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# JORDAN LEFT DERIVATIONS ON SEMIPRIME RINGS

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Throughout, $R$ will represent an associative ring. As usual we write $[x, y] = xy - yx$ for all $x, y \in R$. Let $D\colon R \to R$ be an additive mapping. $D$ is called a derivation (resp. a left derivation) if $D(xy) = D(x)y + xD(y)$ (resp. $D(xy) = xD(y) + yD(x)$) holds for all $x, y \in R$. And $D$ is called a Jordan derivation (resp. a left Jordan derivation) in case $D(x^2) = D(x)x + xD(x)$ (resp. $D(x^2) = 2xD(x)$) for all $x \in R$. Obviously, every derivation is a Jordan derivation. But in general, the converse is not true. A well known result of Herstein [6] states that every Jordan derivation on a prime ring $R$ of char$\neq 2$ is a derivation. A brief proof of this result can be found in [1]. Cusack has generalized this result to 2-torsion free semiprime rings [4](see also [2]). One can easily prove that in a noncommutative prime ring any left derivation is zero. Moreover, Brešar and the author have proved that the existence of a nonzero Jordan left derivation on a prime ring $R$ of char$\neq 2, 3$ forces $R$ to be commutative. This result can be considered as an extension of the well known result of Posner [9]. It should be mentioned that the result of [3] concerning Jordan left derivations has been improved by Deng [5]. In [3] one can find an example which shows that in a noncommutative semiprime ring there exists a nonzero Jordan left derivation.

In [8], Giambruno and Herstein have proved that if $D$ is a derivation on a semiprime ring $R$ such that for some positive integer $n$ the relation $D(x)^n = 0$ holds for all $x \in R$, then $D = 0$. It is well known that if $D$ and $G$ are derivations on a 2-torsion free semiprime ring $R$, such that $D^2(x) = G(x)$ holds for all $x \in R$, then $D = 0$.

Our present objective is to prove the following theorems which modify the results noting about to Jordan left derivations on semiprime rings.

**Theorem 1.** *Let $R$ be a 2-torsion free semiprime ring and $D\colon R \to R$ a Jordan left derivation. If there exists a positive integer $n$ such that $D(x)^n = 0$ for all $x \in R$, then $D = 0$.*

**Theorem 2.** *Let $R$ be a 2-torsion free and 3-torsion free semiprime ring. If $R$ admits Jordan derivations $D$ and $G\colon R \to R$ such that $D^2(x) =$*

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*$G(x)$ for all $x \in R$, then $D = 0$.*

In preparation for proving our theorems, we state the following three lemmas.

**Lemma 3.** *Let $R$ be a 2-torsion free ring. If $D\colon R \to R$ is a Jordan left derivation then for all $x, y \in R$:*
(a) $D(xy + yx) = 2xD(y) + 2yD(x)$,
(b) $D(xyx) = x^2D(y) + 3xyD(x) - yxD(x)$.

*Proof.* A special case of Proposition 1.1 in [3].

**Lemma 4.** *Let $R$ be a 2-torsion free and 3-torsion free ring, and $D\colon R \to R$ a Jordan left derivation. If $D\big([[D(x), x], x]\big) = 0$ holds for all $x \in R$, then $[D(x), x]D(x) = 0$ is fulfilled for all $x \in R$.*

*Proof.* Using Lemma 3 we obtain

$$\begin{aligned} 0 &= D\big([[D(x), x], x]\big) \\ &= D\big(D(x)x^2 + x^2D(x)\big) - 2D\big(xD(x)x\big) \\ &= 4D(x)xD(x) + 2x^2D^2(x) - 2x^2D^2(x) - 6xD(x)^2 + 2xD(x)^2 \\ &= 6[D(x), x]D(x). \end{aligned}$$

Thus we have $6[D(x), x]D(x) = 0$, which completes the proof, since $R$ is 2-torsion free and 3-torsion free.

**Lemma 5.** *Let $R$ a noncommutative prime ring of characteristic defferent from two, and let $D\colon R \to R$ be a left Jordan derivation. In this case $D = 0$.*

*Proof.* A special case of Theorem 1 in [5].

*Proof of Theorem 1.* Since $R$ is semiprime, $\bigcap P = (0)$ where the intersection runs over all prime ideals of $R$. We intend to prove that $D(P) \subset P$ for every prime ideal $P$ of $R$. Let $a \in P$, $x \in R$; then

$$0 = D(ax + xa)^n = 2^n\big(aD(x) + xD(a)\big)^n,$$

which gives

$$\big(aD(x) + xD(a)\big)^n = 0,$$

since $R$ is 2-torsion free. In other words

$$\big(xD(a)\big)^n \equiv 0 \bmod P.$$

Thus, in the prime ring $R' = R/P$, we have

$$\big(x'D(a)'\big)^n = 0 \quad \text{for all } x' \in R'.$$

By a well known result of Levitzki (see Lemma 1.1 in [7]) we have $x'D(a)' = 0$ for all $x' \in R'$, whence it follows $D(a)' = 0$ since $R'$ is prime. In other words $D(a) \in P$, and so $D(P) \subset P$. Therefore $D(P) \subset P$ for all prime ideals $P$ of $R$, and so $D$ induces a left Jordan derivation $D'$ on the prime ring $R' = R/P$. Let us first assume that $R'$ is commutative. In this case $D'$ is derivation and we have also $D'(x')^n = 0$, whence it follows $D' = 0$. In case $R'$ is noncommutative it follows by Lemma 5 that $D' = 0$. Thus, in any case $D'(R') = (0)$, that is, $D(R) \subset P$ for all prime ideals $P$ of $R$. Since $\bigcap P = (0)$, we obtain $D(R) = (0)$, hence $D = 0$. The proof of the theorem is complete.

*Proof of Theorem 2.* Putting $x^2$ for $x$ in $D^2(x) = G(x)$ we obtain

(1) $$D\big(xD(x)\big) = xG(x), \quad x \in R.$$

Let us prove that for all $x \in R$,

(2) $$D\big(D(x)x\big) = 2D(x)^2 + xG(x).$$

Using Lemma 3 and (1) we obtain

$$\begin{aligned} D\big(D(x)x\big) &= D\big(D(x)x + xD(x)\big) - D\big(xD(x)\big) \\ &= 2D(x)^2 + 2xD^2(x) - xG(x) \\ &= 2D(x)^2 + xG(x). \end{aligned}$$

From (1) and (2) we obtain

(3) $$D\big([D(x), x]\big) = 2D(x)^2, \quad x \in R.$$

The linearization of (3) gives

$$D\big([D(x),y]+[D(y),x]\big)=2D(x)D(y)+2D(y)D(x),\quad x,y\in R.$$

Putting in the above relation $y=x^2$, using Lemma 3 and (3), we obtain

$$\begin{aligned}0&=D\big([D(x),x^2]+[D(x^2),x]\big)-2D(x)D(x^2)-2D(x^2)D(x)\\&=D\big(([D(x),x]x+x[D(x),x]\big)\\&\quad+2D\big(x[D(x),x]\big)-4D(x)xD(x)-4xD(x)^2\\&=2[D(x),x]D(x)+2xD\big([D(x),x)]\big)\\&\quad+2D\big(x[D(x),x]\big)-4D(x)xD(x)-4xD(x)^2\\&=2D(x)xD(x)-2xD(x)^2+4xD(x)^2\\&\quad+2D\big(x[D(x),x]\big)-4D(x)xD(x)-4xD(x)^2\\&=-2D(x)xD(x)-2xD(x)^2+2D\big(x[D(x),x]\big).\end{aligned}$$

Thus we have

$$D\big(x[D(x),x]\big)=D(x)xD(x)+xD(x)^2,\quad x\in R.\tag{4}$$

Let us Prove the identity

$$D\big([D(x),x]x\big)=D(x)xD(x)+xD(x)^2,\quad x\in R.\tag{5}$$

Using Lemma 3 and (3) we have

$$\begin{aligned}D\big([D(x),x]x+x[D(x),x]\big)&=2[D(x),x]D(x)+2xD\big([D(x),x]\big)\\&=2[D(x),x]D(x)+4xD(x)^2.\end{aligned}$$

Now applying (4) we obtain

$$\begin{aligned}D\big([D(x),x]x\big)&=2[D(x),x]D(x)+4xD(x)^2-D\big(x[D(x),x]\big)\\&=2[D(x),x]D(x)+4xD(x)^2-D(x)xD(x)-xD(x)^2\\&=D(x)xD(x)+xD(x)^2,\end{aligned}$$

which completes the proof of (5).
From (4) and (5) we obtain

$$D\big([[D(x),x],x]\big)=0,\quad x\in R.\tag{6}$$

From (6) and Lemma 4 it follows

$$[D(x), x]D(x) = 0, \quad x \in R, \tag{7}$$

Using (7) and (3) we obtain

$$\begin{aligned} D\big(D(x)[D(x), x]\big) &= D\big(D(x)[D(x), x] + [D(x), x]D(x)\big) \\ &= 2D(x)D\big([D(x), x]\big) + 2[D(x), x]G(x). \end{aligned}$$

Thus we have

$$D\big(D(x)[D(x), x]\big) = 4D(x)^3 + 2[D(x), x]G(x), \quad x \in R. \tag{8}$$

Let us prove the relation

$$D\big(D(x)[D(x), x]\big) = -6[D(x), x]G(x), \quad x \in R. \tag{9}$$

Using (7) and Lemma 3 we obtain

$$\begin{aligned} 0 &= D\big([D(x), x]D(x)\big) \\ &= D\big(D(x)xD(x)\big) - D\big(xD(x)^2\big) \\ &= D(x)^3 + 3D(x)xG(x) - xD(x)G(x) - D\big(xD(x)^2\big). \end{aligned}$$

Thus we have

$$D\big(xD(x)^2\big) = D(x)^3 + 3D(x)xG(x) - xD(x)G(x), \quad x \in R. \tag{10}$$

Now we have

$$D\big(D(x)^2x + xD(x)^2\big) = 2D(x)^3 + 4xD(x)G(x), \quad x \in R.$$

From the above relation and (10) it follows

$$D\big(D(x)^2x\big) = D(x)^3 + 5xD(x)G(x) - 3D(x)xG(x), \quad x \in R. \tag{11}$$

From (10) and (11) we obtain

$$D\big([D(x)^2, x]\big) = 6[x, D(x)]G(x).$$

Thus we have according to (7),

$$\begin{aligned} 6[x, D(x)]G(x) &= D\big([D(x)^2, x]\big) \\ &= D\big([D(x), x]D(x) + D(x)[D(x), x]\big) \\ &= D\big(D(x)[D(x), x]\big), \end{aligned}$$

which completes the proof of (9). Combining (8) with (9) we arrive at

$$D(x)^3 + 2[D(x), x]G(x) = 0, \quad x \in R \tag{12}$$

Now starting from (7) and using Lemma 3 we obtain

$$\begin{aligned} 0 &= D\big(D(x)[D(x), x]D(x)\big) \\ &= D(x)^2 D\big([D(x), x]\big) + 3D(x)[D(x), x]D^2(x) - [D(x), x]D(x)D^2(x) \\ &= 2D(x)^4 + 3D(x)[D(x), x]G(x). \end{aligned}$$

Thus we have

$$2D(x)^4 + 3D(x)[D(x), x]G(x) = 0, \quad x \in R \tag{13}$$

From (12) and (13) it follows $D(x)^4 = 0$, $x \in R$, which completes the proof of the theorem, since all the assumptions of Theorem 1 are fulfilled.

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