Math. J. Okayama Univ. **39** (1997), 41–44

NOTE ON SCHREIER SEMIGROUP RINGS

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Let D be an integral domain with the quotient field $q(D)$. Let c be an element of D. Assume that, if c is a divisor of a_1a_2 (for $a_1, a_2 \in D$), then c is a product of a divisor of a_1 and a divisor of a_2. Then c is called a primal element of D. If each divisor of c is a primal element of D, then c is called a completely primal element of D. D is called a Schreier ring if D is an integrally closed ring in which every element is primal ([2]). Let S be a semigroup $\ni 0$ of a torsion-free abelian (additive) group. Then S is called a grading monoid (or a g-monoid) ([6]). Let $D[X;S] = \{\sum_{finite} a_sX^s \mid a_s \in D, s \in S\}$ be the semigroup ring of S over D (X is a symbol). [3] is a general reference on $D[X;S]$. For various ring-theoretic properties Π, conditions for $D[X;S]$ to have the property Π have been obtained (cf. [1, 3, 5]). The aim of this note is to obtain conditions for $D[X;S]$ to be a Schreier ring.

Lemma 1 ([2]). *Let D be an integrally closed domain, and let T be a multiplicative system of D generated by completely primal elements. If the quotient ring D_T is a Schreier ring, then D is a Schreier ring.*

For elements s, t of a g-monoid S, if $t = s + s'$ for some $s' \in S$, then s is called a divisor of t. For elements $s, t_1, \cdots, t_n$ of S, if s is a divisor of $t_1, \cdots, t_n$, then s is called a common divisor of $t_1, \cdots, t_n$. The group $\{s_1 - s_2 \mid s_1, s_2 \in S\}$ is called the quotient group of S, and is denoted by $q(S)$. We note that $q(S)$ is a totally ordered abelian group ([3, COROLLARY 3.4]). An element x of $q(S)$ is called integral over S, if $nx \in S$ for some $n \in \mathbf{N}$. If every integral element of $q(S)$ is contained in S, then S is called an integrally closed semigroup. Let G be a torsion-free abelian (additive) group, and Γ a totally ordered abelian group. A homomorphism v of G to Γ is called a valuation on G. The semigroup $\{x \in G \mid v(s) \geq 0\}$ is called the valuation semigroup of G associated with v. A valuation semigroup of $q(S)$ which contains S is called a valuation oversemigroup of S. Let c be an element of S. Assume that, if c is a divisor of $a_1 + a_2$ (for $a_1, a_2 \in S$), then c is a sum of a divisor of a_1 and a divisor of a_2. Then c is called a primal element of S. S is called a Schreier semigroup if S is an integrally closed semigroup in which every element is primal. We consider the following condition:

(∗) For every finite subsets $\{s_1, \cdots, s_n\}, \{t_1, \cdots, t_m\}$ of S and an element s of S, if s is a common divisor of $s_1+t_1, s_1+t_2, \cdots, s_i+t_j, \cdots, s_n+t_m$, then s is a sum of a common divisor of $s_1, \cdots, s_n$, and a common divisor of $t_1, \cdots, t_m$.

If S is integrally closed and satisfies the condition (∗), then S is a Schreier semigroup.

Lemma 2 ([3, THEOREM 12.8]). *S is integrally closed if and only if S is the intersection of all the valuation oversemigroups of S.*

Let v be a valuation on $q(S)$. Let $f = \sum_1^n a_i X^{s_i}$ be an element of $D[X;S]$, where each $a_i \neq 0$ and $s_i \neq s_j$ for $i \neq j$. We set $v^*(f) = \inf_i v(s_i)$.

Lemma 3. (1)([3, THEOREM 15.7]) *v^* naturally induces a valuation on $q(D[X;S])$.*

(2)([3, COROLLARY 12.11]) *$D[X;S]$ is integrally closed if and only if D is integrally closed and S is integrally closed.*

Lemma 4. (1)([4, (4.5)PROPOSITION]) *Let G be a torsion-free abelian group. Then $D[X;G]$ is a Schreier ring if and only if D is a Schreier ring.*

(2)([4, (4.6)PROPOSITION]) *$D[X;S]$ is a Schreier ring if and only if D and $K[X;S]$ are Schreier rings and S is a Schreier semigroup, where $K = q(D)$.*

Lemma 5. *Let k be a field. If $k[X;S]$ is a Schreier ring, then S satisfies the condition (∗).*

Proof. Let $s, s_1, \cdots, s_n, t_1, \cdots, t_m$ be a finite number of elements of S such that s is a common divisor of $s_1+t_1, s_1+t_2, \cdots, s_i+t_j, \cdots, s_n+t_m$. Set $f = X^{s_1}+\cdots+X^{s_n}$ and $g = X^{t_1}+\cdots+X^{t_m}$. Then X^s is a divisor of fg in $k[X;S]$. Hence there exist a divisor f_1 of f and a divisor g_1 of g such that $X^s = f_1g_1$. Noting that S is a subsemigroup of a totally ordered abelian group $q(S)$, we may assume that $f_1 = X^a$ and $g_1 = X^b$ for $a, b \in S$. It follows that a is a common divisor of $s_1, \cdots, s_n$, and b is a common divisor of $t_1, \cdots, t_m$, and $a+b = s$. Therefore S satisfies the condition (∗).

Lemma 6. *Let k be a field, and let S be an integrally closed semigroup which satisfies the condition $(*)$. Then $k[X;S]$ is a Scheier ring.*

Proof. By Lemma 3(2), $k[X;S]$ is integrally closed. Let $s \in S$. We will show that X^s is a primal element of $k[X;S]$. Thus let f, g be non-zero elements of $k[X;S]$ such that $fg = X^s h$ for some $h \in k[X;S]$. Set $f = \sum_1^n a_i X^{s_i}, g = \sum_1^m b_i X^{t_i}$, where each a_i and b_j are non-zero elements of k, $s_i \neq s_j$ for $i \neq j$, and $t_k \neq t_l$ for $k \neq l$. Let $1 \leq k \leq n$, and $1 \leq l \leq m$. Let V be a valuation oversemigroup of S, and let v be the valuation on $q(S)$ associated with V. Then we have

$$v(s_k) + v(t_l) \geq v^*(f) + v^*(g) = v^*(X^s h) = v(s) + v^*(h).$$

It follows that $v(s_k + t_l) \geq 0$, and hence $s_k + t_l - s \in V$. Since V is arbitrary, $s_k + t_l - s \in S$ by Lemma 2. Hence s is a divisor of $s_k + t_l$. Since k and l are arbitrary, s is a common divsor of $s_1 + t_1, s_1 + t_2, \cdots, s_i + t_j, \cdots, s_n + t_m$. Hence there exist a common divisor a of $s_1, \cdots, s_n$, and a common divisor b of $t_1, \cdots, t_m$ such that $s = a + b$. Then X^a is a divisor of f, X^b is a divisor of g, and $X^a X^b = X^s$. Therefore X^s is a primal element of $k[X;S]$. Since s is arbitrary, we see that, for every element s of S, X^s is a completely primal element of $k[X;S]$. $T = \{X^s \mid s \in S\}$ is a multiplicative system of $k[X;S]$ generated by completely primal elements, and we have $k[X;S]_T = k[X;G]$, where $G = q(S)$. By Lemma 4(1), $k[X;G]$ is a Schreier ring. By Lemma 1, $k[X;S]$ is a Schreier ring.

Lemmas 5 and 6 imply the following,

Proposition 7. *Let k be a field. Then $k[X;S]$ is a Schreier ring if and only if S is an integrally closed semigroup which satisfies the condition $(*)$.*

Lemma 4 (2) and Proposition 7 imply the following,

Theorem 8. *$D[X;S]$ is a Schreier ring if and only if D is a Schreier ring, S is an integrally closed semigroup which satisfies the condition $(*)$.*

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(Received November 28, 1997)