

## ON RINGS HAVING A FAITHFUL NOETHERIAN MODULE

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It is well known that every commutative ring with an identity element and having a faithful Noetherian module is Noetherian (cf. [1, p.261]). In this paper, we consider a noncommutative ring with or without an identity element which has a faithful Noetherian or Artinian module, and we shall show that every nil ideal of bounded index of such a ring is nilpotent.

For a nil ideal  $I$  of a ring  $R$ ,  $n(I)$  is defined to be the supremum of the nilpotency indices of the elements in  $I$ .

**Theorem 1.** *If a ring  $R$  has a faithful Noetherian left  $R$ -module  $M$ , then every nil ideal  $I$  of  $R$  with  $n(I) < \infty$  is nilpotent.*

*Proof.* For a  $R$ -submodule  $N$  of  $M$  and an ideal  $A$  of  $R$ ,  $A * N$  is defined to be the set of all  $m \in M$  satisfying  $Am \subseteq N$ . It is easy to see the following properties.

- (a)  $A * N$  is a  $R$ -submodule of  $M$  containing  $N$ .
- (b)  $(AB) * N = B * (A * N)$  for any ideals  $A, B$ .

We shall prove the theorem by induction on  $n(I)$ . If  $n(I) = 1$  then  $I = 0$ . Let  $k$  be a positive integer and assume that every nil ideal  $I$  with  $n(I) \leq k$  is nilpotent. Let  $I$  be a nil ideal of  $R$  with  $n(I) = k + 1$ . Let  $J$  be the ideal of  $R$  generated by all  $x^k$  where  $x \in I$ . Since  $M$  is Noetherian, there exists a positive integer  $n$  such that  $J^n * 0 = J^{n+1} * 0$ .

Now suppose that  $J^n * 0 \neq M$ . Then the set

$$\{(J^n K) * 0 \mid K \text{ is an ideal of } R, (J^n K) * 0 \neq M\} \quad (\ni J)$$

has a maximal member  $(J^n L) * 0$ , where  $L$  is an ideal of  $R$ . Let  $x$  be an arbitrary element in  $I$ . For any  $r \in R$ , we see that  $0 = (x^k r + x)^{k+1} = x^k r x^k + x^k r x^k y$  for some  $y \in I$ . Since  $y$  is nilpotent, we have  $x^k r x^k = 0$ . Therefore, the ideal  $(x^k)$  of  $R$  generated by  $x^k$  is nilpotent. It is obvious that  $(x^k) * ((J^n L) * 0) \supseteq (J^n L) * 0$ . If  $(x^k) * ((J^n L) * 0) = (J^n L) * 0$ , then we have  $(x^k)^2 * ((J^n L) * 0) = (x^k) * \left( (x^k) * ((J^n L) * 0) \right) = (x^k) * ((J^n L) * 0) = (J^n L) * 0$ . Continuing this method, we have  $(x^k)^\nu * ((J^n L) * 0) = (J^n L) * 0$  for any positive integer  $\nu$ , which implies a contradiction  $(J^n L) * 0 = M$ , because  $(x^k)$  is nilpotent. Hence, we have  $(x^k) * ((J^n L) * 0) \supsetneq (J^n L) * 0$ .

Since  $(x^k) * ((J^n L) * 0) = (J^n L(x^k)) * 0$ , the maximality of  $(J^n L) * 0$  implies that  $(J^n L(x^k)) * 0 = M$ , i.e.,  $J^n L(x^k) = 0$ . Hence, we have  $J^n L J = 0$ . Let  $A$  denote the annihilator of  $M / ((J^n L) * 0)$ . Then we have  $J \subseteq A$ , and so  $n((I + A)/A) \leq k$ . Hence, by the assumption of our induction, there exists a positive integer  $m$  such that  $I^m \subseteq A$ , i.e.,  $I^m * ((J^n L) * 0) = M$ . Therefore, we have  $I * ((J^n L) * 0) \supseteq (J^n L) * 0$ , and again by the maximality of  $(J^n L) * 0$ , we have  $(J^n L I) * 0 = M$ , i.e.,  $J^n L I = 0$ . Then, for any  $x \in I$  and  $a \in J^n L$ , we see that  $0 = (x+a)^{k+1} = x^k a$ . Hence, we have  $J^{n+1} L = 0$ . This implies that  $(J^n L) * 0 = L * (J^n * 0) = L * (J^{n+1} * 0) = (J^{n+1} L) * 0 = M$ , a contradiction. Thus we have shown that  $J^n * 0 = M$ , i.e.,  $J^n = 0$ .

Let  $A_i$  denote the annihilator of  $J^{i-1} M / J^i M$  ( $i = 1, \dots, n$ ). Since  $J \subseteq A_i$ , by the assumption of our induction, there exist positive integers  $m_i$  such that  $I^{m_i} J^{i-1} M \subseteq J^i M$  ( $i = 1, \dots, n$ ). Putting  $l = m_1 + \dots + m_n$ , we have  $I^l M = 0$ , that is,  $I^l = 0$ .

**Theorem 2.** *If a ring  $R$  has a faithful Artinian left  $R$ -module  $M$ , then every nil ideal  $I$  of  $R$  with  $n(I) < \infty$  is nilpotent.*

*Proof.* We shall prove the theorem by induction on  $n(I)$ . If  $n(I) = 1$  then  $I = 0$ . Let  $k$  be a positive integer and assume that every nil ideal  $I$  with  $n(I) \leq k$  is nilpotent. Let  $I$  be a nil ideal of  $R$  with  $n(I) = k + 1$ . Let  $J$  be the ideal of  $R$  generated by all  $x^k$  where  $x \in I$ . Since  $M$  is Artinian, there exists a positive integer  $n$  such that  $J^n M = J^{n+1} M$ . Now suppose that  $J^n \neq 0$ . Then the set

$$\{K J^n M \mid K \text{ is an ideal of } R, K J^n M \neq 0\} \quad (\ni J)$$

has a minimal member  $L J^n M$ , where  $L$  is an ideal of  $R$ . Let  $x$  be an arbitrary element in  $I$ . As was shown in the proof of Theorem 1, the ideal  $(x^k)$  of  $R$  generated by  $x^k$  is nilpotent. Since  $L J^n M \neq 0$ , it follows that  $(x^k) L J^n M \subsetneq L J^n M$ , and so  $(x^k) L J^n = 0$  by the minimality of  $L J^n M$ . Hence, we have  $J L J^n = 0$ . Let  $A$  denote the annihilator of  $L J^n M$ . Then we have  $J \subseteq A$ , and so  $n((I + A)/A) \leq k$ . Hence, by the assumption of our induction, there exists a positive integer  $m$  such that  $I^m \subseteq A$ , i.e.,  $I^m L J^n = 0$ . Therefore, we have  $I L J^n M \subsetneq L J^n M$ , and the minimality of  $L J^n M$  implies that  $I L J^n = 0$ . Then, for any  $x \in I$  and  $a \in L J^n$ , we see that  $0 = (a + x)^{k+1} = a x^k$ . Hence, we have  $L J^{n+1} = 0$ . This implies that  $L J^n M = L J^{n+1} M = 0$ , a contradiction. Thus we have shown that  $J^n = 0$ . The rest of the proof is quite similar to the proof of Theorem 1.

In [2], the second author of this paper established amen rings. According to [2], a ring  $R$  is said to be *amen* if every Artinian left  $R$ -module is Noetherian and every Noetherian left  $R$ -module is Artinian. By making use of Theorems 1 and 2, we can give an easy proof to [2, Proposition 2].

**Corollary 3** ([2, Proposition 2]). *Let  $R$  be a ring and  $I$  a nil ideal of  $R$  with  $n(I) < \infty$ . If  $R/I$  is amen then  $R$  is amen.*

*Proof.* Let  $M$  be an Artinian left  $R$ -module. By Theorem 2, we have  $I^n M = 0$  for some positive integer  $n$ . Since  $R/I$  is amen, all  $I^{i-1}M/I^iM$  are Noetherian ( $i = 1, \dots, n$ ). Hence,  $M$  is Noetherian. Similarly, every Noetherian left  $R$ -module is Artinian.

#### REFERENCES

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