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CHARACTERIZATIONS OF HEREDITARY MODULES AND V-MODULES

Dedicated to the memory of Professor Hisao Tominaga

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In this paper R is an associative ring with identity and all modules are unitary right R-modules. We freely use the terminology and notation of Andrson and Fuller [1] and Wisbauer [13], and let *Mod-R* denote the category of all right R-modules.

This paper consists of three sections. In the first section, we characterize semisimple modules, hereditary modules, and cohereditary modules using endomorphisms of modules which are related to injective modules and projective modules. These generalize some results of Nicholson and Varadarajan [8] and Shrikhande [11].

Let M be a module. An M-generated module is a module which is isomorphic to a factor module of $M^{(I)}$ for some index set I, where $M^{(I)}$ is the direct sum of $|I|$-copies of M. We denote by $\sigma[M]$ the full subcategory of *Mod-R* whose objects are all submodules of M-generated modules. Following Hirano [5] and Tominaga [12] we call M a V-module if every proper submodule of M is an intersection of maximal submodules, equivalently if every simple module (in $\sigma[M]$ or *Mod-R*) is M-injective (see [5, Proposition 3.1] or [13, p.190, 23.1]). V-modules are called "co-semisimple" by Fuller [4] and Wisbauer [13, p.190]. In Section 2, we establish the equivalences of the following statements: (1) M is a V-module; (2) The category $\sigma[M]$ has a semisimple module W which cogenerated every cyclic module in $\sigma[M]$; (In this case W is a cogenerator in $\sigma[M]$.) (3) Every proper submodule of M maximal with respect to exclusion of some nonzero element of M is a maximal submodule. These results generalize the main theorem of Faith and Menal [3] and some of Wu and Hu [14] and Faith [2].

In Section 3 the final section, we apply a theorem of Liu [6] to give characterizations of noetherian V-modules, right hereditary rings, right perfect rings, and semisimple rings using their modules. These improve some results of Liu [6, 7], and Xue [15].

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1. Characterizations of semisimple and hereditary modules. Recently, Nicholson and Varadarajan [8] give interesting characterizations of semisimple rings and hereditary rings using endomorphisms of injective modules and projective modules. Using their idea we are able to characterize semisimple modules, (semi-)hereditary modules, and cohereditary modules via endomorphisms of modules which are related to injective modules and projective modules.

The injective envelope of a module X is denoted by $E(X)$.

Theorem 1.1. *The following are equivalent for a module M:*

(a) *M is semisimple;*

(b) *If U is an injective module and $f \in \mathrm{End}(U)$, then $\mathrm{Ker}(f)$ is M-injective;*

(c) *If P is a projective module and $f \in \mathrm{End}(P)$, then $P/\mathrm{Im}(f)$ is M-projective.*

Proof. (a)$\Rightarrow$(b) and (a)$\Rightarrow$(c). These are obvious.
(b)$\Rightarrow$(a). Let $N \leq M$. To show N is a summand of M, we prove that N is M-injective. Take the injective module $U = E(N) \oplus E\big(E(N)/N\big)$ and let

$$f \in \mathrm{End}(U) \quad \text{via } f\colon (x, y) \mapsto (0, x + N).$$

Then $\mathrm{Ker}(f) = N \oplus E\big(E(N)/N\big)$, which is M-injective by the assumption of (b). Hence N is M-injective.
(c)$\Rightarrow$(a). Let $N \leq M$. To show N is a summand of M, it suffices to show that M/N is M-projective. Let P be a projective module with an epimorphism $P \to M$. Let $P' \leq P$ such that $P/P' \cong M/N$. Let Q be a projective module with an epimorphism $h\colon Q \to P'$. Define

$$f \in \mathrm{End}(P \oplus Q) \quad \text{via } f\colon (p, q) \mapsto (h(q), 0).$$

Then $\mathrm{Im}(f) = P' \oplus 0$ and

$$(P \oplus Q)/\mathrm{Im}(f) \cong (P/P') \oplus Q$$

which is M-projective by the hypothesis of (c).
Hence P/P' is M-projective, so is M/N.

The next corollary follows immediately from Theorem 1.1. Note that the condition (c) of this corollary is weaker than the condition (3) of [8, Theorem 1].

Corollary 1.2 (Cf. [8, Theorem 1]). *The following are equivalent for the ring R:*

(a) *R is semisimple;*

(b) *For each injective module U and $f \in \mathrm{End}(U)$, $\mathrm{Ker}(f)$ is injective;*

(c) *For each projective module P and $f \in \mathrm{End}(P)$, $P/\mathrm{Im}(f)$ is R-projective.*

According to [11], a module P is called hereditary in case every submodule of P is projective. Using [11, Theorem 3.2], we are able to generalize it as follows.

Theorem 1.3. *The following are equivalent for a projective module P:*

(a) *P is hereditary;*

(b) *If U is a P-injective module and $f \in \mathrm{End}(U)$, then $\mathrm{Im}(f)$ is P-injective;*

(c) *If U is an injective module and $f \in \mathrm{End}(U)$, then $\mathrm{Im}(f)$ is P-injective.*

Proof. (a)$\Rightarrow$(b). Since $f\colon U \to \mathrm{Im}(f)$ is an epimorphism, $\mathrm{Im}(f)$ is P-injective by [11, Theorem 3.2].
(b)$\Rightarrow$(c). This is obvious.
(c)$\Rightarrow$(a). Let U be an injective module and $V \leq U$. We want to prove that U/V is P-injective. (Then P is hereditary by [11, Theorem 3.2].) Define

$$f \in \mathrm{End}\big(U \oplus E(U/V)\big) \quad \text{via } f\colon (u,x) \mapsto (0, u+V).$$

Since $U \oplus E(U/V)$ is injective, $\mathrm{Im}(f) = 0 \oplus (U/V)$ is P-injective by the assumption of (c). Hence U/V is P-injective.

Recall that a module P is called semihereditary [11] in case every finitely generated submodule of P is projective. A module U is called weakly P-injective [13] in case every homomorphism from a finitely generated submodule of $P^{(I)}$ to U can be extended to one from $P^{(I)}$ to U where I is any index set. Modifying the above proof and using [13, p.331, 39.4], we have the following analogous result.

Theorem 1.4. *The following are equivalent for a projective module P:*

(a) *P is semihereditary;*

(b) *If U is a weakly P-injective module and $f \in \mathrm{End}(U)$, then* $\mathrm{Im}(f)$ *is weakly P-injective;*

(c) *If U is an injective module and $f \in \mathrm{End}(U)$, then* $\mathrm{Im}(f)$ *is weakly P-injective.*

A weakly R-injective module is called FP-injective [13].

Corollary 1.5. *The following are equivalent for the ring R:*

(a) *R is right semihereditary;*

(b) *If U is an FP-injective module and $f \in \mathrm{End}(U)$, then* $\mathrm{Im}(f)$ *is FP-injective;*

(c) *If U is an injective module and $f \in \mathrm{End}(U)$, then* $\mathrm{Im}(f)$ *is FP-injective.*

A module U is called cohereditary [11] in case every factor module of U is injective. It is well-known that R is right hereditary if and only if every injective module is cohereditary. Dualizing the proof of Theorem 1.3 and using [11, Theorem 3.2′], we have the following generalization of [11, Theorem 3.2′].

Theorem 1.6. *The following are equivalent for an injective module U:*

(a) *U is cohereditary;*

(b) *If P is an U-projective module and $f \in \mathrm{End}(P)$, then* $\mathrm{Im}(f)$ *is U-projective;*

(c) *If P is a projective module and $f \in \mathrm{End}(P)$, then* $\mathrm{Im}(f)$ *is U-projective.*

According to [9, Theorem], R is semisimple if and only if every cyclic module is injective. Hence right cohereditary rings are precisely the semisimple rings.

A module U is called semi-cohereditary [11] in case every finitely cogenerated factor module of U is injective. A dual result of Theorem 1.4 can be obtained to characterize semi-cohereditary modules. Recall that R is a right V-ring in case every simple module is injective. It follows from [11, Proposition 4.6] that right semi-cohereditary rings are precisely the right V-rings. We shall study V-modules and V-rings in the next section.

2. Characterizations of V-Modules.

In this section we obtain some new characterizations of V-module as stated in the introduction. The M-injective envelope of a module $X \in \sigma[M]$ is denoted by $E_M(X)$.

Theorem 2.1. *The following are equivalent for a module M:*

(a) *M is a V-module;*

(b) *$\sigma[M]$ has a semisimple cogenerator;*

(c) *$\sigma[M]$ has a semisimple module which cogenerates every cyclic module in $\sigma[M]$;*

(d) *$\sigma[M]$ has a semisimple module W which cogenerates every cyclic module in $\sigma[M]$ with simple essential socle. (In this case W is a cogenerator in $\sigma[M]$)*

Proof. (a)$\Leftrightarrow$(b). This is in [13, p.190, 23.1].
(b)$\Rightarrow$(c)$\Rightarrow$(d). These are obvious.
(d)$\Rightarrow$(a). Let T be any simple module in $\sigma[M]$. To show $E_M(T) = T$ we suppose $0 \neq x \in E_M(T)$. Since $\sigma[M]$ has a cyclic module xR with simple essential socle T, W cogenerates xR by (d). By [1, Corollary 10.3], W finitely cogenerates xR, hence xR must be semisimple (since W is semisimple). Therefore $E_M(T)$ is semisimple. But $E_M(T)$ has a simple essential socle T. Hence we have $E_M(T) = T$.

Let $W = \bigoplus_{i \in I} T_i$ be a direct sum of simple modules. Since W cogenerates every simple module in $\sigma[M]$, $\{T_i\}_{i \in I}$ is a representative set of the simple modules in $\sigma[M]$. By (a), each T_i is M-injective, so W is a cogenerator in $\sigma[M]$ by [13, p.143, 17.12].

Faith and Menal [3] said that a module W satisfies the double annihilator condition (= d.a.c.) with respect to right ideals provided that $I = \mathrm{r}_R \mathrm{l}_W(I)$ for each right ideal I of R. By [1, Lemma 24.4(2)], we see that W satisfies the d.a.c. with respect to right ideals if and only if W cogenerates every cyclic module. Hence the equivalence (a)$\Leftrightarrow$(c) of the following corollary is the V-Ring Theorem established in [3], where the ring R is a right V-ring if R is a V-module when considered as a right module over itself, i.e., every simple module is injective.

Corollary 2.2. *The following are equivalent for the ring R:*

(a) *R is a right V-ring;*

(b) *Mod-R has a semisimple cogenerator;*

(c) *Mod-R has semisimple module which cogenerates every cyclic module;*

(d) *Mod-R has a semisimple module W which cogenerates every cyclic module with simple essential socle. (In this case W is a cogenerator in Mod-R.)*

Recall that a module M is a V-module if and only if every proper submodule of M is an intersection of maximal submodules.

Theorem 2.3. *A module M is a V-module if and only if every proper submodule of M maximal with respect to exclusion of some non-zero element of M is a maximal submodule.*

Proof. Let N be a proper submodule of M.
($\Rightarrow$). Suppose N is maximal with respect to exclusion of some non-zero element $m \in M$. Then $(N + mR)/N$ must be a simple module and it is M-injective. So there is a module homomorphism

$$f \colon M \to (N + mR)/N$$

such that

$$f|_{(N+mR)} \colon N + mR \to (N + mR)/N$$

is the natural epimorphism. Hence f is an epimorphism, $N \subseteq \operatorname{Ker}(f)$, and $m \notin \operatorname{Ker}(f)$. By the maximality of N we have $N = \operatorname{Ker}(f)$. Hence

$$M/N = M/\operatorname{Ker}(f) \cong (N + mR)/N$$

which is simple. So N is a maximal submodule of M.
($\Leftarrow$). For each $m \in M \setminus N$, the Zorn's lemma asserts that

$$\{L \mid N \leq L \leq M,\ m \notin L\}$$

has a maximal member, say L_m. By the assumption, each L_m is a maximal submodule. Since $N \subseteq \bigcap_{m \in M \setminus N} L_m$ and $m \notin L_m$ for each $m \in M \setminus N$, we must have the equality

$$N = \bigcap_{m \in M \setminus N} L_m$$

which is an intersection of the maximal submodules L_m's.

Corollary 2.4 ([14]). *The ring R is a right V-ring if and only if every proper right ideal of R maximal with respect to exclusion of some non-zero element of R is a maximal right ideal.*

A module is a Bass module [2] if every proper submodule is contained in a maximal submodule. Since every proper submodule of a V-module is an intersection of maximal submodules, a V-module is a Bass module.

Let $\{T_i\}_{i\in I}$ be a minimal representative set of the simple modules in $\sigma[M]$ and $W = \bigoplus_{i\in I} T_i$. Then $E_M(W)$ is the minimal M-injective cogenerator in $\sigma[M]$ (see [13, p.143]).

Modifying the proofs of [2, Theorem 8.2], we generalize it as follows.

Theorem 2.5. *Let $S = \mathrm{End}(E_M(W))$. If $E_M(W)$ is a Bass module and $J(S) = 0$, then M is a V-module (and $E_M(W) = W$ is semisimple).*

Proof. If $E_M(W) = W$ then every submodule of $E_M(W)$ is a direct summand, hence is M-injective. So M is a V-module.
We assume $E_M(W) \neq W$. Since $E_M(W)$ is a Bass module, W is contained in a maximal submodule V of $E_M(W)$. The monomorphism $E_M(W)/V \hookrightarrow W$ induces an endomorphism s of $E_M(W)$ such that $\mathrm{Ker}(s) = V$. Since the M-injective module $E_M(W) \in \sigma[M]$, $E_M(W)$ is $E_M(W)$-injective by [1, Proposition 16.13]. Since V is an essential submodule of $E_M(W)$, we have $s \in J(S)$ by [13, p.185, 22.1(1)], contradicting the $J(S) = 0$ assumption.

3. Characterizations of rings by their modules. Let c be any cardinal. A module M is called c-limited [6] in case every direct sum of non-zero submodules of M contains at most c direct summands. E.g., every module M is $|M|$-limited. A module is called an ES-module [6] in case it has an essential socle. E.g., every finitely cogenerated module is an ES-module. In this final section we apply the following interesting theorem of [6, Theorem 1] to characterize noetherian V-modules, right hereditary rings, right perfect rings, and semisimple rings using their modules.

Theorem 3.1 ([6, Theorem 1]). *Let P and Q be properties of modules such that P is preserved by direct sums and Q is inherited by direct summands. If there exists a cardinal c such that every module with property P is the direct sum of a module with property Q and a c-limited ES-module, then every module with property P has property Q.*

According to [10, Corollary 1.4], we know that a module M is a noetherian V-module if and only if every semisimple module is M-injective. If we let P denote the property of being a semisimple module and Q denote the property of being an M-injective module, then by Theorem 3.1 we have the following result.

Theorem 3.2. *A module M is a noetherian V-module if and only if there exists a cardinal c such that every semisimple module is the direct sum of an M-injective module and a c-limited module.*

Proof. If M is a semisimple module and $M = U \oplus N$ where U is M-injective and N is c-limited, then N is a semisimple module which is automatically an ES-module.

A module M is called direct-projective [13, 15] in case given any summand N of M with the projection $p\colon M \to N$ and any epimorphism $f\colon M \to N$, there exists $g \in \mathrm{End}(M)$ such that $fg = p$. E.g., every quasi-projective module is direct-projective, but the converse is false [15, Example A]. Recently, the author [15, Theorem 4] has proved that R is right hereditary if every submodule of a projective module is direct-projective. Using this and Theorem 3.1 we can generalize [7, Theorem 2.1] as follows.

Theorem 3.3. *The ring R is right hereditary if and only if there exists a cardinal c such that every submodule of a projective module is the direct sum of a direct-projective module and a c-limited ES-module.*

Proof. ($\Rightarrow$). This is obvious.
($\Leftarrow$). Using Theorem 3.1, we let P denote the property of being a submodule of a projective module and Q denote the property of being a direct-projective module. Clearly, P is preserved by direct sums and Q is inherited by direct summands [13]. It follows from Theorem 3.1 that every submodule of a projective module is direct-projective. Hence R is right hereditary by [15, Theorem 4].

Recall that R is right perfect if and only if every flat module is projective. A generalization of [6, Corollary 3] is given as follows.

Theorem 3.4. *The ring R is right perfect if and only if there exists a cardinal c such that every flat module is the direct sum of a direct-projective module and a c-limited ES-module.*

Proof. ($\Rightarrow$). This is obvious.
($\Leftarrow$). If we let P denote the property of being a flat module and Q denote the property of being a direct-projective module, then it follows from Theorem 3.1 that every flat module is direct-projective. Now let M be a flat module. Choose a projective module P with an epimorphism $P \to M$. Since $P \oplus M$ is flat, it is direct-projective by the above proof. It follows from [15, Lemma 1] that M is a projective module.

As the dual of direct-projectivity, we call a module M direct-injective [13, 15] in case given any summand N of M with the injection $i\colon N \to M$ and any monomorphism $f\colon N \to M$, there exists $g \in \mathrm{End}(M)$ such that $gf = i$. E.g., every quasi-injective module is direct-injective, but the converse is false [15, Example B]. By [15, Theorem 9], the ring R is semisimple if every module is direct-projective (direct-injective). Using [15, lemmas 1 and 2] and modifying the proof of [6, Corollary 6] and Theorem 3.4 we have our concluding theorem, which partially generalize [6, Corollary 6] and [15, Theorem 9].

Theorem 3.5. *The following are equivalent for the ring R:*

(a) *R is semisimple;*

(b) *There exists a cardinal c such that every module is the direct sum of a direct-projective module and a c-limited ES-module;*

(c) *There exists a cardinal c such that every module is the direct sum of a direct-injective module and a c-limited ES-module.*

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