

ON RINGS OF CERTAIN TYPE ASSOCIATED WITH SIMPLE RING-EXTENSIONS

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Let R be a Noetherian domain (which is commutative and has a unit), let $R[X]$ be a polynomial ring, let α be an element of an algebraic extension field of the quotient field K of R and let $\pi: R[X] \rightarrow R[\alpha]$ be the R -algebra homomorphism sending X to α . Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Then $\eta_i \in K$ ($1 \leq i \leq d$) are uniquely determined by α . Put $d = [K(\alpha) : K]$, $I_{\eta_i} := R :_R \eta_i$ and $I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i}$. If $\text{Ker}(\pi) = I_{[\alpha]} \varphi_\alpha(X) R[X]$, we say that α is *anti-integral* over R (cf. [3]). Put $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \dots, \eta_d)$. Then $J_{[\alpha]} = c(I_{[\alpha]} \varphi_\alpha(X))$, where $c(\)$ denotes the ideal generated by the coefficients of the polynomials in $(\)$, that is, the content ideal of $(\)$. If $J_{[\alpha]} \not\subseteq p$ for all $p \in \text{Dp}_1(R) := \{p \in \text{Spec}(R) \mid \text{depth} R_p = 1\}$, the element α is called a *super-primitive element* over R . A super-primitive element over R is anti-integral over R (cf. [3, Theorem 1.12]). It is also known that any algebraic element over a Krull domain R is super-primitive over R (cf. [3, Theorem 1.13]), and hence α is anti-integral over R . We also note here that $I_{[\alpha]} = R \Leftrightarrow R[\alpha]$ is integral over R and that $J_{[\alpha]} = R \Leftrightarrow R[\alpha]$ is flat over R , provided that α is anti-integral over R .

Assume that α is an anti-integral element of degree $d \geq 2$ over R . We have seen in [1] that $R[\alpha] \cap R[\alpha^{-1}] = R \oplus I_{[\alpha]} \zeta_1 \oplus \cdots \oplus I_{[\alpha]} \zeta_{d-1}$ as an R -module, where $\zeta_i := \alpha^i + \eta_1 \alpha^{i-1} + \cdots + \eta_i$ ($1 \leq i \leq d$). Note that $\zeta_d = \varphi_\alpha(\alpha) = 0$. Put $\zeta_0 := 1$ and $\eta_0 := 1$ for convenience. It is easy to see that $\alpha \zeta_i = \zeta_{i+1} - \eta_{i+1}$ ($0 \leq i \leq d-1$) by definition. Note that $R[\alpha] \cap R[\alpha^{-1}]$ and $R[\alpha]$ are birational and their quotient fields are equal to $K(\alpha)$. Let H be an ideal of R and put $C_H := R + H\zeta_1 + \cdots + H\zeta_{d-1}$, which is an R -submodule in $K(\alpha) = K \oplus K\zeta_1 \oplus \cdots \oplus K\zeta_{d-1}$. Our objective of this paper is to investigate when C_H forms a subring of $K(\alpha)$ (R -algebra).

Throughout this paper, we use the above notation and conventions unless otherwise specified. Our general reference for unexplained terminology is [2].

We start with the following lemma, which is obviously seen.

Lemma 1. *Assume that α is an anti-integral element of degree d*

over R . Let H denote an ideal of R . Then C_H forms a subring of $K(\alpha)$ if and only if $H^2\zeta_i\zeta_j \subseteq C_H$ for all $i, j \in \{1, 2, \dots, d-1\}$ with $j \leq i$.

Proposition 2. Let H denote an ideal of R . Assume that α is an anti-integral element over R of degree $d = 2$. Then C_H forms a subring of $K(\alpha)$ if and only if $H^2\eta_1 \subseteq H$ and $H^2\eta_2 \subseteq R$.

Proof. Note $\zeta_1 = \alpha + \eta_1$ and hence $\zeta_1^2 = \zeta_1(\alpha + \eta_1) = \alpha\zeta_1 + \eta_1\zeta_1 = \zeta_2 - \eta_2 + \eta_1\zeta_1 = \eta_1\zeta_1 - \eta_2$ (here $\zeta_2 = 0$). Assume that C_H is a subring of $K(\alpha)$. Then $H^2\zeta_1^2 \in C_H$ yields that for any $h \in H$, $h^2\eta_1\zeta_1 - h^2\eta_2 \in R + H\zeta_1 \subseteq K \oplus K\zeta_1$. Consequently, $H^2\eta_1 \subseteq H$ and $H^2\eta_2 \subseteq R$. Conversely, take $a + h_1\zeta_1, b + h_2\zeta_1 \in C_H$ with $h_1, h_2 \in H$ and $a, b \in R$. Then $(a + h_1\zeta_1)(b + h_2\zeta_1) = ab + (ah_2 + bh_1)\zeta_1 + h_1h_2\zeta_1^2 \in R + H\zeta_1$ because $h_1h_2\zeta_1^2 \in H^2\zeta_1^2 = H^2(\eta_1\zeta_1 - \eta_2) \subseteq R + H\zeta_1 = C_H$.

We consider the case $d = 3$, and compute $\zeta_i\zeta_j$ by a definite calculation.

Example 3. Assume that α is an anti-integral element of degree $d = 3$ over R . Note $\zeta_1 = \alpha + \eta_1$, $\zeta_2 = \alpha\zeta_1 + \eta_2$ and $\zeta_3 = \alpha\zeta_2 + \eta_3$. Thus

$$\begin{aligned}\zeta_1^2 &= \zeta_1(\alpha + \eta_1) = \alpha\zeta_1 + \eta_1\zeta_1 = \zeta_2 - \eta_2 + \eta_1\zeta_1, \\ \zeta_2\zeta_1 &= \zeta_2(\alpha + \eta_1) = \alpha\zeta_2 + \zeta_2\eta_1 = \zeta_3 - \eta_3 + \eta_1\zeta_2 = -\eta_3 + \eta_1\zeta_2 \quad \text{and} \\ \zeta_2^2 &= \zeta_2(\alpha\zeta_1 + \eta_2) = (\alpha\zeta_2)\zeta_1 + \zeta_2\eta_2 = (\zeta_3 - \eta_3)\zeta_1 + \zeta_2\eta_2 = -\eta_3\zeta_1 + \eta_2\zeta_2\end{aligned}$$

because $\zeta_3 = 0$.

Assume that C_H is a subring of $K(\alpha)$. Then we have $H^2\eta_1 \subseteq H$, $H^2\eta_2 \subseteq H$ and $H^2\eta_3 \subseteq H$ since $\zeta_1^2, \zeta_2\zeta_1$ and ζ_2^2 are in C_H .

Conversely, if $H^2\eta_1 \subseteq H$, $H^2\eta_2 \subseteq H$ and $H^2\eta_3 \subseteq H$, then $\zeta_1^2, \zeta_2\zeta_1$ and ζ_2^2 are in C_H . So we conclude that C_H is a subring of $k(\alpha)$ by Lemma 1.

Lemma 4. Assume that α is an anti-integral element of degree $d(\geq 3)$ over R and that $1 \leq j \leq i < d$.

(i) If $i + j < d$, then

$$(*) \quad \zeta_i\zeta_j = \zeta_{i+j} - \sum_{t=1}^j \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-1} \eta_{j-s}\zeta_{i+s}.$$

(ii) If $i + j \geq d$, then

$$(**) \quad \zeta_i\zeta_j = - \sum_{t=1}^{d-i} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{d-i-1} \eta_{j-s}\zeta_{i+s}.$$

(Note that $d - i > 0$ and $d - i - 1 \geq 0$ because $1 \leq j \leq i < d$.)

Proof. Note first that $\eta_0 := 1$, $\zeta_0 := 1$, $\zeta_d = 0$ and $\zeta_{i+1} = \alpha\zeta_i + \eta_{i+1}$. Now we compute $\zeta_i\zeta_j$ as follows :

$$\begin{aligned} \zeta_i\zeta_j &= \zeta_i(\alpha\zeta_{j-1} + \eta_j) \\ &= \alpha\zeta_i\zeta_{j-1} + \eta_j\zeta_i \\ &= (\zeta_{i+1} - \eta_{i+1})\zeta_{j-1} + \eta_j\zeta_i \\ &= \zeta_{i+1}\zeta_{j-1} - \eta_{i+1}\zeta_{j-1} + \eta_j\zeta_i \\ &= \zeta_{i+2}\zeta_{j-2} - \eta_{i+2}\zeta_{j-2} + \eta_{j-1}\zeta_{i+1} - \eta_{i+1}\zeta_{j-1} + \eta_j\zeta_i \\ &= \zeta_{i+2}\zeta_{j-2} - (\eta_{i+2}\zeta_{j-2} + \eta_{i+1}\zeta_{j-1}) + (\eta_j\zeta_i + \eta_{j-1}\zeta_{i+1}) \\ &\quad \dots\dots\dots \\ &\quad \dots\dots\dots . \end{aligned}$$

(i) Repeat the above process, we have

$$\zeta_i\zeta_j = \zeta_{i+j}\zeta_0 - \sum_{t=1}^j \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-1} \eta_{j-s}\zeta_{i+s},$$

that is,

$$\zeta_i\zeta_j = \zeta_{i+j} - \sum_{t=1}^j \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-1} \eta_{j-s}\zeta_{i+s}.$$

(ii) Put $\ell := i + j - d$. Then $j \leq i < d$ yields $\ell < d - 1$ and $j - \ell \geq 1$. Thus continuing the above process yields

$$\zeta_i\zeta_j = \zeta_d\zeta_\ell - \sum_{t=1}^{j-\ell} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-\ell-1} \eta_{j-s}\zeta_{i+s},$$

that is,

$$(**) \quad \zeta_i\zeta_j = - \sum_{t=1}^{d-i} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{d-i-1} \eta_{j-s}\zeta_{i+s}.$$

Remark 5. Assume that α is an anti-integral element of degree $d(\geq 3)$ over R and that $1 \leq j \leq i < d$. For (i, j) with $i + j < d$, let $\Delta_1^{(j)} := \{ j - t \mid t = 1, \dots, j \}$ and $\Delta_1^{(i,j)} := \{ i + s \mid s = 0, \dots, j - 1 \}$. For (i, j) with $i + j \geq d$, let $\Delta_2^{(j,i)} := \{ j - t \mid t = 1, \dots, d - i \}$ and $\Delta_2^{(i)} := \{ i + s \mid s = 0, \dots, d - i - 1 \}$.

- (i) If $i + j < d$, then $\Delta_1^{(i,j)} \cap \Delta_1^{(j)} = \emptyset$ because $j - t < i + s$, and $\bigcup_{i+j < d} (\Delta_1^{(i,j)} \cup \Delta_1^{(j)}) = \{0, 1, 2, \dots, d-1\}$.
- (ii) Assume $i + j \geq d$. Then $\Delta_2^{(i)} \cap \Delta_2^{(j,i)} = \emptyset$ because $j - t < i + s$. Put $e_k := -1$ if $k \in \Delta_2^{(j,i)}$ and $e_k := 1$ if $k \in \Delta_2^{(i)}$. Then

$$(**) \quad \zeta_i \zeta_j = \sum_{k \in \Delta_2^{(j,i)} \cup \Delta_2^{(i)}} e_k \eta_{i+j-k} \zeta_k.$$

Consider ζ_{d-1}^2 . Then we have

$$\begin{aligned} \zeta_{d-1}^2 &= \zeta_{d-1}(\alpha \zeta_{d-2} + \eta_{d-1}) \\ &= \alpha \zeta_{d-1} \zeta_{d-2} + \eta_{d-1} \zeta_{d-1} \\ &= (\zeta_d - \eta_d) \zeta_{d-2} + \eta_{d-1} \zeta_{d-1} \\ &= -\eta_d \zeta_{d-2} + \eta_{d-1} \zeta_{d-1}. \end{aligned}$$

Hence if $\zeta_{d-1}^2 \in C_H$, then $H^2 \eta_d \in H$.

Theorem 6. *Assume that α is an anti-integral element of degree $d(\geq 3)$ over R . Let H be an ideal of R . Then $C_H := R + H\zeta_1 + \dots + H\zeta_{d-1}$ is a subring of $K(\alpha)$ if and only if $H^2 \eta_i \subseteq H$ for all i ($1 \leq i \leq d$).*

Proof. Our conclusion follows Lemmas 1 and 4 and Remark 5.

Corollary 7. *Assume that α is an anti-integral element of degree $d(\geq 3)$ over R . Let H be an ideal of R . If C_H is a subring of $K(\alpha)$, then every element in $H\eta_i$ is integral over R for all i ($1 \leq i \leq d$).*

Proof. We have $H^2 \eta_i \subseteq H$ for all i ($1 \leq i \leq d$) by Theorem 6. So $H\eta_i \subseteq H :_K H$. Since H is a finitely generated ideal, any element in $H :_K H$ is integral over R . Thus every element of $H\eta_i$ is integral over R for all i ($1 \leq i \leq d$).

Corollary 8. *Assume that R is normal and that α is of degree $d(\geq 3)$ over R . Let H be an ideal of R . If C_H is a subring of $K(\alpha)$, then $H \subseteq I_{[\alpha]}$ and hence $C_H \subseteq R[\alpha] \cap R[\alpha^{-1}]$.*

Proof. Since R is normal, the element α is anti-integral over R by [3]. We have $H\eta_i \subseteq H :_K H = R$ for all i ($1 \leq i \leq d$), that is, $H \subseteq I_{[\alpha]}$.

Theorem 9. *Let H be an ideal of R . Assume that α is an anti-integral element of degree $d(\geq 2)$ over R .*

- (1) *If C_H is a subring of $K(\alpha)$, then $H^2 \subseteq I_{[\alpha]}$.*
- (2) *Assume that $\text{grade}(H) > 1$. Then C_H is a subring of $K(\alpha)$ if and only if $\eta_i \in R$ for all i if and only if α is integral over R .*
- (3) *If $I_{[\alpha]} \neq R$ and if C_H is a subring of $K(\alpha)$, then $\text{grade}(H) \leq 1$.*
- (4) *When H is an invertible ideal and $d \geq 3$, $H \subseteq I_{[\alpha]}$ if and only if C_H is a subring of $K(\alpha)$.*

Proof. (1) Assume that $d \geq 3$. Since C_H is a ring, we have $H^2\eta_i \subseteq H$ for all $(1 \leq i \leq d)$ by Theorem 6. Hence $H^2\eta_i \subseteq H \subseteq R$, which means that $H^2 \subseteq I_{\eta_i}$ for all $(1 \leq i \leq d)$. Thus $H^2 \subseteq I_{[\alpha]}$. Next assume that $d = 2$. Then Proposition 2 yields our conclusion.

(2) Assume that C_H is a subring of $K(\alpha)$. Then $H^2 \subseteq I_{[\alpha]}$ by (1). Hence $1 < \text{grade}(H) \leq \text{grade}(I_{[\alpha]})$, which means that $I_{[\alpha]} = R$ because $I_{[\alpha]}$ is a divisorial ideal of R . So we have $I_{\eta_i} = R$ for all i ($1 \leq i \leq d$), that is, $\eta_i \in R$ for all i ($1 \leq i \leq d$). Furthermore α is integral over R by [3]. Conversely, assume that α is integral over R . Then $I_{[\alpha]} = R$ (cf. [3]), and hence $\eta_i \in R$ for all $(1 \leq i \leq d)$. So if $d \geq 3$, $H^2\eta_i \subseteq H$ yields that C_H is a ring by Theorem 6. If $d = 2$, C_H is a ring by Proposition 2.

(3) Our conclusion follows the result (2).

(4) Since $d \geq 3$, C_H is a subring if and only if $H^2\eta_i \subseteq H$ for all i ($1 \leq i \leq d$). Since H is invertible, $H^2\eta_i \subseteq H$ for all i ($1 \leq i \leq d$) if and only if $H\eta_i \subseteq R$ for all i ($1 \leq i \leq d$) if and only if $H \subseteq I_{[\alpha]}$.

Theorem 10. *Assume that α is an anti-integral element of degree $d(\geq 2)$ over R . Let H be an ideal of R . If $H \subseteq I_{[\alpha]}$, then C_H is a subring of $K(\alpha)$.*

Proof. The inclusion $H \subseteq I_{[\alpha]}$ yields that $H^2\eta_i \subseteq H(\subseteq R)$ for all i ($1 \leq i \leq d$). Hence C_H is a ring by Proposition 2 and Theorem 6.

Corollary 11. *Assume that R is a locally factorial domain and that α is an element of degree $d(\geq 2)$. The collection $\Delta := \{C_H \mid H \text{ is an ideal of } R \text{ and } C_H \text{ is a subring of } K(\alpha)\}$ has the maximum member $C_{I_{[\alpha]}} (= R[\alpha] \cap R[\alpha^{-1}])$.*

Proof. Note first that α is anti-integral over R because R is a Krull domain (cf. [3]). Take $C_H \in \Delta$. Then $H^2 \subseteq I_{[\alpha]}$ by Theorem 9(1). Since $I_{[\alpha]}$ is a divisorial ideal, we have $\text{grade}(H) = \text{grade}(H^2) = 1$. Since R is locally factorial, every non-zero ideal of grade one is invertible. So H is invertible. Hence $H \subseteq I_{[\alpha]}$ by Theorem 9(4).

Finally we close this paper by the following results concerned with the ring $R[\alpha] \cap R[\alpha^{-1}]$.

Proposition 12. *Assume that α is an anti-integral element of degree d . If an element a in R is a non-zero-divisor on $R/I_{[\alpha]}$, then $R[a\alpha] \cap R[\alpha^{-1}] = R + I_{[\alpha]}(a\zeta_1) + \cdots + I_{[\alpha]}(a^{d-1}\zeta_{d-1})$.*

Proof. We have only to show the inclusion (\subseteq) because $a^i I_{[\alpha]} \zeta_i \subseteq R[a\alpha] \cap R[\alpha^{-1}]$. Take an element $\beta \in R[a\alpha] \cap R[\alpha^{-1}]$ and write $\beta = x_n(a\alpha)^n + \cdots + x_1(a\alpha) + x_0 = y_0 + y_1(\alpha^{-1}) + \cdots + y_m(\alpha^{-1})^m$ with $x_i, y_j \in R$. Then we have $\alpha^m \beta = x_n a^n \alpha^{n+m} + \cdots + x_0 \alpha^m = y_0 \alpha^m + y_1 \alpha^{m-1} + \cdots + y_m$. Put $f(X) := x_n a^n X^{n+m} + \cdots + x_0 X^m - (y_0 X^m + y_1 X^{m-1} + \cdots + y_m) \in \text{Ker}(\pi)$, where $\pi: R[X] \rightarrow R[\alpha]$ denotes the canonical R -homomorphism. Since α is an anti-integral element over R , $\text{Ker}(\pi) = I_{[\alpha]} \varphi_\alpha(X) R[X]$. Hence $x_n a^n \in I_{[\alpha]}$. Since a is a non-zero-divisor of $R/I_{[\alpha]}$, we have $x_n \in I_{[\alpha]}$. Put $\beta_*(X) := x_n a^n X^n + \cdots + x_1 a X + x_0$. Since $x_n \varphi_\alpha(X) \in R[X]$, considering $\deg(\beta_*(X) - x_n \varphi_\alpha(X) X^{n-d}) < n$ if $n \geq d$, we can use the induction on n and assume that $n < d$. So considering $\beta - x_n a^n \zeta_n$ instead of β , we can conclude that $\beta \in R + I_{[\alpha]}(a\zeta_1) + \cdots + I_{[\alpha]}(a^{d-1}\zeta_{d-1})$ by induction on n .

Proposition 13. *Assume that α is an anti-integral element of degree d over R . If an element a in R is a non-zero-divisor on $R/I_{[\alpha^{-1}]}$, then $R[a^{-1}\alpha] \cap R[\alpha^{-1}] = R[\alpha] \cap R[\alpha^{-1}]$.*

Proof. We have only to show the inclusion (\subseteq). Take an element $\beta \in R[a^{-1}\alpha] \cap R[\alpha^{-1}]$ and write $\beta = x_n(a^{-1}\alpha)^n + \cdots + x_1(a^{-1}\alpha) + x_0 = y_0 + y_1(\alpha^{-1}) + \cdots + y_m(\alpha^{-1})^m$ with $x_i, y_j \in R$. Then $a^n \beta = x_n \alpha^n + \cdots + a^n x_0 = a^n y_0 + a^n y_1(\alpha^{-1}) + \cdots + a^n y_m(\alpha^{-1})^m$. Since α is anti-integral over R , so is α^{-1} (cf. [1]). Putting $f(X) := a^n y_m X^{m+n} + \cdots + a^n X^n - (a^n x_0 X^n + \cdots + x_n)$, we have $f(\alpha^{-1}) = 0$ and hence $f(X) \in I_{[\alpha^{-1}]} \varphi_{\alpha^{-1}}(X)$. Thus we have $a^n y_m \in I_{[\alpha^{-1}]}$. Since a is a non-zero-divisor on $R/I_{[\alpha^{-1}]}$, we have $y_m \in I_{[\alpha^{-1}]}$. Hence $y_m \varphi_{\alpha^{-1}}(X) \in R[X]$. Put $\beta_*(X) := y_0 + y_1 X + \cdots + y_m X^m$. Considering $\deg(\beta_*(X) - y_m \varphi_{\alpha^{-1}}(X)) < m$, we can assume that $m < d$ by induction. Let $\varphi_{\alpha^{-1}}(X) := X^d + \eta'_1 X^{d-1} + \cdots + \eta'_d$ be the monic minimal

polynomial of α^{-1} over K and let $\zeta'_i := (\alpha^{-1})^i + \eta'_1(\alpha^{-1})^{i-1} + \cdots + \eta'_i$ ($1 \leq i \leq d$). Considering $\beta - y_m \zeta'_m$ instead of β , we obtain that $\beta - y_m \zeta'_m \in R + I_{[\alpha^{-1}]} \zeta'_1 + \cdots + I_{[\alpha^{-1}]} \zeta'_{d-1} = R[\alpha^{-1}] \cap R[\alpha]$ (cf. [1]) by induction on m . Thus $\beta \in R[\alpha^{-1}] \cap R[\alpha]$. Therefore we have $R[a^{-1}\alpha] \cap R[\alpha^{-1}] \subseteq R[\alpha] \cap R[\alpha^{-1}]$.

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