dot{-}\, \underline{\beta} \,\dot{-}\, \underline{\gamma}) \\ &= (\underline{\gamma} \circ \omega) \dotplus (\underline{\alpha} \circ \omega) \dotplus (\underline{\beta} \circ \omega) \,\dot{-}\, (\underline{\alpha} \circ \omega) \,\dot{-}\, (\underline{\beta} \circ \omega) \,\dot{-}\, (\underline{\gamma} \circ \omega) \\ &= \underline{\gamma \circ \omega} \dotplus \underline{\alpha \circ \omega} \dotplus \underline{\beta \circ \omega} \,\dot{-}\, \underline{\alpha \circ \omega} \,\dot{-}\, \underline{\beta \circ \omega} \,\dot{-}\, \underline{\gamma \circ \omega} \\ &= (\Gamma^* i)_*([\alpha \circ \omega, \beta \circ \omega]^{\gamma \circ \omega}_{\Gamma^*}). \end{aligned}$$

Then we have $[\alpha, \beta]^{\gamma}_{\Gamma^*} \circ \omega = [\alpha \circ \omega, \beta \circ \omega]^{\gamma \circ \omega}_{\Gamma^*}$.

(ii) We prove the case $\gamma \in [W, \Gamma^* X]$. The other case is proved similarly. Consider the following commutative diagram.

$$\begin{array}{ccccc} 0 & \longrightarrow & [W, \Gamma^*(X \flat Y)] & \xrightarrow{(\Gamma^* i)_*} & [W, \Gamma^*(X \vee Y)] \\ & & \Big\downarrow \{\Gamma^*(\delta \flat \varepsilon)\}_* & & \Big\downarrow \{\Gamma^*(\delta \vee \varepsilon)\}_* \\ 0 & \longrightarrow & [W, \Gamma^*(X' \flat Y')] & \xrightarrow{(\Gamma^* i')_*} & [W, \Gamma^*(X' \vee Y')] \end{array}$$

Let $j_1' \colon X' \to X' \vee Y'$ and $j_2' \colon Y' \to X' \vee Y'$ be the inclusion maps. We write $(\underline{\zeta})' = \Gamma^* j_1' \circ \zeta$ and $(\underline{\eta})' = \Gamma^* j_2' \circ \eta$ for any maps $\zeta \colon W \to \Gamma^* X'$ and

$\eta\colon W \to \Gamma^*Y'$. Now we see

$$
\begin{aligned}
(\Gamma i')_*\{\Gamma^*(\delta \flat \varepsilon)\circ[\alpha,\beta]^{\gamma}_{\Gamma^*}\} &= (\Gamma^* i')_*\{(\Gamma^*(\delta \flat \varepsilon))_*([\alpha,\beta]^{\gamma}_{\Gamma^*})\}\\
&= (\Gamma^*(\delta\vee\varepsilon))_*\{(\Gamma^* i)_*([\alpha,\beta]^{\gamma}_{\Gamma^*})\}\\
&= (\Gamma^*(\delta\vee\varepsilon))_*(\underline{\gamma}\dotplus\underline{\alpha}\dotplus\underline{\beta}\mathbin{\dot{-}}\underline{\alpha}\mathbin{\dot{-}}\underline{\beta}\mathbin{\dot{-}}\underline{\gamma})\\
&= \{\Gamma^*(\delta\vee\varepsilon)\circ\underline{\gamma}\}\dotplus\{\Gamma^*(\delta\vee\varepsilon)\circ\underline{\alpha}\}\dotplus\{\Gamma^*(\delta\vee\varepsilon)\circ\underline{\beta}\}\\
&\qquad \mathbin{\dot{-}}\{\Gamma^*(\delta\vee\varepsilon)\circ\underline{\alpha}\}\mathbin{\dot{-}}\{\Gamma^*(\delta\vee\varepsilon)\circ\underline{\beta}\}\mathbin{\dot{-}}\{\Gamma^*(\delta\vee\varepsilon)\circ\underline{\gamma}\}\\
&= (\underline{\Gamma^*\delta\circ\gamma})'\dotplus(\underline{\Gamma^*\delta\circ\alpha})'\dotplus(\underline{\Gamma^*\varepsilon\circ\beta})'\\
&\qquad \mathbin{\dot{-}}(\underline{\Gamma^*\delta\circ\alpha})'\mathbin{\dot{-}}(\underline{\Gamma^*\varepsilon\circ\beta})'\mathbin{\dot{-}}(\underline{\Gamma^*\delta\circ\gamma})'\\
&= (\Gamma^* i')_*([\Gamma^*\delta\circ\alpha,\Gamma^*\varepsilon\circ\beta]^{\gamma'}_{\Gamma^*}),
\end{aligned}
$$

since we have the following commutative diagrams for α and β and similar one for γ:

$$
\begin{array}{ccccc}
W & \xrightarrow{\alpha} & \Gamma^*X & \xrightarrow{\Gamma^*j_1} & \Gamma^*(X\vee Y)\\
 & & \Big\downarrow{\scriptstyle\Gamma^*\delta} & & \Big\downarrow{\scriptstyle\Gamma^*(\delta\vee\varepsilon)}\\
 & & \Gamma^*X' & \xrightarrow{\Gamma^*j_1'} & \Gamma^*(X'\vee Y')
\end{array}
$$

$$
\begin{array}{ccccc}
W & \xrightarrow{\beta} & \Gamma^*Y & \xrightarrow{\Gamma^*j_2} & \Gamma^*(X\vee Y)\\
 & & \Big\downarrow{\scriptstyle\Gamma^*\varepsilon} & & \Big\downarrow{\scriptstyle\Gamma^*(\delta\vee\varepsilon)}\\
 & & \Gamma^*Y' & \xrightarrow{\Gamma^*j_2'} & \Gamma^*(X'\vee Y')
\end{array}
$$

Proposition 3.2. *If W is a co-Hopf space, then $[\alpha,\beta]^{\gamma}_{\Gamma^*}=0$ for any elements $\alpha\in[W,\Gamma^*X]$, $\beta\in[W,\Gamma^*Y]$ and $\gamma\in[W,\Gamma^*X]$ (or $\gamma\in[W,\Gamma^*Y]$).*

Proof. If W is a co-Hopf space, then $[W,\Gamma^*(X\vee Y)]$ is an abelian group. It follows that $[\underline{\alpha},\underline{\beta}]^{\underline{\gamma}}=\underline{\gamma}\dotplus\underline{\alpha}\dotplus\underline{\beta}\mathbin{\dot{-}}\underline{\alpha}\mathbin{\dot{-}}\underline{\beta}\mathbin{\dot{-}}\underline{\gamma}=0\in[W,\Gamma^*(X\vee Y)]$ in (3.2) or (3.3). Then we have the result by the definition of Γ^*-Whitehead product.

Proposition 3.3. *Let Γ_1 be a co-grouplike space and Γ_2 a co-Hopf space. If $\Gamma_2^*\colon [W,\Gamma_1^*(X \flat Y)] \to [\Gamma_2^* W, \Gamma_2^*\Gamma_1^*(X \flat Y)]$ is the Γ_2-loop homomorphism, then $\Gamma_2^*([\alpha,\beta]^{\gamma}_{\Gamma_1^*}) = 0$ for any $\alpha \in [W,\Gamma^* X]$, $\beta \in [W,\Gamma^* Y]$ and $\gamma \in [W,\Gamma^* X]$ (or $\gamma \in [W,\Gamma^* Y]$).*

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc}
[\Gamma_2^* W, \Gamma_2^*\Gamma_1^*(X \flat Y)] & \xrightarrow{(\Gamma_2^*\Gamma_1^* i)_*} & [\Gamma_2^* W, \Gamma_2^*\Gamma_1^*(X \vee Y)] \\
\tau^* \downarrow & & \downarrow \tau^* \\
[\Gamma_2\Gamma_2^* W, \Gamma_1^*(X \flat Y)] & \xrightarrow{(\Gamma_1 i)_*} & [\Gamma_2\Gamma_2^* W, \Gamma_1^*(X \vee Y)]
\end{array}$$

The map $\tau^*\colon [A,\Gamma_2^* B] \to [\Gamma_2 A, B]$ in the above diagram is the adjoint isomorphism. We remark that $(\Gamma_1 i)_*\colon [\Gamma_2\Gamma_2^* W, \Gamma_1^*(X \flat Y)] \to [\Gamma_2\Gamma_2^* W, \Gamma_1^*(X \vee Y)]$ is a monomorphism and hence

$$(\Gamma_2^*\Gamma_1^* i)_*\colon [\Gamma_2^* W, \Gamma_2^*\Gamma_1^*(X \flat Y)] \to [\Gamma_2^* W, \Gamma_2^*\Gamma_1^*(X \vee Y)]$$

is a monomorphism.
Now we see $\Gamma_2^*([\underline{\alpha},\underline{\beta}]^{\underline{\gamma}}) = 0$ in $[\Gamma_2^* W, \Gamma_2^*\Gamma_1^*(X \vee Y)]$, since the group is an abelian group. Then we have

$$\begin{aligned}
(\Gamma_2^*\Gamma_1^* i)_*\{\Gamma_2^*([\alpha,\beta]^{\gamma}_{\Gamma_1^*})\} &= \Gamma_2^*\Gamma_1^* i \circ \Gamma_2^*([\alpha,\beta]^{\gamma}_{\Gamma_1^*}) \\
&= \Gamma_2^*(\Gamma_1^* i \circ [\alpha,\beta]^{\gamma}_{\Gamma_1^*}) \\
&= \Gamma_2^*([\underline{\alpha},\underline{\beta}]^{\underline{\gamma}}) = 0
\end{aligned}$$

for the inclusion map $i\colon X \flat Y \to X \vee Y$. It follows that $\Gamma_2^*([\alpha,\beta]^{\gamma}_{\Gamma_1^*}) = 0$.

Theorem 3.4. *Let $\tilde{T}\colon\ X \flat Y \to Y \flat X$ be a "switching" map of induced fibres. Let $\alpha \in [W,\Gamma^* X]$, $\beta \in [W,\Gamma^* Y]$ and $\gamma \in [W,\Gamma^* X]$ (or $\gamma \in [W,\Gamma^* Y]$) be any elements. Then the Γ^*-Whitehead product $[\alpha,\beta]^{\gamma}_{\Gamma^*}$ satisfies*

$$(\Gamma^*\tilde{T})_*([\alpha,\beta]^{\gamma}_{\Gamma^*}) = \underset{\cdot}{-}[\beta,\alpha]^{\gamma}_{\Gamma^*}.$$

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc} [W,\Gamma^*(X \flat Y)] & \xrightarrow{\quad (\Gamma^* i)_* \quad} & [W,\Gamma^*(X \vee Y)] \\ {\scriptstyle (\Gamma^*\tilde{T})_*}\downarrow & & \downarrow{\scriptstyle (\Gamma^* T')_*} \\ [W,\Gamma^*(Y \flat X)] & \xrightarrow{\quad (\Gamma^* i')_* \quad} & [W,\Gamma^*(Y \vee X)] \end{array}$$

In the above diagram, the maps $i\colon X \flat Y \to X \vee Y$ and $i'\colon Y \flat X \to Y \vee X$ are the inclusion maps and $T'\colon X \vee Y \to Y \vee X$ is a switching map. Then we have

$$\begin{aligned} (\Gamma^* i')_*\{(\Gamma^*\tilde{T})_*([\alpha,\beta]^{\gamma}_{\Gamma^*})\} &= (\Gamma^* T')_*\{(\Gamma^* i)_*([\alpha,\beta]^{\gamma}_{\Gamma^*})\} \\ &= (\Gamma^* T')_*(\underline{\gamma} \dotplus \underline{\alpha} \dotplus \underline{\beta} \mathbin{\dot{-}} \underline{\alpha} \mathbin{\dot{-}} \underline{\beta} \mathbin{\dot{-}} \underline{\gamma}) \\ &= \underline{\gamma}' \dotplus \underline{\alpha}' \dotplus \underline{\beta}' \mathbin{\dot{-}} \underline{\alpha}' \mathbin{\dot{-}} \underline{\beta}' \mathbin{\dot{-}} \underline{\gamma}' \\ &= \dot{-}(\underline{\gamma}' \dotplus \underline{\beta}' \dotplus \underline{\alpha}' \mathbin{\dot{-}} \underline{\beta}' \mathbin{\dot{-}} \underline{\alpha}' \mathbin{\dot{-}} \underline{\gamma}') \\ &= (\Gamma i')_*(\dot{-}[\beta,\alpha]^{\gamma}_{\Gamma^*}), \end{aligned}$$

where $\underline{\alpha}' = \Gamma^* j_2' \circ \alpha\colon W \to \Gamma^* X \to \Gamma^*(Y \vee X)$ and $\underline{\beta}' = \Gamma^* j_1' \circ \beta\colon W \to \Gamma^* Y \to \Gamma^*(Y \vee X)$ and $\underline{\gamma}'$ is defined similarly. Since $(\Gamma^* i')_*$ is a monomorphism, we have the result.

Theorem 3.5. (i) *Let $\alpha,\beta \in [W,\Gamma^* X]$ and $\gamma \in [W,\Gamma^* Y]$ be any elements. Then we have*

$$[\alpha \dotplus \beta,\gamma]_{\Gamma^*} = [\beta,\gamma]^{\alpha}_{\Gamma^*} \dotplus [\alpha,\gamma]_{\Gamma^*}.$$

(ii) *Let $\alpha \in [W,\Gamma^* X]$ and $\beta,\gamma \in [W,\Gamma^* Y]$ be any elements. Then we have*

$$[\alpha,\beta \dotplus \gamma]_{\Gamma^*} = [\alpha,\beta]_{\Gamma^*} \dotplus [\alpha,\gamma]^{\beta}_{\Gamma^*} .$$

Proof. These are obtained by Lemma 2.7 and the definition of the Γ^*-Whitehead product.

We need further study on the theory of cocategory and flat product to prove biadditivity and to consider suitable formulation for Jacobi identity for Γ^*-Whitehead product, namely dual results of Theorems 2.9 and 2.11.

4. Samelson product and dual Samelson product. Let G be a grouplike space. Then there exists a short exact sequence of groups

$$0 \longrightarrow [X \wedge Y, G] \xrightarrow{q^*} [X \times Y, G] \xrightarrow{j^*} [X \vee Y, G] \longrightarrow 0$$

induced by a cofibration sequence

$$X \vee Y \xrightarrow{j} X \times Y \xrightarrow{q} X \wedge Y \xrightarrow{\delta} \Sigma(X \vee Y) \xrightarrow{\Sigma j} \Sigma(X \times Y).$$

(cf. 1.3.5 Lemma of Zabrodsky [10].) For any elements $\alpha \in [X, G]$ and $\beta \in [Y, G]$, we define $\widetilde{\alpha} = \alpha \circ p_1$ and $\widetilde{\beta} = \beta \circ p_2$ for projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$. Then we define the commutator

$$[\widetilde{\alpha}, \widetilde{\beta}] = \widetilde{\alpha} \dotplus \widetilde{\beta} \dot{-} \widetilde{\alpha} \dot{-} \widetilde{\beta} \in [X \times Y, G].$$

We see $j^*([\widetilde{\alpha}, \widetilde{\beta}]) = 0$ and hence there exists a unique element $\langle \alpha, \beta \rangle \in [X \wedge Y, G]$ such that $q^*(\langle \alpha, \beta \rangle) = [\widetilde{\alpha}, \widetilde{\beta}]$. The element $\langle \alpha, \beta \rangle$ is called the *generalized Samelson product* (cf. Section 5 of Chapter 10 of Whitehead [9]). This is the *commutator product* of Arkowitz [2].

If $\gamma \in [X, G]$ or $\gamma \in [Y, G]$, then γ acts on the Samelson product $\langle \alpha, \beta \rangle$ by

$$q^*(\langle \alpha, \beta \rangle^{\gamma}) = \widetilde{\gamma} \dotplus [\widetilde{\alpha}, \widetilde{\beta}] \dot{-} \widetilde{\gamma} = [\widetilde{\alpha}, \widetilde{\beta}]^{\widetilde{\gamma}}.$$

Now consider the adjoint isomorphism

$$\tau \colon [\Gamma A, Z] \cong [A, \Gamma^* Z].$$

If Γ is a co-grouplike space, then $\Gamma^* Z = \mathrm{map}^*(\Gamma, Z)$ is a grouplike space. Then putting $G = \Gamma^* Z$ in the definition of the Samelson product, we can define $\langle \tau(\alpha), \tau(\beta) \rangle \in [X \wedge Y, \Gamma^* Z]$ for any elements $\alpha \in [\Gamma X, Z]$ and $\beta \in [\Gamma Y, Z]$. The following result is a generalization of a result of Arkowitz [2].

Proposition 4.1. *For any elements* $\alpha \in [\Gamma X, Z]$, $\beta \in [\Gamma Y, Z]$ *and* $\gamma \in [\Gamma X, Z]$ *(or* $\gamma \in [\Gamma Y, Z]$*), we have*

$$\tau([\alpha, \beta]_{\Gamma}^{\gamma}) = \langle \tau(\alpha), \tau(\beta) \rangle^{\tau(\gamma)}$$

Proof. Consider the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [\Gamma(X \wedge Y), Z] & \xrightarrow{(\Gamma q)^*} & [\Gamma(X \times Y), Z] & \xrightarrow{(\Gamma j)^*} & [\Gamma(X \vee Y), Z] & \longrightarrow & 0 \\ & & \downarrow \tau & & \downarrow \tau & & \downarrow \tau & & \\ 0 & \longrightarrow & [X \wedge Y, \Gamma^* Z] & \xrightarrow{q^*} & [X \times Y, \Gamma^* Z] & \xrightarrow{j^*} & [X \vee Y, \Gamma^* Z] & \longrightarrow & 0 \end{array}$$

Vertical arrows τ's are group isomorphisms. We remark that $\tau(\overline{\alpha}) = \tau(\alpha \circ \Gamma p_1) = \tau(\alpha) \circ p_1$, etc. Then we have the result by definitions of the Γ-Whitehead product and the Samelson product.

Now we consider the dual Samelson product. Let C be a co-grouplike space. Then there exists a short exact sequence of groups

$$0 \longrightarrow [C, X \flat Y] \xrightarrow{i_*} [C, X \vee Y] \xrightarrow{j_*} [C, X \times Y] \longrightarrow 0$$

induced by a fibration sequence

$$\cdots \longrightarrow \Omega(X \vee Y) \xrightarrow{\Omega j} \Omega(X \times Y) \xrightarrow{\partial} X \flat Y \xrightarrow{i} X \vee Y \xrightarrow{j} X \times Y.$$

For any elements $\alpha \in [C, X]$ and $\beta \in [C, Y]$, we define $\underset{\sim}{\alpha} = j_1 \circ \alpha \colon C \to X \to X \vee Y$ and $\underset{\sim}{\beta} = j_2 \circ \beta \colon C \to Y \to X \vee Y$ for the inclusion maps $j_1 \colon X \to X \vee Y$ and $j_2 \colon Y \to X \vee Y$. We define the commutator

$$[\underset{\sim}{\alpha}, \underset{\sim}{\beta}] = \underset{\sim}{\alpha} \dotplus \underset{\sim}{\beta} \dot{-} \underset{\sim}{\alpha} \dot{-} \underset{\sim}{\beta} \in [C, X \vee Y].$$

We see $j_*([\underset{\sim}{\alpha}, \underset{\sim}{\beta}]) = 0$ and hence there exists a unique element $\langle \alpha, \beta \rangle_* \in [C, X \flat Y]$ such that

$$i_*(\langle \alpha, \beta \rangle_*) = [\underset{\sim}{\alpha}, \underset{\sim}{\beta}].$$

The element $\langle \alpha, \beta \rangle_*$ is the *dual Samelson product.* This is the *flat product* of Arkowitz [3].

If $\gamma \in [C, X]$ or $\gamma \in [C, Y]$, then γ acts on the dual Samelson product $\langle \alpha, \beta \rangle_*$ by

$$i_*(\langle \alpha, \beta \rangle_*^{\gamma}) = \underset{\sim}{\gamma} \dotplus [\underset{\sim}{\alpha}, \underset{\sim}{\beta}] \dot{-} \underset{\sim}{\gamma}.$$

Now consider the adjoint isomorphism

$$\tau^* \colon [A, \Gamma^* Z] \cong [\Gamma A, Z].$$

If Γ is a co-grouplike space, then ΓA is a co-grouplike space. Then we can define $\langle \tau^*(\alpha), \tau^*(\beta) \rangle_* \in [\Gamma W, X \flat Y]$ for any elements $\alpha \in [W, \Gamma^* X]$ and $\beta \in [W, \Gamma^* Y]$.

We have the following result.

Proposition 4.2. *If* $\alpha \in [W, \Gamma^* X]$, $\beta \in [W, \Gamma^* Y]$ *and* $\gamma \in [W, \Gamma^* X]$ (*or* $\gamma \in [W, \Gamma^* Y]$), *then*

$$\tau^*([\alpha, \beta]_{\Gamma^*}^{\gamma}) = \langle \tau^*(\alpha), \tau^*(\beta) \rangle_*^{\tau^*(\gamma)}.$$

Proof. Consider the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [W,\Gamma^*(X \flat Y)] & \xrightarrow{(\Gamma^* i)_*} & [W,\Gamma^*(X \vee Y)] & \xrightarrow{(\Gamma^* j)_*} & [W,\Gamma^*(X \times Y)] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau^*} & & \downarrow{\scriptstyle \tau^*} & & \downarrow{\scriptstyle \tau^*} & & \\ 0 & \longrightarrow & [\Gamma W, X \flat Y] & \xrightarrow{i_*} & [\Gamma W, X \vee Y] & \xrightarrow{j_*} & [\Gamma W, X \times Y] & \longrightarrow & 0 \end{array}$$

The vertical arrows τ^*'s are adjoint isomorphisms. We remark that $\tau^*(\underset{\sim}{\alpha}) = \tau^*(\Gamma^* j_1 \circ \alpha) = j_1 \circ \tau^*(\alpha)$, etc. Then the result follows from the definitions of the Γ^*-Whitehead product and the dual Samelson product.

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