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QUASI-HAMSHER MODULES AND QUASI-MAX RINGS

WEIMIN XUE

Throughout rings are associative with identity and modules are unitary. We freely use the terminology and notations of Anderson and Fuller [1].

Faith [5] said that a module M is Hamsher if every non-zero submodule of M has a maximal submodule. It is well-known (see, e.g., [1, §11]) that M has finite length if and only if M is Hamsher and artinian. A ring R is called right max [5] if every non-zero right R-module has a maximal submodule. The class of right max rings includes right perfect rings, and right V-rings as well. As generalizations, we call a module M quasi-Hamsher if every non-zero artinian submodule of M has a maximal submodule, and call a ring R right quasi-max if every right R-module is quasi-Hamsher, i.e., every non-zero artinian right R-module has a maximal submodule. In this paper, we characterize quasi-Hamsher modules and quasi-max rings, which are shown to be proper generalizations of Hamsher modules and max rings, respectively. The dual notions of quasi-Hamsher modules and quasi-max rings are also considered.

1. Quasi-Hamsher modules and quasi-max rings. It is easy to see that the class of (quasi-)Hamsher modules is closed under submodules. The following two propositions show that this class is also closed under extensions, direct products, and direct sums. In this and the next section, R is a fixed ring and modules are right R-modules when not specified.

Proposition 1.1. *Let* $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ *be an exact sequence of modules. If both* M_1 *and* M_2 *are (resp. quasi-Hamsher) Hamsher, then so is* M.

Proof. Let $0 \neq A$ be a (resp. artinian) submodule of M. If $g(A) \neq 0$, being a (resp. artinian) submodule of the (resp. quasi-Hamsher) Hamsher module M_2, $g(A)$ has a maximal submodule B. Then $A \cap g^{-1}(B)$ is a maximal submodule of A. If $g(A) = 0$, $A \subseteq \mathrm{Ker}(g) = \mathrm{Im}(f) \cong M_1$ and so A has a maximal submodule since M_1 is (resp. quasi-Hamsher) Hamsher.

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Proposition 1.2. *The following are equivalent for a family* $\{M_i\}_{i\in I}$ *of modules*:

(a) *Every* M_i *is Hamsher (resp. quasi-Hamsher);*

(b) $\prod_{i\in I} M_i$ *is Hamsher (resp. quasi-Hamsher);*

(c) $\bigoplus_{i\in I} M_i$ *is Hamsher (resp. quasi-Hamsher);*

Proof. (a)⇒(b). Let $0 \neq A$ be a (resp. artinian) submodule of $\prod_{i\in I} M_i$. Let $p_i\colon \prod_{i\in I} M_i \to M_i$ be the canonical projections. We have an M_i such that $p_i(A) \neq 0$. Now $p_i(A)$ is a (resp. artinian) submodule of the (resp. quasi-Hamsher) Hamsher module M_i so $p_i(A)$ has a maximal submodule B. The $A \cap p_i^{-1}(B)$ is a maximal submodule of A.
(b)⇒(c)⇒(a). These are obvious, since the class of (quasi-)Hamsher modules is closed under submodules.

Cai and Xue [4] called a module strongly artinian if each of its proper submodule has finite length. It is easy to see that a non-zero strongly artinian module has finite length if and only if it has a maximal submodule, if and only if it is finitely generated.

Proposition 1.3. *The following are equivalent for a module* M:

(a) M *is quasi-Hamsher;*

(b) *Every artinian submodule of* M *has finite length;*

(c) *Every artinian submodule of* M *is finitely generated;*

(d) *Every strongly artinian submodule of* M *is finitely generated;*

(e) *Every non-zero strongly artinian submodule of* M *has a maximal submodule (hence has finite length).*

Proof. (a)⇒(b). Let A be a non-zero artinian submodule of M. Since each submodule of A is still artinian, A is an artinian Hamsher module, which has finite length.
(b)⇒(c)⇒(d) ⇔(e) and (c)⇒(a). These are obvious.
(e)⇒(b). If A is an artinian submodule of M and A has infinite length, then the non-empty family

$$\{B \leqq A \mid B \text{ has infinite length}\}$$

has a minimal member, say B. It is eary to see that B is strongly artinian and B has infinite length.

As a generalization of right max rings we call a ring R right quasi-max if every right R-module is quasi-Hamsher. The next characterizations of right quasi-max rings follow immediately from the above proposition.

Theorem 1.4. *The following are equivalent for a ring R:*

(a) *R is right quasi-max;*

(b) *Every non-zero (strongly) artinian right R-module has a maximal submodule;*

(c) *Every (strongly) artinian right R-module has finite length;*

(d) *Every (strongly) artinian right R-module is finitely generated.*

Camillo and Xue [3] called a ring R right quasi-perfect if every artinian right R-module has a projective cover. Using Theorem 1.4 and [3, Theorem 1] we see that a ring R is right quasi-perfect if and only if it is semiperfect and right quasi-max. Hence the next result follows immediately from [3, Proposition 6].

Proposition 1.5. *If R is commutative semiperfect ring with nil $J(R)$ then R is quasi-max.*

It is known (see [5, p.203]) that a ring R is right max if and only if $R/J(R)$ is right max and $J(R)$ is right T-nilpotent. The ring R in [3, Example 7] is a local commutative ring with nil $J(R)$ which is not T-nilpotent. Hence R is not max, but R is quasi-max by Proposition 1.5. Therefore there is a quasi-Hamsher R-module which is not Hamsher. We conclude that quasi-Hamsher modules and right quasi-max rings are proper generalizations of Hamsher modules and right max rings, respectively.

Example 1.6. Let D be a divison ring. Let R be the ring of all countablely infinite upper triangular matrices over D with constant on the main diagonal and having non-zero entries in only finitely many rows above the main diagonal. Then R is a local right perfect ring which is not left perfect. Miller and Turnidge [6] constructed and artinian left R-module M which is not noetherian. Hence R is not left quasi-max. This shows that the notion of (quasi-)max rings is not left-right symmetric.

In view of the above example and Proposition 1.5, we mention the following result, which follows immediately from [8, Proposition 2].

Proposition 1.7 ([8]). *Let R be a semiperfect ring with nil $J(R)$. If $J(R)$ is of bounded index n, i.e., $j^n = 0$ for each $j \in J(R)$, then R is (two-sided) quasi-max, equivalently, quasi-perfect.*

Modifying the proof of [5, Theorem 1] we have an analogous result.

Theorem 1.8. *The following are equivalent for a ring R:*

(a) *R is right quasi-max;*

(b) *The category Mod-R has a cogenerator C which is quasi-Hamsher;*

(c) *The injective envelope $E(T)$ of T is quasi-Hamsher for each simple right R-module T.*

Proof. (a)⇒(b). This is obvious.
(b)⇒(c). Since C is a cogenerator there is a monomorphism $E(T) \to C$ for each simple right R-module T. Hence $E(T)$ must be quasi-Hamsher since C is.
(c)⇒(a). Let T range over all simple right R-modules. Then $\bigoplus E(T)$ is a cogenerator of *Mod-R* and $\bigoplus E(T)$ is quasi-Hamsher by Proposition 1.2. Let A be a non-zero artinian right R-module. We have a non-zero homomorphism $f\colon A \to \bigoplus E(T)$. Since $f(A)$ is a non-zero artinian submodule of $\bigoplus E(T)$, which is quasi-Hamsher, $f(A)$ has a maximal submodule B. Then $f^{-1}(B)$ is a maximal submodule of A.

2. Quasi-Loewy modules and quasi-Loewy rings. A module M is called Loewy (resp. quasi-Loewy) if every non-zero (resp. non-zero noetherian) factor module of M has non-zero socle. It is well-known (see, e.g., [1, §11]) that M has finite length if and only if M is Loewy and noetherian. The next two propositions show that the class of (quasi-) Loewy modules is closed under extensions and direct sums.

Proposition 2.1. *Let $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ be an exact sequence of modules. If both M_1 and M_2 are Loewy (resp. quasi-Loewy) then M is Loewy (resp. quasi-Loewy).*

Proof. Let $0 \neq M/N$ be a (resp. noetherian) factor module of M. We have an exact sequence

$$0 \to M_1/N_1 \to M/N \to M_2/N_2 \to 0.$$

If $M_1/N_1 \neq 0$, $\mathrm{Soc}(M_1/N_1) \neq 0$ and then $\mathrm{Soc}(M/N) \neq 0$. If $M_1/N_1 = 0$, $M_2/N_2 \cong M/N \neq 0$. Then $\mathrm{Soc}(M_2/N2) \neq 0$, and so $\mathrm{Soc}(M/N) \neq 0$.

Proposition 2.2. *Let $\{M_i\}_{i\in I}$ be a family of modules. Then $\bigoplus_{i\in I} M_i$ is Loewy (resp. quasi-Loewy) if and only if each M_i is Loewy (resp. quasi-Loewy).*

Proof. ($\Rightarrow$). The class of (quasi-)Loewy modules is closed under factor modules.
($\Leftarrow$). Let $j_i\colon M_i \to \bigoplus_{i\in I} M_i$ be the canonical injection. If $(\bigoplus_{i\in I} M_i)/N$ is a non-zero (noetherian) factor module of $\bigoplus_{i\in I} M_i$ then there is an $i \in I$ such that $0 \neq pj_i\colon M_i \to (\bigoplus_{i\in I} M_i)/N$ where $p\colon \bigoplus_{i\in I} M_i \to (\bigoplus_{i\in I} M_i)/N$ is the natural epimorphism. Since $0 \neq \mathrm{Im}(pj_i)$ which is isomorphic to a (noetherian) factor module of M_i we have $0 \neq \mathrm{Soc}(\mathrm{Im}(pj_i)) \leqq \mathrm{Soc}((\bigoplus_{i\in I} M_i)/N)$.

If $R = \prod_{i=1}^{\infty} F_i$ is an infinite product of the fields F_i's then R is not a Loewy R-module by [2, p.354, Remark 3(2)]. Since each F_i is a Loewy R-module, this shows that the class of Loewy modules is not closed under direct products. We do not know if the class of quasi-Loewy modules is closed under direct products.

A module is called strongly noetherian [4] if each of its proper factor module has finite length. It is easy to see that a non-zero strongly noetherian module has finite length if and only if it has non-zero socle, if and only if it is finitely cogenerated.

Proposition 2.3. *The following are equivalent for a module M:*

(a) *M is quasi-Loewy;*

(b) *Every noetherian factor module of M has finite length;*

(c) *Every noetherian factor module of M is finitely cogenerated;*

(d) *Every strongly noetherian factor module of M is finitely cogenerated;*

(e) *Every non-zero strongly noetherian factor module of M has non-zero socle (hence has finite length).*

Proof. (a)$\Rightarrow$(b). Let $0 \neq M/N$ be a noetherian factor module of M. Since each factor module of M/N is still noetherian, M/N is a noetherian Loewy module, which has finite length.
(b)$\Rightarrow$(c)$\Rightarrow$ (d)$\Leftrightarrow$(e) and (c)$\Rightarrow$(a). These are obvious.
(e)$\Rightarrow$(b). If M/N is a noetherian factor module of M and M/N has infinite length, then the non-empty family

$$\{N \leqq N' \leqq M \mid M/N' \text{ has infinite length}\}$$

has a maximal member, say N'. It is easy to see that M/N' is strongly noetherian and M/N' has infinite length.

A ring R is called right quasi-Loewy if every right R-module is quasi-Loewy. The next characterizations of right quasi-Loewy rings follow immediately from the above proposition.

Theorem 2.4. *The following are equivalent for a ring R:*

(a) *R is right quasi-Loewy;*

(b) *Every non-zero (strongly) noetherian right R-module has non-zero socle;*

(c) *Every (strongly) noetherian right R-module has finite length;*

(d) *Every (strongly) noetherian right R-module is finitely cogenerated;*

It follows from Theorems 1.4 and 2.4 that the rings studied by Tanabe [8] are precisely left quasi-max and left quasi-Loewy rings. An analogous result of Theorem 1.8 is the following

Theorem 2.5. *A ring R is right quasi-Loewy if and only if Mod-R has a generator G which is quasi-Loewy.*

Proof. ($\Rightarrow$). This is clear.
($\Leftarrow$). If M is a noetherian right R-module, $M \cong G^n/H$. Now G^n is quasi-Loewy by Proposition 2.2, so G^n/H has finite length by Proposition 2.3. Hence R is right quasi-Loewy by Theorem 2.4.

The next proposition gives a class of commutative quasi-Loewy rings.

Proposition 2.6. *If R is a commutative semiperfect ring with nil $J(R)$ then R is quasi-Loewy.*

Proof. By Theorem 2.5, it suffices to show that R is a quasi-Loewy R-module. Let I be an ideal of R such that R/I is a noetherian R-module. Then the commutative semiperfect noetherian ring R/I has nil $J(R/I)$. Hence R/I is an artinian ring. Then R/I has finite length as an R-module.

A ring R is right Loewy if every right R-module is Loewy, i.e., every non-zero right R-module has non-zero socle, equivalently, the right R-module R_R is Loewy. Every left perfect ring is right Loewy. By [7], R is right Loewy if and only if $R/J(R)$ is right Loewy and $J(R)$ is left

T-nilpotent. The ring R in [3, Example 7] is a local commutative ring with nil $J(R)$ which is not T-nilpotent. Hence R is not Loewy. But R is quasi-Loewy by the above proposition. Therefore there is a quasi-Loewy R-module which is not Loewy. We conclude that quasi-Loewy modules and right quasi-Loewy rings are proper generalizations of Loewy modules and right Loewy rings, respectively.

Let R be the ring in Example 1.6. Then R is a local right perfect ring which is not left perfect. Miller and Turnidge [6] constructed a noetherian right R-module M which is not artinian. Hence R is not right quasi-Loewy. This shows that the notion of (quasi-)Loewy rings is not left-right symmetric. In view of this fact and Proposition 2.6, we state the next result, which follows from [8, Proposition 2].

Proposition 2.7 ([8]). *Let R be a semiperfect ring with nil $J(R)$. If $J(R)$ is of bounded index n then R is (two-sided) quasi-Loewy.*

Since a commutative regular ring need not be Loewy (see $R = \prod_{i=1}^{\infty} F_i$ preceding Proposition 2.3) we recall the following result which follows from [8, Theorem 1]. Here we give a simple proof.

Proposition 2.8 ([8]). *Every strongly regular ring R is a (two-sided) quasi-Loewy ring.*

Proof. Let $M = \sum_{i=1}^{n} m_i R$ be a noetherian right R-module. To show M has finite length, it suffices to show each $m_i R$ has finite length. We have $m_i R \cong R/I$ for some ideal I of R. Since R/I is a right noetherian regular ring it is semisimple. So R/I ($\cong m_i R$) has finite length as a right R-module.

3. Morita duality. A bimodule ${}_S U_R$ defines a Morita duality if ${}_S U_R$ is faithfully balanced and both U_R and ${}_S U$ are injective cogenerators. In this case, both R and S are semiperfect rings. A presentation of Morita duality can be found in [1, §23,§24] and [9, Chapter 1].

Using properties of of Morita duality and [3, Thorems 10 and 11] we conclude this paper with the following two results.

Proposition 3.1. *Let ${}_S U_R$ define a Morita duality. If M_R is a U-reflexive right R-module then*

(a) *M_R is Hamsher (resp. quasi-Hamsher) if and only if the left S-module ${}_S\mathrm{Hom}_R(M_R, {}_S U_R)$ is Loewy (resp. quasi-Loewy).*

(b) *M_R is Loewy (resp. quasi-Loewy) if and only if the left S-module ${}_S\mathrm{Hom}_R(M_R, {}_SU_R)$ is Hamsher (resp. quasi-Hamsher).*

Theorem 3.2. *If ${}_SU_R$ defines a Morita duality the following are equivalent:*

(a) *R is right quasi-max;*

(b) *S is left quasi-max;*

(c) *R is right quasi-Loewy;*

(d) *S is left quasi-Loewy.*

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REFERENCES

[1] F. W. ANDERSON and K. R. FULLER: Rings and Categories of Modules, 2nd edition, Springer, New York, 1992.

[2] V. P. CAMILLO and K. R. FULLER: On Loewy length of rings, Pacific J. Math. **53** (1974), 347–354.

[3] V. P. CAMILLO and W. XUE: On quasi-perfect rings, Comm. Algebra **19** (1991), 2841–2850, Addendum, Comm. Algebra **20** (1992), 1839–1840.

[4] Y. CAI and W. XUE: Strongly noetherian modules and rings, Kobe J. Math. **9** (1992), 33–37.

[5] C. FAITH: Rings whose modules have maximal submodules, Publ. Mate. **39** (1995), 201–214.

[6] R. W. MILLER and D. R. TURNIDGE: Some examples from infinite matrix rings, Proc. Amer. Math. Soc. **38** (1973), 65–67.

[7] C. NASTASESCU and N. POPESCU: Anneaux semi-artiniens, Bull. Soc. Math. France **96** (1968), 357–368.

[8] K. TANABE: On rings whose artinian modules are precisely noetherian modules, Comm. Algebra **22** (1994), 4023–4032.

[9] W. XUE: Rings with Morita Duality, Lect. Notes Math. Vol. 1523, Springer, Berlin, 1992.

DEPARTMENT OF MATHEMATICS
FUJIAN NORMAL UNIVERSITY
FUZHOU, FUJIAN 350007
PEOPLE'S REPUBLIC OF CHINA

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CURRENT ADDRESS:
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF IOWA
IOWA CITY, IA 52242-1419
U.S.A.
E-mail: wxue@math.uiowa.edu