Math. J. Okayama Univ. **39** (1997), 89–112

THE STEENROD ALGEBRA AND BRAID GROUPS

MIZUHO HIKIDA

1. Introduction. Let $q \neq 0$ be an element of $\mathbf{Z}/(p)$ for a prime p. In this paper, we define a map $\phi_q \colon B_t \to \mathbf{Z}/(p)$ from the braid group B_t to $\mathbf{Z}/(p)$ related to a cohomology of the Steenrod algebra with

$$\phi_q(\sigma\tau) = \phi_q(\tau) + (-q)^{|\tau|}\phi_q(\sigma), \tag{1.1}$$

where $|\tau| = \sum_i \epsilon_i$ for $\tau = \sigma_{i_1}^{\epsilon_1}\sigma_{i_2}^{\epsilon_2}\cdots\sigma_{i_n}^{\epsilon_n}$ (σ_i are generators of B_t). This map emerged from computation of the stable homotopy groups of spheres. In this paper, we argue the relation between this map and a cohomology of the Steenrod algebra. In the forthcoming paper, we shall argue the stable homotopy groups of spheres by using this map.

The braid group B_t of degree t is generated by σ_i $(1 \leq i \leq t-1)$ and relations

$$(R1) \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} (1 \leq i \leq t-2)$$

$$(R2) \qquad \sigma_i\sigma_j = \sigma_j\sigma_i \qquad (|i-j| \geq 2).$$

Hence B_t has a presentation $F_t/\langle R1, R2\rangle$ for a free group $F_t = \langle \sigma_1, \cdots, \sigma_{t-1}\rangle$. We notice that the symmetric group S_t has a presentation $F_t/\langle R1, R2, \sigma_i^2 = e\rangle = B_t/\langle \sigma_i^2 = e\rangle$ (e is a unit). We denote $\bar{\sigma} \in S_t$ for $\sigma \in F_t$. For example, $\bar{\sigma_i}$ is a transposition $(i, i+1)$. Then we denote $\sigma N = (n_{\bar{\sigma}^{-1}(1)}, \cdots, n_{\bar{\sigma}^{-1}(t)})$.

Definition 1.1. Inductively we define $\bar{\phi}_q : F_t \to \mathbf{Z}/(p)$ by taking

$$\bar{\phi}_q(e) = 0, \quad \bar{\phi}_q(\sigma_i) = 1, \quad \bar{\phi}_q(\sigma_i^{-1}) = q^{-1} \quad \text{and}$$

$$\bar{\phi}_q(\sigma_i\sigma) = \bar{\phi}_q(\sigma) + (-q)^{|\sigma|}, \quad \bar{\phi}_q(\sigma_i^{-1}\sigma) = \bar{\phi}_q(\sigma) - (-q)^{|\sigma|-1}.$$

Then $\bar{\phi}_q(\gamma\sigma) = \bar{\phi}_q(\sigma) = \bar{\phi}_q(\sigma\gamma)$ for $\gamma \in \langle R1, R2\rangle$. $\phi_q : B_t \to \mathbf{Z}/(p)$ is induced from $\bar{\phi}_q$.

At Proposition 5.2, we show (1.1) and that ϕ_q is well defined.

Let A_* be the dual of Steenrod algebra for a prime p. Then

$$(1.2)\quad A_* = \begin{cases} \mathbf{Z}/(2)[\xi_1, \xi_2, \cdots] & \text{for } p=2, \\ E(\tau_0, \tau_1, \cdots) \otimes_{\mathbf{Z}/(p)} \mathbf{Z}/(p)[\xi_1, \xi_2, \cdots] & \text{for odd prime } p. \end{cases}$$

where $E(\tau_0, \tau_1, \cdots)$ is the exterior algebra, $\deg \xi_i = 2^i - 1$ for $p = 2$, $= 2(p^i - 1)$ for an odd prime p, and $\deg \tau_i = 2p^i - 1$. We have a subalgebra

$$(1.3)\qquad P_* = \begin{cases} \mathbf{Z}/(2)[\xi_1^2, \xi_2^2, \cdots] & \text{for } p=2, \\ \mathbf{Z}/(p)[\xi_1, \xi_2, \cdots] & \text{for odd prime } p. \end{cases}$$

Let $c : A_* \to A_*$ be a conjugation. In this paper, we use elements

$$(1.4)\qquad \eta_i = \begin{cases} c(\xi_{i+1}) & \text{for } p=2 \\ c(\tau_i) & \text{for } p>2 \end{cases} \quad \text{and} \quad t_i = \begin{cases} c(\xi_i^2) & \text{for } p=2 \\ c(\xi_i) & \text{for } p>2 \end{cases}.$$

For a coproduct $\Delta : A_* \to A_* \otimes_{\mathbf{Z}/(p)} A_*$,

(1.5)

$$\Delta\tau_n = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \tau_i \quad \text{and} \quad \Delta\xi_n = \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i \quad (\xi_0 = 1).$$

Hence,

$$(1.6)\quad \Delta\eta_n = 1 \otimes \eta_n + \sum_{i=0}^{n} \eta_i \otimes t_{n-i}^{p^i} \quad \text{and} \quad \Delta t_n = \sum_{i=0}^{n} t_i \otimes t_{n-i}^{p^i} \quad (t_0 = 1).$$

Let $C = \{C^t; \delta : C^t \to C^{t+1}\}$ be a cochain complex defined by

$$C^t = A_*^t \otimes_{\mathbf{Z}/(p)} P_* = \underbrace{A_* \otimes_{\mathbf{Z}/(p)} \cdots \otimes_{\mathbf{Z}/(p)} A_*}_{t \text{ times}} \otimes_{\mathbf{Z}/(p)} P_* \qquad \text{and}$$

$$(1.7)\quad \delta(x_t \otimes \cdots \otimes x_2 \otimes x_1 \otimes m) = x_t \otimes \cdots \otimes x_2 \otimes x_1 \otimes \Delta m + \sum_{i=1}^{t} (-1)^i x_t \otimes \cdots \otimes \Delta x_i \otimes \cdots \otimes x_1 \otimes m \; (-1)^{t+1} 1 \otimes x_t \otimes \cdots \otimes x_1 \otimes m,$$

in which $x_i \in A_*$, $m \in P_*$ and $\Delta : N \to A_* \otimes N$ for $N = A_*$ or P_*. This cochain complex has the cohomology

$$(1.8)\qquad \begin{aligned} H^t(C^*; \delta) &= \mathrm{Ext}_{A_*}^t(\mathbf{Z}/(p), P_*) \\ &= \mathbf{Z}/(p)[a_0, a_1, \cdots] \quad (a_i \in \mathrm{Ext}^1) \quad (\text{see } (2.1\text{-}2)). \end{aligned}$$

At Definition 2.2, we define $X(N) \in C^t$ representing $a_N = a_{n_1} \cdots a_{n_t}$ for any sequence $N = (n_1, \cdots, n_t)$ of integers $n_i \geq 0$ (see Proposition 2.3 iii)). Then, for any $\sigma \in F_t$, we define $Y_\sigma(N) \in C^{t-1}$ in Definition 3.1 such that

(1.9) $\delta Y_\sigma(N) = X(N) - X(\sigma N)$ (see Proposition 3.3 i)) and

(1.10) $Y_{\sigma\tau}(N) = Y_\tau(N) + Y_\sigma(\tau N)$ (see Proposition 3.4 i)).

Moreover we define $R_\gamma(N) \in C^{t-2}$ for $\gamma \in \langle R1, R2\rangle$ in Definition 4.1 such that

(1.11) $\delta R_\gamma(N) = Y_\gamma(N)$ (see Proposition 4.2 i)).

Definition 1.2. (1) We define submodules of C^t, C^{t-1} and C^{t-2} by taking

$$C_X^t = \langle X(N)\rangle, \quad C_Y^{t-1}(N) = \langle Y_\sigma(\tau N) | \sigma, \tau \in F_t\rangle,$$
$$C_R^{t-2}(N) = \langle R_\gamma(N) | \gamma \in \langle R1, R2\rangle\rangle,$$
$$C_Y^{t-1} = \cup_N C_Y^{t-1}(N) \quad \text{and} \quad C_R^{t-2} = \cup_N C_R^{t-2}(N)$$

(2) We define a map $\bar{\phi}_{q,N} : F_t \to C_Y^{t-1}(N)$ by taking

$$\bar{\phi}_{q,N}(e) = 0, \quad \bar{\phi}_{q,N}(\sigma_i) = Y_{\sigma_i}(N), \quad \bar{\phi}_{q,N}(\sigma_i^{-1}) = q^{-1} Y_{\sigma_i}(N) \quad \text{and}$$
$$\bar{\phi}_{q,N}(\sigma_i^{\pm 1}\sigma) = \bar{\phi}_{q,N}(\sigma) + q^{|\sigma|}\bar{\phi}_{q,\sigma N}(\sigma_i^{\pm 1}).$$

At Proposition 5.3, we see that $\bar{\phi}_{q,N}$ is well defined and

(1.12) $$\bar{\phi}_{q,N}(\sigma\tau) = \bar{\phi}_{q,N}(\tau) + q^{|\tau|}\bar{\phi}_{q,\tau N}(\sigma).$$

The definitions of ϕ_q, $\bar{\phi}_{q,N}$ and the following theorem are the main purposes in this paper.

Theorem 1.3.
i) $C_X^t / \delta C_Y^{t-1} = H^t(C^*; \delta) = \mathrm{Ext}^t_{A_*}(\mathbf{Z}/(p), P_*) = \mathbf{Z}/(p)[a_0, a_1, \cdots]$.

ii) $\bar{\phi}_{1,N}(\sigma) = Y_\sigma(N)$, *and so* $\bar{\phi}_{1,N}(\langle R1, R2\rangle) = \delta C_R^{t-2}(N)$.

iii) $\bar{\phi}_{q,N}(\langle R1, R2\rangle)$ *is a submodule of* $C_Y^{t-1}(N)$.

iv) *We assume that* $n_i \neq n_j$ *for any* $i \neq j$. *If* $t \geq 5$ *and* $q^2 \not\equiv 1$ (p) *then*

$$C_Y^{t-1}(N)/\bar{\phi}_{q,N}(\langle R1, R2\rangle) = \mathbf{Z}/(p) \quad and$$

$\bar{\phi}_{q,N}$ *induces*

$$\phi_q : B_t = F_t/\langle R1, R2\rangle \to C_Y^{t-1}(N)/\bar{\phi}_{q,N}(\langle R1, R2\rangle) = \mathbf{Z}/(p)$$

in Definition 1.1.

The cochain complex C and its cohomology are useful in the field of the stable homotopy theory. Let $H\mathbf{Z}/(p)$ be the Eilenberg-MacLane spectrum and MU_* a ring spectrum representing the complex cobordism theory. Then the Brown-Peterson spectrum BP is a minimal wedge summand of MU localized at a prime p. Now $H\mathbf{Z}/(p)_*(BP) = P_*$ and the Adams spectral sequence $E_2 = \mathrm{Ext}_{A_*}^{s,t}(\mathbf{Z}/(p), P_*) \Rightarrow \pi_*(BP)$ collapses and converges (see [9]). The Adams spectral sequence $\{E(H\mathbf{Z}/(p))_r^{s,t}, d_r^{H\mathbf{Z}/(p)}\}$ with

$$E(H\mathbf{Z}/(p))_2^{s,t} = \mathrm{Ext}_{A_*}^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p))$$

and the Novikov-Adams spectral sequence $\{E(BP)_r^{s,t}, d_r^{BP}\}$ with

$$E(BP)_2^{s,t} = \mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*)$$

are used to calculate the stable homotopy groups of the spheres. For these E_2-terms, we have the Mahowald and May spectral sequences

$$\{E(\mathrm{Mah})_{u,r}^{s,t}, d_r^{\mathrm{Mah}}\} \quad \text{and} \quad \{E(\mathrm{May})_{u,r}^{s,t}, d_r^{\mathrm{May}}\} \quad \text{with}$$

$$E(\mathrm{Mah})_{u,2}^{s,t} = \mathrm{Ext}_{P_*}^{s,u}(\mathbf{Z}/(p), \mathrm{Ext}_{A_*}^{t,*}(\mathbf{Z}/(p), P_*)) \Rightarrow E(H\mathbf{Z}/(p))_2^{s+t,u} \quad \text{and}$$
$$E(\mathrm{May})_{u,1}^{s,t} = \mathrm{Ext}_{P_*}^{s,u}(\mathbf{Z}/(p), \mathrm{Ext}_{A_*}^{t,*}(\mathbf{Z}/(p), P_*)) \Rightarrow E(BP)_2^{s,u-t}.$$

By [4, 6], we can calculate $d_r^{H\mathbf{Z}/(p)}$ and d_r^{BP} from d_r^{May} and d_r^{Mah}, respectively. Moreover d_r^{Mah} is calculated by using δ of (1.7), $X(N)$, $Y_\sigma(N)$ and $R_\gamma(N)$ by [4]. We shall use the argument in this paper to calculate d_r^{BP} for $p = 2$ in the forthcoming paper.

This paper is organized as follows. We define $X(N)$, $Y_\sigma(N)$ and $R_\gamma(N)$ in § 2, § 3 and § 4, respectively. The maps ϕ_q and $\bar{\phi}_{q,N}$ are argued in § 5.

2. Definition of $X(N)$. In this section, we define an element $X(N) \in C^t$. Then we argue the properties of this element.

Let $C_A = \{C_A^t; \delta^A : C_A^t \to C_A^{t+1}\}$ be a cochain complex defined by

$$C_A^t = A_*^t \otimes_{\mathbf{Z}/(p)} A_* \qquad \text{and}$$

$$\delta^A(x_t \otimes \cdots \otimes x_2 \otimes x_1 \otimes x_0) = \sum_{i=0}^{t}(-1)^i x_t \otimes \cdots \otimes \Delta x_i \otimes \cdots \otimes x_0 + (-1)^{t+1} 1 \otimes x_t \otimes \cdots \otimes x_0 \quad (x_i \in A_*).$$

Then we notice that

(2.1) $\quad C_A \supset C, \quad \delta^A | C = \delta,$

(2.2)

$$H^t(C_A^*; \delta^A) = \mathrm{Ext}_{A_*}^t(\mathbf{Z}/(p), A_*) = \begin{cases} 0 & \text{for} \quad t > 0, \\ \mathbf{Z}/(p) & \text{for} \quad t = 0 \end{cases} \quad \text{and}$$

(2.3)

$$\delta^A(x \otimes y) = -\delta^A x \otimes y + x \otimes \Delta y \quad \text{for} \quad x \in A_*^t = C_A^{t-1}, y \in A_*.$$

Hence we have the following lemma.

Lemma 2.1. *For $x \in C^t$ $(t > 0)$, $\delta x = 0$ if and only if there exists an element $y^A \in C_A^{t-1}$ such that $\delta^A y^A = x$*

Let $B_* = A_* // P_*$ be a quotient algebra, $pr : A_* \to B_*$ a projection and $\Delta_B : B_* \to B_* \otimes_{\mathbf{Z}/(p)} B_*$ an induced coproduct. We have a cochain complex

$$C_B = \{C_B^t = B_*^t; \delta^B : C_B^t \to C_B^{t+1}\}$$

defined by

$$\delta^B(x_t \otimes \cdots \otimes x_1) = x_t \otimes \cdots \otimes x_1 \otimes 1 + \sum_{i=1}^{t}(-1)^i x_t \otimes \cdots \otimes \Delta_B x_i \otimes \cdots \otimes x_1 + (-1)^{t+1} 1 \otimes x_t \otimes \cdots \otimes x_1, \quad (x_i \in B_*).$$

Now pr induces a cochain homomorphism $pr_* : C = A^t \otimes_{\mathbf{Z}/(p)} P_* \to C_B = B_*^t \otimes_{\mathbf{Z}/(p)} \mathbf{Z}/(p)$ and an isomorphism

(2.4)

$$pr_* : \mathrm{Ext}_{A_*}^t(\mathbf{Z}/(p), P_*) = H^t(C^*) \xrightarrow{\cong} \mathrm{Ext}_{B_*}^t(\mathbf{Z}/(p), \mathbf{Z}/(p)) = H^t(C_B^*)$$

by the change-of-ring theorem. Then

$$\mathrm{Ext}^*_{B_*}(\mathbf{Z}/(p),\mathbf{Z}/(p)) = \mathbf{Z}/(p)[a_0, a_1, \cdots] \tag{2.5}$$

and $a_i \in \mathrm{Ext}^1_{B_*}(\mathbf{Z}/(p),\mathbf{Z}/(p))$ is represented by $pr(\eta_n) \in B_* = C^1_B$, where $\eta_n \in A_*$ is the one of (1.4). Consider elements $x(n) = \eta_n \in C^0_A = A_*$ and $X(n) = \delta^A x(n) = \sum_{i=0}^{n} \eta_i \otimes t^{p^i}_{n-i} \in C^1$. $X(n)$ represents a_n since $pr_*(X(n)) = \eta_n \in B_* = C^1_B$. Let $N = (n_1, n_2, \cdots, n_t)$ be a sequence of integers with $n_i \geq 0$. We shall define an element $x(N) \in C^{t-1}_A = A^t_*$ so that $\delta^A x(N)$ is included in C^t and represents a monomial $a_N = a_{n_1} a_{n_2} \cdots a_{n_t}$ as follows:

Definition 2.2. Inductively we define $x(N) \in C^{t-1}_A$ by taking $x(n_1) = \eta_{n_1}$ and

$$x(N, n_{t+1}) = x(n_1, \cdots, n_t, n_{t+1}) = \sum_{I=0}^{N} x(I) \otimes t^{p^I}_{N-I} \eta_{n_{t+1}},$$

where

$$I = (i_1, i_2, \cdots, i_t),$$
$$\sum_{I=0}^{N} = \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} \cdots \sum_{i_t=0}^{n_t}$$

and

$$t^{p^I}_{N-I} = t^{p^{i_1}}_{n_1-i_1} t^{p^{i_2}}_{n_2-i_2} \cdots t^{p^{i_t}}_{n_t-i_t}.$$

Moreover we define

$$X(N) = \sum_{I=0}^{N} x(I) \otimes t^{p^I}_{N-I} \in C^t,$$

and so

$$x(N, n) = X(N)\eta_n.$$

We shall prove that $\delta^A x(N) = X(N)$. For this purpose, we notice the following:

(2.6)

$$(a \otimes x)(b \otimes y) = (-1)^{\deg x \cdot \deg b} ab \otimes xy \quad \text{(see [11, Theorem 17.8])},$$

$$\Delta t_{N-I}^{p^I} = \sum_{J=I}^{N} t_{J-I}^{p^I} \otimes t_{N-J}^{p^J} \quad \text{and} \tag{2.7}$$

$$\Delta t_{N-I}^{p^I} \eta_n = \sum_{J=I}^{N} t_{J-I}^{p^I} \otimes t_{N-J}^{p^J} \eta_n + \sum_{J=I}^{N} \sum_{i=0}^{n} t_{J-I}^{p^I} \eta_i \otimes t_{N-J}^{p^J} t_{n-i}^{p^i} \tag{2.8}$$

by (1.6). For any expression $f(I)$,

$$\begin{aligned} \sum_{I=0}^{N} \sum_{J=I}^{N} f(I) \otimes t_{J-I}^{p^I} \otimes t_{N-J}^{p^J} &= \sum_{J=0}^{N} \sum_{I=0}^{J} f(I) \otimes t_{J-I}^{p^I} \otimes t_{N-J}^{p^J} \\ &= \sum_{I=0}^{N} \sum_{J=0}^{I} f(J) \otimes t_{I-J}^{p^J} \otimes t_{N-I}^{p^I}. \end{aligned} \tag{2.9}$$

Proposition 2.3. *For the above $X(N)$, we have the following.*

i) $\delta^A x(N) = X(N)$.

ii) $\Delta_* X(N) = \sum_{I=0}^{N} X(I) \otimes t_{N-I}^{p^I} \in C^t \otimes P_*$, *in which* $\Delta_* \colon C^t \to C^t \otimes_{\mathbf{Z}/(p)} P_*$ *is the one induced by* $\Delta : P_* \to P_* \otimes_{\mathbf{Z}/(p)} P_* \subset A_* \otimes_{\mathbf{Z}/(p)} P_*$.

iii) $X(N)$ *represents a monomial* a_N.

Proof. i) We shall use induction on t. For the case of $t = 1$, (1.6) implies this part. By (2.3), (2.8), (2.9) and induction,

$$\begin{aligned} \delta^A x(N, n) &= \delta^A \sum_{I=0}^{N} x(I) \otimes t_{N-I}^{p^I} \eta_n \\ &= -\sum_{I=0}^{N} \delta^A x(I) \otimes t_{N-I}^{p^I} \eta_n + \sum_{I=0}^{N} x(I) \otimes \Delta t_{N-I}^{p^I} \eta_n \\ &= -\sum_{I=0}^{N} X(I) \otimes t_{N-I}^{p^I} \eta_n + \sum_{I=0}^{N} \sum_{J=0}^{I} x(J) \otimes t_{I-J}^{p^J} \otimes t_{N-I}^{p^I} \eta_n \end{aligned}$$

$$+\sum_{I=0}^{N}\sum_{J=0}^{I}\sum_{i=0}^{n} x(J)\otimes t_{I-J}^{p^J}\eta_i\otimes t_{N-I}^{p^I}t_{n-i}^{p^i}\,.$$

By definition,

$$X(I)=\sum_{J=0}^{I} x(J)\otimes t_{I-J}^{p^J}$$

and

$$x(I,i)=\sum_{J=0}^{I} x(J)\otimes t_{I-J}^{p^J}\eta_i.$$

Hence

$$\delta^A x(N,n)=\sum_{I=0}^{N}\sum_{i=0}^{n} x(I,i)\otimes t_{N-I}^{p^I}t_{n-i}^{p^i}=X(N,n).$$

ii) By the definition, (2.7) and (2.9),

$$\begin{aligned}\Delta_* X(N)&=\sum_{I=0}^{N} x(I)\otimes\sum_{J=I}^{N} t_{J-I}^{p^I}\otimes t_{N-J}^{p^J}=\sum_{I=0}^{N}\sum_{J=0}^{I} x(J)\otimes t_{I-J}^{p^J}\otimes t_{N-I}^{p^I}\\&=\sum_{I=0}^{N} X(I)\otimes t_{N-I}^{p^I}.\end{aligned}$$

iii) Let $pr_*=pr\otimes\cdots\otimes pr: C_A^{t-1}=A_*^t\to C_B^t=B_*^t$ be a homomorphism. By induction, we see that $pr_* x(N,n)=pr_* x(N)\otimes\eta_n=\eta_{n_1}\otimes\cdots\eta_{n_t}\otimes\eta_n$. This implies this proposition.

3. Definition of $Y_\sigma(N)$. In this section, we define an element $Y_\sigma(N)\in C^{t-1}$ for $\sigma\in F_t$. Then we argue the properties of this element.

Definition 3.1. (1) For $t\geq 0$, we define $y_{t+1}(N,n,m)\in C_A^t$ for $N=(n_1,\cdots,n_t)$ by taking

$$y_1(n,m)=-\eta_n\eta_m \qquad \text{and}$$

$$y_{t+1}(N,n,m)=-X(N)\eta_n\eta_m=-\sum_{I=0}^{N} x(I)\otimes t_{N-I}^{p^I}\eta_n\eta_m.$$

Inductively we define $y_i(N,n) \in C_A^{t-1}$ for $1 \le i \le t-1$ by taking

$$y_i(N,n) = -\sum_{I=0}^{N} y_i(I) \otimes t_{N-I}^{p^I}\eta_n.$$

(2) Next we denote $y_e(N) = 0$, $y_{\sigma_i}(N) = y_i(N)$ and $y_{\sigma_i^{-1}}(N) = y_i(N)$ for generators $\sigma_i \in F_t$ and define y_σ inductively by $y_{\sigma_i^{\pm 1}\sigma}(N) = y_\sigma(N) + y_i(\sigma N)$ for $\sigma \in F_t$. Now we take $Y_a(N) = \sum_{I=0}^{N} y_a(I) \otimes t_{N-I}^{p^I} \in C^{t-1}$ for $a = i$ or σ.

Remark 3.2. In the above definition, we notice that $\eta_n\eta_m = -\eta_m\eta_n$, and so

$$y_i(\sigma_i N) = -y_i(N) \tag{3.1}$$

by induction on t for a fixed i. Then

$$y_{\sigma_i^{-1}\sigma_i\sigma}(N) = y_\sigma(N) + y_i(\sigma N) + y_i(\sigma_i\sigma N) = y_\sigma(N).$$

Hence $y_\sigma(N)$ for any $\sigma \in F_t$ is well-defined.

By (1.6), (2.6) and (2.7), we see that

$$\begin{aligned} \Delta\eta_n\eta_m &= (1 \otimes \eta_n + \sum_{i=0}^{n} \eta_i \otimes t_{n-i}^{p^i})(1 \otimes \eta_m + \sum_{j=0}^{m} \eta_j \otimes t_{m-j}^{p^j}) \\ &= 1 \otimes \eta_n\eta_m + \sum_{i=0}^{n} \eta_i \otimes t_{n-i}^{p^i}\eta_m \\ &\quad - \sum_{j=0}^{m} \eta_j \otimes \eta_n t_{m-j}^{p^j} + \sum_{i=0}^{n}\sum_{j=0}^{m} \eta_i\eta_j \otimes t_{n-i}^{p^i} t_{m-j}^{p^j}, \end{aligned} \tag{3.2}$$

and

(3.3)

$$\Delta t^{p^I}_{N-I}\eta_n\eta_m = \sum_{J=I}^{N} t^{p^I}_{J-I} \otimes t^{p^J}_{N-J}\eta_n\eta_m + \sum_{J=I}^{N}\sum_{i=0}^{n} t^{p^I}_{J-I}\eta_i \otimes t^{p^J}_{N-J}t^{p^i}_{n-i}\eta_m$$
$$- \sum_{J=I}^{N}\sum_{j=0}^{m} t^{p^I}_{J-I}\eta_j \otimes t^{p^J}_{N-J}\eta_n t^{p^j}_{m-j}$$
$$+ \sum_{J=I}^{N}\sum_{i=0}^{n}\sum_{j=0}^{m} t^{p^I}_{J-I}\eta_i\eta_j \otimes t^{p^J}_{N-J}t^{p^i}_{n-i}t^{p^j}_{m-j}.$$

Proposition 3.3. *For the above elements, we have the following.*
i) [(1.9)] $\delta Y_\sigma(N) = X(N) - X(\sigma N)$.

ii) $\Delta_* Y_\sigma(N) = \sum_{I=0}^{N} Y_\sigma(I) \otimes t^{p^I}_{N-I}$.

Proof. i) In the first place, we prove that

$$\delta^A y_i(N) = -x(N) + x(\sigma_i N) + Y_i(N). \tag{3.4}$$

For example, by (2.3), (2.9) and (3.3),

$$\delta^A y_{t+1}(N,n,m) = \sum_{I=0}^{N} \delta^A x(I) \otimes t^{p^I}_{N-I}\eta_n\eta_m - \sum_{I=0}^{N} x(I) \otimes \Delta t^{p^I}_{N-I}\eta_n\eta_m$$
$$= \sum_{I=0}^{N} X(I) \otimes t^{p^I}_{N-I}\eta_n\eta_m$$
$$- \sum_{I=0}^{N}\sum_{J=0}^{I} x(J) \otimes t^{p^J}_{I-J} \otimes t^{p^I}_{N-I}\eta_n\eta_m$$
$$- \sum_{I=0}^{N}\sum_{J=0}^{I}\sum_{i=0}^{n} x(J) \otimes t^{p^J}_{I-J}\eta_i \otimes t^{p^I}_{N-I}t^{p^i}_{n-i}\eta_m$$
$$+ \sum_{I=0}^{N}\sum_{J=0}^{I}\sum_{j=0}^{m} x(J) \otimes t^{p^J}_{I-J}\eta_j \otimes t^{p^I}_{N-I}\eta_n t^{p^j}_{m-j}$$
$$- \sum_{I=0}^{N}\sum_{J=0}^{I}\sum_{i=0}^{n}\sum_{j=0}^{m} x(J) \otimes t^{p^J}_{I-J}\eta_i\eta_j \otimes t^{p^I}_{N-I}t^{p^i}_{n-i}t^{p^j}_{m-j} .$$

By the definitions of $X(I)$, $X(I,i)$, $X(I,j)$ and $y_{t+1}(I,i,j)$,

$$\begin{aligned}\delta^A y_{t+1}(N,n,m) &= -\sum_{I=0}^{N}\sum_{i=0}^{n} x(I,i)\otimes t_{N-I}^{p^I}t_{n-i}^{p^i}\eta_m \\ &+\sum_{I=0}^{N}\sum_{j=0}^{m} x(I,j)\otimes t_{N-I}^{p^I}\eta_n t_{m-j}^{p^j} \\ &+\sum_{I=0}^{N}\sum_{i=0}^{n}\sum_{j=0}^{m} y_{t+1}(I,i,j)\otimes t_{N-I}^{p^I}t_{n-i}^{p^i}t_{m-j}^{p^j} \\ &= -x(N,n,m)+x(N,m,n)+Y_{t+1}(N,n,m)\ .\end{aligned}$$

Then (3.4) for any i is proved by induction on t. In fact, by (2.3), (2.8) and (2.9),

$$\begin{aligned}\delta^A y_i(N,n) =& \sum_{I=0}^{N}\delta^A y_i(I)\otimes t_{N-I}^{p^I}\eta_n - \sum_{I=0}^{N} y_i(I)\otimes \Delta t_{N-I}^{p^I}\eta_n \\ =& \sum_{I=0}^{N}\{-x(I)+x(\sigma_i I)+Y_i(I)\}\otimes t_{N-I}^{p^I}\eta_n \\ &-\sum_{I=0}^{N}\sum_{J=0}^{I} y_i(J)\otimes t_{I-J}^{p^J}\otimes t_{N-I}^{p^I}\eta_n \\ &-\sum_{I=0}^{N}\sum_{J=0}^{I}\sum_{a=0}^{n} y_i(J)\otimes t_{I-J}^{p^J}\eta_a\otimes t_{N-I}^{p^I}t_{n-a}^{p^a} \\ =& -x(N,n)+x(\sigma_i N,n)+Y_i(N,n).\end{aligned}$$

Next, by (2.3), (2.9) and (3.4),

$$\begin{aligned}\delta Y_i(N) &= -\sum_{I=0}^{N}\delta^A y_i(I)\otimes t_{N-I}^{p^I} + \sum_{I=0}^{N} y_i(I)\otimes\Delta t_{N-I}^{p^I} \\ &= -\sum_{I=0}^{N}\{-x(I)+x(\sigma_i I)+Y_i(I)\}\otimes t_{N-I}^{p^I} \\ &+\sum_{I=0}^{N}\sum_{J=0}^{I} y_i(J)\otimes t_{I-J}^{p^J}\otimes t_{N-I}^{p^I}\end{aligned}$$

$$= X(N) - X(\sigma_i N).$$

In the last place, we assume that i) of this proposition holds for $\sigma \in F_t$, then

$$\begin{aligned}\delta Y_{\sigma_i^{\pm 1}\sigma}(N) &= \delta\left\{Y_\sigma(N) + Y_{\sigma_i^{\pm 1}}(\sigma N)\right\}\\ &= X(N) - X(\sigma N) + X(\sigma N) - X(\sigma_i^{\pm 1}\sigma N)\\ &= X(N) - X(\sigma_i^{\pm 1}\sigma N).\end{aligned}$$

By induction, we see i) of this proposition.
ii) This is proved by the same manner as ii) of Proposition 2.3.

Next we prepare the following proposition.

Proposition 3.4.
i) $y_{\sigma\tau}(N) = y_\tau(N) + y_\sigma(\tau N)$, $Y_{\sigma\tau}(N) = Y_\tau(N) + Y_\sigma(\tau N)$, $y_i(\sigma_i N) = -y_i(N)$ *and* $Y_i(\sigma_i N) = -Y_i(N)$.

ii) $Y_\sigma(N) + Y_{\sigma^{-1}}(\sigma N) = 0$ *and* $Y_{\sigma^{-1}}(N) = -Y_\sigma(\sigma^{-1}N)$.

iii) *If* $\tau\sigma N = \sigma N$ *then* $Y_{\sigma^{-1}\tau\sigma}(N) = Y_\tau(\sigma N)$.

iv) $Y_{\sigma_i^n}(N) = (\frac{1-(-1)^n}{2})Y_i(N)$.

Proof. i) This is trivial by induction on a length of the word σ.
ii) By i), $Y_\sigma(N) + Y_{\sigma^{-1}}(\sigma N) = Y_{\sigma^{-1}\sigma}(N) = Y_e(N) = 0$.
iii) By i), ii) and the assumption,

$$\begin{aligned}Y_{\sigma^{-1}\tau\sigma}(N) &= Y_{\tau\sigma}(N) + Y_{\sigma^{-1}}(\tau\sigma N)\\ &= Y_\sigma(N) + Y_\tau(\sigma N) + Y_{\sigma^{-1}}(\sigma N)\\ &= Y_\tau(\sigma N).\end{aligned}$$

iv) By i) and induction,

$$\begin{aligned}Y_{\sigma_i^{n+1}}(N) &= Y_{\sigma_i}(N) + Y_{\sigma_i^n}(\sigma_i N)\\ &= Y_i(N) + (\frac{1-(-1)^n}{2})Y_i(\sigma_i N)\\ &= \{1 - (\frac{1-(-1)^n}{2})\}Y_i(N)\\ &= (\frac{1-(-1)^{n+1}}{2})Y_i(N).\end{aligned}$$

Definition 3.5. We define elements $\gamma_i, \gamma_{i,j} \in F_t$ ($|i-j| \geq 2$) by

$$\gamma_i = \sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{and} \quad \gamma_{i,j} = \sigma_i^{-1}\sigma_j^{-1}\sigma_i\sigma_j,$$

and normal subgroups

$$\langle R1\rangle = \langle \gamma_i\rangle, \quad \langle R2\rangle = \langle \gamma_{i,j}\rangle \quad \text{and} \quad \langle R1, R2\rangle = \langle \gamma_i, \gamma_{i,j}\rangle$$

generated by $\gamma_i, \gamma_{i,j}$ for $i+1<j$.

Remark 3.6. We notice that $\gamma_{i,j}^{-1} = \gamma_{j,i}$.

Consider $Y_\gamma(N)$ for these elements $\gamma \in \langle R1, R2\rangle$. The following lemma is shown by the definitions and Proposition 3.4.

Lemma 3.7. *For the above elements, we have the following.*

i) $\gamma N = N$ *for* $\gamma \in < R1, R2 >$, *and so* $Y_{\gamma\gamma'}(N) = Y_{\gamma'\gamma}(N) = Y_\gamma(N) + Y_{\gamma'}(N)$ *for* $\gamma\gamma' \in \langle R1, R2\rangle$ *and* $Y_{\sigma^{-1}\gamma\sigma}(N) = Y_\gamma(\sigma N)$ *for* $\gamma \in \langle R1, R2\rangle$ *and* $\sigma \in F_t$.

ii) $y_{\gamma_i}(N) = y_{i+1}(N) - y_{i+1}(\sigma_i N) + y_{i+1}(\sigma_i\sigma_{i+1}N) - y_i(N) + y_i(\sigma_{i+1}N) - y_i(\sigma_{i+1}\sigma_i N)$ *and* $Y_{\gamma_i}(N) = Y_{i+1}(N) - Y_{i+1}(\sigma_i N) + Y_{i+1}(\sigma_i\sigma_{i+1}N) - Y_i(N) + Y_i(\sigma_{i+1}N) - Y_i(\sigma_{i+1}\sigma_i N)$.

iii) $y_{\gamma_{i,j}}(N) = y_j(N) - y_i(N) + y_j(\sigma_i N) + y_i(\sigma_j N)$ *and* $Y_{\gamma_{i,j}}(N) = Y_j(N) - Y_i(N) + Y_j(\sigma_i N) + Y_i(\sigma_j N)$.

4. Definition of $R_\gamma(N)$. In this section, we define $R_\gamma(N) \in C^{t-2}$ for $\gamma \in \langle R1, R2\rangle$ and prove (1.11).

Definition 4.1. For $n, m, l \geq 0$, $N = (n_1, \cdots, n_t)$ and i ($1 \leq i \leq t-1$), we define elements $r_{t+1}(N, n, m, l) \in C_A^t$ and $r_{i,t+1}(N, n, m) \in C_A^{t-1}$ as follows:

$$r_1(n, m, l) = \eta_n\eta_m\eta_l, \ r_{t+1}(N, n, m, l) = X(N)\eta_n\eta_m\eta_l \text{ and}$$

$$r_{i,t+1}(N, n, m) = Y_i(N)\eta_n\eta_m.$$

Then we define $r_i(N) \in C_A^{t-3}$ for $1 \le i \le t-2$ and $r_{i,j}(N) \in C_A^{t-3}$ for $1 \le i \le j-2 \le t-3$ by taking inductively

$$r_i(N,n) = \sum_{I=0}^{N} r_i(I) \otimes t_{N-I}^{p^I} \eta_n \quad \text{for} \quad 1 \le i \le t-2 \quad \text{and}$$
$$r_{i,j}(N,n) = \sum_{I=0}^{N} r_{i,j}(I) \otimes t_{N-I}^{p^I} \eta_n \quad \text{for} \quad 1 \le i \le j-2 \le t-3.$$

Next we take

$$r_{\gamma_i}(N) = r_i(N), r_{\gamma_i^{-1}}(N) = -r_i(N), r_{\gamma_{i,j}}(N) = r_{i,j}(N) \quad \text{and}$$

$$r_{\gamma_{i,j}^{-1}}(N) = -r_{i,j}(N).$$

Then we define $r_{\sigma^{-1}\gamma^{\pm 1}\sigma}(N) = r_{\gamma^{\pm 1}}(\sigma N)$ for $\gamma = \gamma_i$, $\overline{\gamma}_i$, $\gamma_{i,j}$ and $\overline{\gamma}_{i,j}$. Now any element $\rho \in \langle R1, R2 \rangle$ is given by $\rho = \rho_1 \cdots \rho_k \cdots \rho_n \in \langle R1, R2 \rangle$ where $\rho_k = \sigma^{-1}\gamma\sigma$ for some σ and $\gamma = \gamma_i^{\pm 1}, \gamma_{i,j}^{\pm 1}$. We define $r_\rho(N) = \sum_{i=1}^{n} r_{\rho_i}(N)$. Moreover we take

$$R_a(N) = \sum_{I=0}^{N} r_a(I) \otimes t_{N-I}^{p^I},$$

where $r_a = r_i$, $r_{i,j}$ and r_ρ for $\rho \in \langle R1, R2 \rangle$.

We notice that

(4.1)

$$\begin{aligned}
\Delta \eta_n \eta_m \eta_l = {} & 1 \otimes \eta_n \eta_m \eta_l + \sum_{i=0}^{n} \eta_i \otimes t_{n-i}^{p^i} \eta_m \eta_l - \sum_{j=0}^{m} \eta_j \otimes \eta_n t_{m-j}^{p^j} \eta_l \\
& + \sum_{k=0}^{l} \eta_k \otimes \eta_n \eta_m t_{l-k}^{p^k} + \sum_{i=0}^{n} \sum_{j=0}^{m} \eta_i \eta_j \otimes t_{n-i}^{p^i} t_{m-j}^{p^j} \eta_l \\
& - \sum_{i=0}^{n} \sum_{k=0}^{l} \eta_i \eta_k \otimes t_{n-i}^{p^i} \eta_m t_{l-k}^{p^k} + \sum_{j=0}^{m} \sum_{k=0}^{l} \eta_j \eta_k \otimes \eta_n t_{m-j}^{p^j} t_{l-k}^{p^k} \\
& + \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{l} \eta_i \eta_j \eta_k \otimes t_{n-i}^{p^i} t_{m-j}^{p^j} t_{l-k}^{p^k}
\end{aligned}$$

by (1.6), (2.6) and (3.2). Hence, by (2.6) and (2.7),

$$
\begin{aligned}
(4.2)\quad \Delta t_{N-I}^{p^I}\eta_n\eta_m\eta_l = & \sum_{J=I}^{N} t_{J-I}^{p^I} \otimes t_{N-J}^{p^J}\eta_n\eta_m\eta_l \\
& + \sum_{J=I}^{N}\sum_{i=0}^{n} t_{J-I}^{p^I}\eta_i \otimes t_{N-J}^{p^J} t_{n-i}^{p^i}\eta_m\eta_l \\
& - \sum_{J=I}^{N}\sum_{j=0}^{m} t_{J-I}^{p^I}\eta_j \otimes t_{N-J}^{p^J}\eta_n t_{m-j}^{p^j}\eta_l \\
& + \sum_{J=I}^{N}\sum_{k=0}^{l} t_{J-I}^{p^I}\eta_k \otimes t_{N-J}^{p^J}\eta_n\eta_m t_{l-k}^{p^k} \\
& + \sum_{J=I}^{N}\sum_{i=0}^{n}\sum_{j=0}^{m} t_{J-I}^{p^I}\eta_i\eta_j \otimes t_{N-J}^{p^J} t_{n-i}^{p^i} t_{m-j}^{p^j}\eta_l \\
& - \sum_{J=I}^{N}\sum_{i=0}^{n}\sum_{k=0}^{l} t_{J-I}^{p^I}\eta_i\eta_k \otimes t_{N-J}^{p^J} t_{n-i}^{p^i}\eta_m t_{l-k}^{p^k} \\
& + \sum_{J=I}^{N}\sum_{j=0}^{m}\sum_{k=0}^{l} t_{J-I}^{p^I}\eta_j\eta_k \otimes t_{N-J}^{p^J}\eta_n t_{m-j}^{p^j} t_{l-k}^{p^k} \\
& + \sum_{J=I}^{N}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l} t_{J-I}^{p^I}\eta_i\eta_j\eta_k \otimes t_{N-J}^{p^J} t_{n-i}^{p^i} t_{m-j}^{p^j} t_{l-k}^{p^k}\,.
\end{aligned}
$$

Now we have (1.11) as follows:

Proposition 4.2.

i) [(1.11)] $\delta R_\gamma(N) = Y_\gamma(N)$ *for* $\gamma \in \langle R1, R2\rangle$.

ii) $\Delta_* R_\gamma(N) = \sum_{I=0}^{N} R_\gamma(I) \otimes t_{N-I}^{p^I}$ *for* $\gamma \in \langle R1, R2\rangle$.

iii) $R_{\sigma^{-1}\gamma\sigma}(N) = R_\gamma(\sigma N)$

$R_{\gamma'\gamma}(N) = R_{\gamma\gamma'}(N) = R_\gamma(N) + R_{\gamma'}(N)$ *for* $\gamma, \gamma' \in \langle R1, R2\rangle$ *and* $\sigma \in F_t$.

Proof. i) By using Proposition 2.3 i), Lemma 3.7 ii), (2.3), (2.9) and (4.2), we see that

$$
\begin{aligned}
\delta^A r_{t+1}(N,n,m,l) &= -\sum_{I=0}^{N} \delta^A x(I) \otimes t_{N-I}^{p^I} \eta_n \eta_m \eta_l + \sum_{I=0}^{N} x(I) \otimes \Delta t_{N-I}^{p^I} \eta_n \eta_m \eta_l \\
&= \sum_{I=0}^{N} \sum_{i=0}^{n} x(I,i) \otimes t_{N-I}^{p^I} t_{n-i}^{p^i} \eta_m \eta_l \\
&\quad - \sum_{I=0}^{N} \sum_{j=0}^{m} x(I,j) \otimes t_{N-I}^{p^I} \eta_n t_{m-j}^{p^j} \eta_l \\
&\quad + \sum_{I=0}^{N} \sum_{k=0}^{l} x(I,k) \otimes t_{N-I}^{p^I} \eta_n \eta_m t_{l-k}^{p^k} \\
&\quad + \sum_{I=0}^{N} \sum_{i=0}^{n} \sum_{j=0}^{m} y_{t+1}(I,i,j) \otimes t_{N-I}^{p^I} t_{n-i}^{p^i} t_{m-j}^{p^j} \eta_l \\
&\quad - \sum_{I=0}^{N} \sum_{i=0}^{n} \sum_{k=0}^{l} y_{t+1}(I,i,k) \otimes t_{N-I}^{p^I} t_{n-i}^{p^i} \eta_m t_{l-k}^{p^k} \\
&\quad + \sum_{I=0}^{N} \sum_{j=0}^{m} \sum_{k=0}^{l} y_{t+1}(I,j,k) \otimes t_{N-I}^{p^I} \eta_n t_{m-j}^{p^j} t_{l-k}^{p^k} \\
&\quad + \sum_{I=0}^{N} \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{l} r_{t+1}(I,i,j,k) \otimes t_{N-I}^{p^I} t_{n-i}^{p^i} t_{m-j}^{p^j} t_{l-k}^{p^k} \\
&= -y_{t+2}(N,n,m,l) + y_{t+2}(N,m,n,l) \\
&\quad - y_{t+2}(N,l,n,m) + y_{t+1}(N,n,m,l) \\
&\quad - y_{t+1}(N,n,l,m) + y_{t+1}(N,m,l,n) \\
&\quad + R_{t+1}(N,n,m,l) \\
&= -y_{\gamma_{t+1}}(N,n,m,l) + R_{t+1}(N,n,m,l).
\end{aligned}
$$

Then, by induction, we can prove that

$$
\delta^A r_i(N) = -y_{\gamma_i}(N) + R_i(N). \tag{4.3}
$$

In fact, by (2.3), (2.8) and (2.9),

$$
\delta^A r_i(N,n) = -\sum_{I=0}^{N} \delta^A r_i(I) \otimes t_{N-I}^{p^I} \eta_n + \sum_{I=0}^{N} r_i(I) \otimes \Delta t_{N-I}^{p^I} \eta_n
$$

$$
\begin{aligned}
&= -\sum_{I=0}^{N} \{-y_{\gamma_i}(I) + R_i(I)\} \otimes t_{N-I}^{p^I}\eta_n \\
&\quad + \sum_{I=0}^{N} R_i(I) \otimes t_{N-I}^{p^I}\eta_n \\
&\quad + \sum_{I=0}^{N}\sum_{a=0}^{n} r_i(I,a) \otimes t_{N-I}^{p^I} t_{n-a}^{p^a} \\
&= -y_{\gamma_i}(N,n) + R_i(N,n).
\end{aligned}
$$

Now,

$$
\begin{aligned}
\delta R_i(N) &= -\sum_{I=0}^{N} \delta^A r_i(I) \otimes t_{N-I}^{p^I} + \sum_{I=0}^{N} r_i(I) \otimes \Delta t_{N-I}^{p^I} \\
&= -\sum_{I=0}^{N} \{-y_{\gamma_i}(I) + R_i(I)\} \otimes t_{N-I}^{p^I} \\
&\quad + \sum_{I=0}^{N}\sum_{J=0}^{I} r_i(J) \otimes t_{I-J}^{p^J} \otimes t_{N-I}^{p^I} \\
&= Y_{\gamma_i}(N).
\end{aligned}
$$

Next, by Lemma 3.7, (2.3), (2.9), (3.3) and (3.4),

$$
\begin{aligned}
\delta^A r_{i,t+1}(N,n,m) &= -\sum_{I=0}^{N} \delta^A y_i(I) \otimes t_{N-I}^{p^I}\eta_n\eta_m + \sum_{I=0}^{N} y_i(I) \otimes \Delta t_{N-I}^{p^I}\eta_n\eta_m \\
&= -\sum_{I=0}^{N} \{-x(I) + x(\sigma_i I) + Y_i(I)\} \otimes t_{N-I}^{p^I}\eta_n\eta_m \\
&\quad + \sum_{I=0}^{N} Y_i(I) \otimes t_{N-I}^{p^I}\eta_n\eta_m \\
&\quad + \sum_{I=0}^{N}\sum_{a=0}^{n} y_i(I,a) \otimes t_{N-I}^{p^I} t_{n-a}^{p^a}\eta_m \\
&\quad - \sum_{I=0}^{N}\sum_{b=0}^{m} y_i(I,b) \otimes t_{N-I}^{p^I}\eta_n t_{m-b}^{p^b}
\end{aligned}
$$

$$+\sum_{I=0}^{N}\sum_{a=0}^{n}\sum_{b=0}^{m} r_{i,t+1}(I,a,b)\otimes t_{N-I}^{p^I}t_{n-a}^{p^a}t_{m-b}^{p^b}$$
$$= -y_{t+1}(N,n,m)+y_{t+1}(\sigma_i N,n,m)$$
$$+\,y_i(N,n,m)-y_i(N,m,n)+R_{i,t+1}(N,n,m)$$
$$= -y_{\gamma_{i,t+1}}(N,n,m)+R_{i,t+1}(N,n,m).$$

Then, by induction, we can prove that

$$\delta^A r_{i,j}(N) = -y_{\gamma_{i,j}}(N)+R_{i,j}(N). \tag{4.4}$$

In fact, by (2.3), (2.8) and (2.9),

$$\delta^A r_{i,j}(N,n) = -\sum_{I=0}^{N}\delta^A r_{i,j}(I)\otimes t_{N-I}^{p^I}\eta_n + \sum_{I=0}^{N} r_{i,j}(I)\otimes \Delta t_{N-I}^{p^I}\eta_n$$
$$= -\sum_{I=0}^{N}\{-y_{\gamma_{i,j}}(I)+R_{i,j}(I)\}\otimes t_{N-I}^{p^I}\eta_n$$
$$+\sum_{I=0}^{N} R_{i,j}(I)\otimes t_{N-I}^{p^I}\eta_n$$
$$+\sum_{I=0}^{N}\sum_{a=0}^{n} r_{i,j}(I,a)\otimes t_{N-I}^{p^I}t_{n-a}^{p^a}$$
$$= -y_{\gamma_{i,j}}(N,n)+R_{i,j}(N,n).$$

Now we see that

$$\delta R_{i,j}(N) = -\sum_{I=0}^{N}\delta^A r_{i,j}(I)\otimes t_{N-I}^{p^I} + \sum_{I=0}^{N} r_{i,j}(I)\otimes \Delta t_{N-I}^{p^I}$$
$$= -\sum_{I=0}^{N}\{-y_{\gamma_{i,j}}(I)+R_{i,j}(I)\}\otimes t_{N-I}^{p^I}$$
$$+\sum_{I=0}^{N}\sum_{J=0}^{I} r_{i,j}(J)\otimes t_{I-J}^{p^J}\otimes t_{N-I}^{p^I}$$
$$= Y_{\gamma_{i,j}}(N).$$

By the definition of $R_\gamma(N)$ and the above results, Lemma 3.7 implies this proposition i).

ii) This is proved by the same manner as ii) of Proposition 2.3.

iii) By the definition, this is trivial.

5. ϕ_q and $\bar{\phi}_{q,N}$. In this section, we argue the properties of ϕ_q and $\bar{\phi}_{q,N}$ and prove Theorem 1.3.

In the first place, we prepare the following lemma.

Lemma 5.1.

i) $\bar{\phi}_q(\sigma_i^{-1}\sigma_i\sigma) = \bar{\phi}_q(\sigma) = \bar{\phi}_q(\sigma_i\sigma_i^{-1}\sigma)$.

ii) $\bar{\phi}_q(\sigma\tau) = \bar{\phi}_q(\tau) + (-q)^{|\tau|}\bar{\phi}_q(\sigma)$.

iii) $\bar{\phi}_q((\sigma\tau)\delta) = \bar{\phi}_q(\sigma(\tau\delta)) = \bar{\phi}_q(\delta) + (-q)^{|\delta|}\bar{\phi}_q(\tau) + (-q)^{|\tau\delta|}\bar{\phi}_q(\sigma)$.

iv) $\bar{\phi}_q(\sigma^{-1}) = -(-q)^{-|\sigma|}\bar{\phi}_q(\sigma)$.

v) $\bar{\phi}_q(\sigma^{-1}\gamma\sigma) = (-q)^{|\sigma|}\bar{\phi}_q(\gamma) + \bar{\phi}_q(\sigma) - (-q)^{|\gamma|}\bar{\phi}_q(\sigma)$.

vi) $\bar{\phi}_q(\gamma_i) = 0$ *and* $\bar{\phi}_q(\gamma_{i,j}) = 0$.

vii) *For* $\gamma \in \langle R1, R2\rangle$, $\bar{\phi}_q(\gamma) = 0$ *and* $\bar{\phi}_q(\gamma\sigma) = \bar{\phi}_q(\sigma) = \bar{\phi}_q(\sigma\gamma)$

Proof. i) By definition,

$$\begin{aligned}\bar{\phi}_q(\sigma_i^{-1}\sigma_i\sigma) &= \bar{\phi}_q(\sigma_i\sigma) + (-q)^{|\sigma|+1}q^{-1} \\ &= \bar{\phi}_q(\sigma) + (-q)^{|\sigma|} + (-q)^{|\sigma|+1}q^{-1} = \bar{\phi}_q(\sigma).\end{aligned}$$

As the same way, we see $\bar{\phi}_q(\sigma_i\sigma_i^{-1}\sigma) = \bar{\phi}_q(\sigma)$.

ii) We assume $\bar{\phi}_q(\sigma\tau) = \bar{\phi}_q(\tau) + (-q)^{|\tau|}\bar{\phi}_q(\sigma)$. Then, by definition,

$$\begin{aligned}\bar{\phi}_q(\sigma_i\sigma\tau) &= \bar{\phi}_q(\sigma\tau) + (-q)^{|\sigma\tau|} \\ &= \bar{\phi}_q(\tau) + (-q)^{|\tau|}\bar{\phi}_q(\sigma) + (-q)^{|\tau|+|\sigma|} = \bar{\phi}_q(\tau) + (-q)^{|\tau|}\bar{\phi}_q(\sigma_i\sigma).\end{aligned}$$

As the same way, $\bar{\phi}_q(\sigma_i^{-1}\sigma\tau) = \bar{\phi}_q(\tau) + (-q)^{|\tau|}\bar{\phi}_q(\sigma_i^{-1}\sigma)$. By induction on the word σ, we see ii).

iii) By ii), this is trivial.

iv) By ii), $\bar{\phi}_q(\sigma^{-1}) + (-q)^{-|\sigma|}\bar{\phi}_q(\sigma) = \bar{\phi}_q(\sigma\sigma^{-1}) = \bar{\phi}_q(e) = 0$.

v) This is easy by iii) and iv).

vi) By ii),

$$\begin{aligned}\bar{\phi}_q(\gamma_i) &= \bar{\phi}_q(\sigma_{i+1}) - q\bar{\phi}_q(\sigma_i) + q^2\bar{\phi}_q(\sigma_{i+1}) \\ &\qquad - q^3\bar{\phi}_q(\sigma_i^{-1}) + q^2\bar{\phi}_q(\sigma_{i+1}^{-1}) - q\bar{\phi}_q(\sigma_i) \\ &= 1 - q + q^2 - q^2 + q - 1 = 0\end{aligned}$$

By the same way, we see the another equation.

vii) is seen by ii), v), vi) and $|\gamma| = 0$.

This lemma i) implies that $\bar{\phi}_q$ is well defined. By vii), $\bar{\phi}_q$ induces

$$\phi_q : B_t = F_t/\langle R1, R2\rangle \to \mathbf{Z}/(p).$$

Hence we have the following.

Proposition 5.2.
i) ϕ_q *is well defined.*

ii) $\phi_q(\sigma\tau) = \phi_q(\tau) + (-q)^{|\tau|}\phi_q(\sigma)$.

iii) $\phi_q(\sigma^{-1}) = -(-q)^{-|\sigma|}\phi_q(\sigma)$.

iv) $\phi_q(\sigma^{-1}\gamma\sigma) = (-q)^{|\sigma|}\phi_q(\gamma) + \phi_q(\sigma) - (-q)^{|\gamma|}\phi_q(\sigma)$.

Next consider the map $\bar{\phi}_{q,N} : F_t \to C_Y^{t-1}(N)$ of Definition 1.2 (2).

Proposition 5.3.
i) $\bar{\phi}_{q,N}(\sigma_i^{-1}\sigma_i\sigma) = \bar{\phi}_{q,N}(\sigma) = \bar{\phi}_{q,N}(\sigma_i\sigma_i^{-1}\sigma)$. *Hence* $\bar{\phi}_{q,N}$ *is well defined.*

ii) $\bar{\phi}_{q,N}(\sigma\tau) = \bar{\phi}_{q,N}(\tau) + q^{|\tau|}\bar{\phi}_{q,\tau N}(\sigma)$.

iii) $\bar{\phi}_{q,N}((\sigma\tau)\delta) = \bar{\phi}_{q,N}(\sigma(\tau\delta)) = \bar{\phi}_{q,N}(\delta) + q^{|\delta|}\bar{\phi}_{q,\delta N}(\tau) + q^{|\tau\delta|}\bar{\phi}_{q,\tau\delta N}(\sigma)$.

iv) $\bar{\phi}_{q,N}(\sigma^{-1}) = -q^{-|\sigma|}\bar{\phi}_{q,\sigma^{-1}N}(\sigma)$.

v) $\bar{\phi}_{q,N}(\sigma^{-1}\gamma\sigma) = q^{|\sigma|}\bar{\phi}_{q,\sigma N}(\gamma) + \bar{\phi}_{q,N}(\sigma) - q^{|\gamma|}\bar{\phi}_{q,\sigma^{-1}\gamma\sigma N}(\sigma)$.

Proof. i) By definition and Proposition 3.4 i),

$$\begin{aligned}\bar{\phi}_{q,N}(\sigma_i^{-1}\sigma_i\sigma) &= \bar{\phi}_{q,N}(\sigma) + q^{|\sigma|}\bar{\phi}_{q,\sigma N}(\sigma_i) + q^{|\sigma|+1}\bar{\phi}_{q,\sigma_i\sigma N}(\sigma_i^{-1}) \\ &= \bar{\phi}_{q,N}(\sigma) + q^{|\sigma|}Y_i(\sigma N) + q^{|\sigma|+1}q^{-1}Y_i(\sigma_i\sigma N) \\ &= \bar{\phi}_{q,N}(\sigma)\end{aligned}$$

The another equation is seen by the same way.
ii) By induction on word σ and

$$\begin{aligned}\bar{\phi}_{q,N}(\sigma_i^{\pm 1}\sigma\tau) &= \{\bar{\phi}_{q,N}(\tau) + q^{|\tau|}\bar{\phi}_{q,\tau N}(\sigma)\} + q^{|\sigma|+|\tau|}\bar{\phi}_{q,\sigma\tau N}(\sigma_i^{\pm 1}) \\ &= \bar{\phi}_{q,N}(\tau) + q^{|\tau|}\bar{\phi}_{q,\tau N}(\sigma_i^{\pm 1}\sigma),\end{aligned}$$

we see ii).
iii) is given by ii).
iv) is given by ii) and $\bar{\phi}_{q,N}(\sigma_i\sigma_i^{-1}) = \bar{\phi}_{q,N}(e) = 0$.
v) is given by iii) and iv).

Now we argue $\bar{\phi}_{q,N}(\langle R1, R2\rangle)$. The following lemma is proved by Proposition 5.3.

Lemma 5.4. *For $\gamma, \gamma' \in \langle R1, R2\rangle$ and $\sigma \in F_t$, we have the following.*

i) $\bar{\phi}_{q,N}(\sigma^{-1}\gamma\sigma) = q^{|\sigma|}\bar{\phi}_{q,\sigma N}(\gamma)$.

ii) $\bar{\phi}_{q,N}(\gamma\gamma') = \bar{\phi}_{q,N}(\gamma)+\bar{\phi}_{q,N}(\gamma')$. *Hence $\bar{\phi}_{q,N}(\langle R1, R2\rangle)$ is a submodule of $C_Y^{t-1}(N)$.*

iii)

$$\begin{aligned}&\bar{\phi}_{q,N}(\sigma\gamma) = \bar{\phi}_{q,N}(\sigma) + \bar{\phi}_{q,N}(\gamma) \quad \textit{and} \\ &\bar{\phi}_{q,N}(\gamma\sigma) = \bar{\phi}_{q,N}(\sigma(\sigma^{-1}\gamma\sigma)) = \bar{\phi}_{q,N}(\sigma) + \bar{\phi}_{q,N}(\sigma^{-1}\gamma\sigma).\end{aligned}$$

Hence $\bar{\phi}_{q,N}$ induces a map

$$\phi_{q,N} : B_t = F_t/\langle R1, R2\rangle \to C_Y^{t-1}(N)/\bar{\phi}_{q,N}(\langle R1, R2\rangle).$$

iv)

$$\begin{aligned}\bar{\phi}_{q,N}(\gamma_i) &= Y_{i+1}(N) - Y_i(N) - q\{Y_{i+1}(\sigma_i N) - Y_i(\sigma_{i+1}N)\} \\ &\qquad + q^2\{Y_{i+1}(\sigma_i\sigma_{i+1}N) - Y_i(\sigma_{i+1}\sigma_i N)\} \quad \textit{and} \\ \bar{\phi}_{q,N}(\gamma_{i,j}) &= Y_j(N) - Y_i(N) - q\{Y_j(\sigma_i N) - Y_i(\sigma_j N)\}.\end{aligned}$$

Next we define elements $Z_{i,\sigma}(N) = Y_1(N) - (-1)^{|\sigma|}Y_i(\sigma N)$ and a module $C_Z^{t-1}(N) = \langle Z_{i,\sigma}(N)\rangle$.

Lemma 5.5.

i) *If* $p=2$ *then* $C_Y^{t-1}(N)/C_Z^{t-1}(N) = \mathbf{Z}/(2)$. *If* $p>2$ *and* $n_i \neq n_j$ *for any* $i \neq j$ *then* $C_Y^{t-1}(N)/C_Z^{t-1}(N) = \mathbf{Z}/(p)$. *If* $p>2$ *and* $n_i = n_j$ *for some* $i \neq j$ *then* $C_Y^{t-1}(N)/C_Z^{t-1}(N) = 0$.

ii) $\bar{\phi}_{q,N}(\langle R1, R2\rangle) \subset C_Z^{t-1}(N)$.

iii) *If* $q^2 \not\equiv 1$ (p) *and* $t \geq 5$ *then* $\bar{\phi}_{q,N}(\langle R1, R2\rangle) = C_Z^{t-1}(N)$.

Proof. i) $\{Y_i(\sigma N) | \sigma \in F_t\}$ is a basis of $C_Y^{t-1}(N)$, and so is $\{Y_1(N)\} \cup \{Z_{i,\sigma}(N) | \sigma \in F_t\}$. Hence $C_Y^{t-1}(N)/C_Z^{t-1}(N)$ has a generator $Y_1(N)$. If $p=2$ then $Y_i(\sigma N) \neq 0$. Therefore $C_Y^{t-1}(N)/C_Z^{t-1}(N) = \mathbf{Z}/(2)$.
We assume $p>3$. If $n_i \neq n_j$ for any $i \neq j$ then $Y_i(\sigma N) \neq 0$ for any i and $\sigma \in F_t$, and so $C_Y^{t-1}(N)/C_Z^{t-1}(N) = \mathbf{Z}/(p)$. If $n_i = n_j$ for some $i \neq j$ then there exists an element $\sigma \in F_t$ so that $n_{\bar{\sigma}^{-1}(1)} = n_{\bar{\sigma}^{-1}(2)}$, i.e., $\sigma_i \sigma N = \sigma N$, and so $Y_1(\sigma N) = Y_1(\sigma_1 \sigma N) = -Y_1(\sigma N)$. Hence $Y_1(\sigma N) = 0$ by $p>2$ and $Y_1(N) = Z_{1,\sigma}(N) \in C_Z^{t-1}(N)$. Thus $C_Y^{t-1}(N)/C_Z^{t-1}(N) = 0$.

ii) By definition, if $|\sigma| = |\tau|$ then $Y_i(\sigma N) - Y_j(\tau N) = (-1)^{|\sigma|}\{Z_{j,\tau}(N) - Z_{i,\sigma}(N)\}$. We see ii) by Lemma 5.4.

iii) By Lemma 5.4,

$$\begin{aligned}\bar{\phi}_{q,N}(\sigma^{-1}\gamma_{i,j}\sigma) &= q^{|\sigma|}\{Y_j(\sigma N) - Y_i(\sigma N)\} \\ &\quad - q^{|\sigma|+1}\{Y_j(\sigma_i \sigma N) - Y_i(\sigma_j \sigma N)\}\end{aligned}$$

and

$$\begin{aligned}\bar{\phi}_{q,N}((\sigma_i\sigma_j\sigma)^{-1}\gamma_{i,j}\sigma_i\sigma_j\sigma) &= q^{|\sigma|+2}\{Y_j(\sigma_i\sigma_j\sigma N) - Y_i(\sigma_i\sigma_j\sigma N)\} \\ &\quad - q^{|\sigma|+3}\{Y_j(\sigma_i\sigma_i\sigma_j\sigma N) - Y_i(\sigma_j\sigma_i\sigma_j\sigma N)\} \\ &= q^{|\sigma|+3}\{Y_j(\sigma N) - Y_i(\sigma N)\} \\ &\quad - q^{|\sigma|+2}\{Y_j(\sigma_i\sigma N) - Y_i(\sigma_j\sigma N)\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\bar{\phi}_{q,N}(\sigma^{-1}\gamma_{i,j}\sigma) - q^{-1}\bar{\phi}_{q,N}((\sigma_i\sigma_j\sigma)^{-1}\bar{\gamma}_{i,j}\sigma_i\sigma_j\sigma) \\ &\qquad = (1-q^2)q^{|\sigma|}\{Y_j(\sigma N) - Y_i(\sigma N)\},\end{aligned}$$

and so $Y_j(\sigma N) \equiv Y_i(\sigma N) \mod \bar{\phi}_{q,N}(\langle R1, R2\rangle)$ for $|i-j| \geq 2$ if $q^2 \not\equiv 1$ (p). Hence $Y_i(\sigma N) \equiv Y_1(\sigma N)$ for $i \geq 3$. By $t \geq 5$, $Y_2(\sigma N) \equiv Y_4(\sigma N) \equiv Y_1(\sigma N)$. Thus

$$(5.1) \qquad Y_i(\sigma N) \equiv Y_1(\sigma N) \mod \bar{\phi}_{q,N}(\langle R1, R2\rangle) \quad \text{for any } i.$$

By this equation,

$$Y_i(\sigma_j^{\pm 1}\sigma N) \equiv Y_1(\sigma_j^{\pm 1}\sigma N) \equiv Y_j(\sigma_j^{\pm 1}\sigma N) = -Y_j(\sigma N) \equiv -Y_1(\sigma N).$$

By induction, $Y_i(\sigma N) \equiv (-1)^{|\sigma|} Y_1(N) \mod \bar{\phi}_{q,N}(\langle R1, R2\rangle)$. Thus

$$Z_{i,\sigma}(N) \in \bar{\phi}_{q,N}(\langle R1, R2\rangle).$$

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. i-ii) are trivial by the definitions, (1.8) and Proposition 3.3.
iii) is Lemma 5.4 ii).
iv) We define a map $\psi : C_Y^{t-1}(N) \to \mathbf{Z}/(p)$ by taking $\psi(Y_i(\sigma N)) = (-1)^{|\sigma|}$. If $n_i \neq n_j$ for any $i \neq j$ then ψ is well defined. By Definition 1.1 and 1.2, $\bar{\phi}_q = \psi\bar{\phi}_{q,N}$. Now $\operatorname{Ker}\psi = C_Z^{t-1}(N) = \bar{\phi}_{q,N}(\langle R1, R2\rangle)$ by Lemma 5.5, and so $\bar{\phi}_{q,N}$ induces ϕ_q.

References

[1] J. F. Adams: Stable Homotopy and Generalised Homology, Univ. of Chicago Press, Chicago Illinois London, 1974.

[2] G. Burde and H. Zieschang: Knots, Walter de Gruyter, Berlin New York, 1985.

[3] H. Cartan and S. Eilenberg: Homological Algebra, Princeton Univ. Press, Princeton New Jersey, 1956.

[4] M. Hikida: Relations between several Adams spectral sequences, Hiroshima Math. J. **19** (1989), 37–76.

[5] M. Hikida and K. Shimomura: An exact sequence related to Adams-Novikov E_2-terms of a cofibering, J. Math. Soc. Japan **46** (1994), 645–661.

[6] H. R. Miller: On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space, J. Pure appl. Algebra **20** (1981), 287–312.

[7] J. W. Milnor: The Steenrod algebra and its dual, Ann. of Math. **67** (1958), 150–171.

[8] J. W. Milnor and J. C. Moore: On the structure of Hopf algebras, Ann. of Math. **81** (1965), 211–264.

[9] D. C. Ravenel: Complex Cobordism and Stable Homotopy Groups of Spheres, Academic Press, New York, 1986.

[10] N. E. STEENROD and D. B. A. EPSTEIN: Cohomology Operations, Annals of Math. Studies No. 50, Princeton Univ. Press, Princeton, 1962.
[11] R. M. SWITZER: Algebraic Topology, Homotopy and Homology, Springer-Verlag, Berlin New York, 1975.

HIROSHIMA PREFECTURAL UNIVERSITY
SHOBARA-SHI, HIROSHIMA 727 JAPAN
E-mail: hikida@bus.hiroshima-pu.ac.jp

(*Received July 8, 1997*)