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# HOPF MAPS AND TRIALITY

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**Introduction:** In this note we prove a relation between three kinds of Hopf maps from the seven sphere, using the concept of triality. Let $S^7$ denote the (unit) sphere in $\mathbb{R}^8 \cong \mathbb{K}$ the Caley number field. Classically, the Hopf projection $h_1 : S^7 \to S^4$ is expressed in terms of quaternions.

In §1 we give an expression for $h_1$ in terms of Cayley numbers, which seems to be quite natural in a sense described in the text.

In §2 we employ the concept of triality [1], to show that a certain relation holds between three kinds of Hopf-type maps from $S^7$.

In §3 we use the principal $S^3$-bundles over $S^7$ and the geometry of the exceptional Lie group $G_2$ to determine a map that generates $\pi_7 Spin(5)$, though not with an explicit formula.

In §4 we use E. Cartan's inclusion of a symmetric space $G/H$, quotient of the symmetric pair $H \subset G$, into $G$ as a totally geodesic submanifold and the result of §2 to give a characterization of the space $G_2/SO(4)$ as a totally geodesic submanifold of $G_2$.

We indicate how the considerations above could lead to an explicit description of a map generating $\pi_7 Sp(2) \cong \mathbb{Z}$ [10].

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## §1. The Hopf map.

Let $S^7$ denote the unitary Cayley numbers $\mathbb{K} \cong \mathbb{R}^8$ and denote its elements by Greek letters $\alpha, \beta, \dots$. We begin by expressing the Hopf projection $S^3 \cdots S^7 \xrightarrow{h_1} S^4$ in the context of Cayley multiplication. Recall [6] that this multiplication is defined on pairs of quaternions $\alpha = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\beta = \begin{pmatrix} c \\ d \end{pmatrix}$ as

$$\alpha\beta = \begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - \bar{d}b \\ da + b\bar{c} \end{pmatrix}.$$

If $\alpha$ is unitary, i.e., $|a|^2 + |b|^2 = 1$, the projection $h_1$ is classically considered as the quotient map of the free $Sp(1) \cong S^3$ action on $S^7$, say from the right by

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} aq \\ bq \end{pmatrix}.$$

An invariant projection on $\mathbb{Q}P^1 \cong S^4$ is usually taken to be $h_1\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} |a|^2 - |b|^2 \\ 2a\bar{b} \end{pmatrix}$, where $2a\bar{b}$ lives in $D^4$, the unit disk of $\mathbb{R}^4 \cong \mathbb{Q}$, the quaternions, while the first coordinate $|a|^2 - |b|^2$ is in $[-1,1] \subset \mathbb{R}$ and can be thought of as representing the "necessary height" for the pair to be in $S^4$, i.e., a unitary vector in $\mathbb{R}^5$.

Given any element $(J,K)$ of $V_{7,2}$, the orthonormal 2-frames in $\mathbb{R}^7 \cong Im\mathbb{K}$ with the euclidean metric, observe that it defines an inclusion of $\mathbb{Q}$ in $\mathbb{K}$ by the correspondence of the usual units $i \longmapsto J$, $j \longmapsto K$ and $k \longmapsto JK$.

The elements $J$ and $K$ can be thought of as belonging to the purely imaginary equator $S^6$ of $S^7$, as they satisfy $J^2 = K^2 = -1$ and $JK = -KJ$, being orthogonal to each other.

**Lemma 1:** The map $\delta : S^7 \longrightarrow S^7$ defined by $\delta(\alpha) = (J\overline{\alpha})(\alpha K)$ satisfies $\langle \delta(\alpha), 1\rangle = 0$, $\langle \delta(\alpha), J\rangle = 0$ and $\langle \delta(\alpha), K\rangle = 0$.

*Proof.* $\langle (J\overline{\alpha})(\alpha K), 1\rangle = -\langle \alpha J, \alpha K\rangle = -\langle J, K\rangle = 0$ by the invariance of the euclidean metric with respect to Cayley multiplication. Similarly, $\langle (J\overline{\alpha})(\alpha K), J\rangle = -\langle (\alpha J)J, \alpha K\rangle = \langle \alpha, \alpha K\rangle = \langle 1, K\rangle = 0$ and analogously for $K$ in place of $J$. QED

**Theorem 1:** (i) The image of $\delta$ lies in the unitary four sphere in the 5-dimensional linear subspace of $\mathbb{K}$ perpendicular to $1, J$ and $K$.

(ii) The map $h : S^7 \longrightarrow S^4$ defined by

$$\text{(h)} \qquad h(\alpha) = (e_1\overline{\alpha})(\alpha e_2)$$

is in a way to be explained in the Remarks following the proof, essentially the Hopf map, where $e_1$ and $e_2$ are $\begin{pmatrix} i \\ 0 \end{pmatrix}$ and $\begin{pmatrix} j \\ 0 \end{pmatrix}$ in $\mathbb{Q} \oplus \mathbb{Q} \cong \mathbb{K}$.

*Proof:* (i) is immediate. To show (ii) observe that $\overline{\alpha} = \begin{pmatrix} \overline{a} \\ -b \end{pmatrix}$ and therefore

$$h(\alpha) = \left[\begin{pmatrix} i \\ 0 \end{pmatrix}\begin{pmatrix} \bar{a} \\ -b \end{pmatrix}\right]\left[\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} j \\ 0 \end{pmatrix}\right] = \begin{pmatrix} i\bar{a} \\ -bi \end{pmatrix}\begin{pmatrix} aj \\ -bj \end{pmatrix}$$
$$= \begin{pmatrix} i\bar{a}aj + j\bar{b}bi \\ -bji\bar{a} + bij\bar{a} \end{pmatrix} = \begin{pmatrix} (|a|^2 - |b|^2)k \\ 2bk\bar{a} \end{pmatrix},$$

where $k = ij$ in $\mathbb{Q}$.

A homotopy between $h$ and $h_1$ is constructed as follows: $h\begin{pmatrix} a \\ b \end{pmatrix}$ is homotopic to $\begin{pmatrix} |a|^2 - |b|^2 \\ 2b\bar{a} \end{pmatrix}$ through $\left(\begin{pmatrix} a \\ b \end{pmatrix}, t\right) \longmapsto \begin{pmatrix} |a|^2 - |b|^2 \\ 2b(\cos(t) + \sin(t)k)\bar{a} \end{pmatrix}$, $0 \leq t \leq \pi/2$, and $\begin{pmatrix} |a|^2 - |b|^2 \\ 2b\bar{a} \end{pmatrix}$ is homotopic to $\begin{pmatrix} |b|^2 - |a|^2 \\ -2a\bar{b} \end{pmatrix} = -h_1\begin{pmatrix} a \\ b \end{pmatrix}$ by observing that these two maps just differ by a change of sign in the first two coordinates. Therefore $h$ is homotopic to $-h_1$. Since both $h$ and $h_1$ are essentially the same Hopf map, modulo choice of orientation, we have the claimed result. QED

**Remarks:** i) The map $-h_1$ is $(-\iota_4) \circ h_1$, where $-\iota_4$ is the antipodal map in $S^4$. Its homotopy class in $\pi_7(S^4)$ is $[h_1] \pm \sum w$ where $[h_1]$ is the class of $h_1$ and $\sum w$ is the suspension of the Blakers-Massey element that generates $\pi_6 S^3$. The ambiguity of the sign depends on the choice of orientation.

ii) The classification of $S^3$-principal bundles over $S^4$, by $\pi_4 BS^3 \cong \pi_3 S^3 \cong \mathbb{Z}$, implies, through the exact homotopy sequence of such a fibration, that there are precisely two total spaces whose third homotopy group is zero and are, therefore, homeomorphic to $S^7$.

These two bundles correspond to 1 and $-1$ in $\mathbb{Z}$ and are represented by $h_1$ and $-h_1$ in an order that depends on the choice of orientation.

iii) From the homotopy ladder of the pullback diagram

$$\begin{array}{ccc} S^3 & & S^3 \\ \vdots & & \vdots \\ P_n & \longrightarrow & ES^3 \\ \downarrow & & \downarrow \\ S^4 & \xrightarrow[n]{} & BS^3, \end{array}$$

where $n$ denotes a map of degree $n$ in the $4^{\text{th}}$ homotopy group, it follows that $\pi_3(P_n) \cong \mathbb{Z}_n$.

iv) The map $h$ is the invariant projection of the following free $Sp(1)$ action on $S^7$:

$$\begin{pmatrix} a \\ b \end{pmatrix} q = \begin{pmatrix} aq \\ bkq\bar{k} \end{pmatrix} = \begin{pmatrix} aq \\ -bkqk \end{pmatrix}.$$

v) The expression (h) for the Hopf map emfasizes that it reflects the non-associativity of the Cayley product. This is analogous to the fact that the Hopf map $h_0 : S^3 \longrightarrow S^2$ defined by $h_0(q) = qi\bar{q}$, reflects the non commutativity of the quaternionic product.

vi) The expression for $h$ in (h) can be seen as part of the classical expression for the "next" Hopf map

$$h_2 : S^{15} \longrightarrow S^8, \quad \text{as follows:}$$

Let $\begin{pmatrix} A \\ B \end{pmatrix}$ be a pair of Cayley numbers with

$$|A|^2 + |B|^2 = 1, \quad \text{i.e.,} \quad \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{in} \quad S^{15} \subseteqq \mathbb{R}^{16} \cong \mathbb{K} \oplus \mathbb{K}.$$

An algebraic expression for $h_2$ is again

$$h_2 \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} |A|^2 - |B|^2 \\ 2\bar{A}B \end{pmatrix}.$$

Consider now the inclusion of $S^7$ in $S^{15}$ by $\alpha \longmapsto \frac{1}{\sqrt{2}} \begin{pmatrix} -\alpha e_1 \\ \alpha e_2 \end{pmatrix}$ and compose with $h_2$.

## §2. Other Hopf maps

Consider the following Hopf-type maps [11],

$$H' : S^7 \times S^6 \longrightarrow S^6 \quad \text{defined by}$$
$$H'(\alpha, J) = \alpha J \bar{\alpha}.$$

Observe that for a fixed $J$, for example $J = e_1$, the map $\alpha \longmapsto \alpha e_1 \bar{\alpha}$ generates $\pi_7(S^6) \cong \mathbb{Z}_2$.

Similarly, the map $H : S^7 \longrightarrow V_{7,2}$ with $H(\alpha) = \{\alpha e_1 \bar{\alpha}, \alpha e_2 \bar{\alpha}\}$ generates $\pi_7(V_{7,2}) \cong \mathbb{Z}_4$. This follows easily from [11] and the exact homotopy sequence of the fibrations $SO(5)\cdots SO(7)\longrightarrow V_{7,2}$ and $S^5\cdots V_{7,2}\longrightarrow S^6$.

In order to relate the above described Hopf maps we consider the bundle

$$S^4\cdots V_{7,3}\xrightarrow{p} V_{7,2}$$

with the obvious projection $p(J,K,L)=(J,K)$. Observe that the map $\varphi$ of Lemma 1 furnishes a family of sections of $p$ parametrized by $S^7$, through the Hopf map:

$$x: S^7\times V_{7,2}\to V_{7,3},\quad \text{by}\quad x(\alpha,(J,K))=(J,K,(J\overline{\alpha})(\alpha K)).$$

For any fixed $(J,K)$ in $V_{7,2}$ the map $\alpha\longmapsto(J\overline{\alpha})(\alpha K)$ is the Hopf map from $S^7$ to the fiber $p^{-1}(J,K)\cong S^4$. Note that the Cayley multiplication also furnishes a section $\chi$ of $p$ as follows:

$$\chi(J,K)=(J,K,JK),$$

where $JK$ is the Cayley product of $J$ and $K$.

Conjugation by elements of $S^7$ preserves $V_{7,2}$, i.e., if $\alpha$ is in $S^7$ then $(\alpha J\overline{\alpha},\alpha K\overline{\alpha})$ is in $V_{7,2}$ for $(J,K)$ in $V_{7,2}$. The non-associativity of Cayley numbers prevents us from calling this an action.

The concept of triality [1], [11] provides a relation between the Hopf maps defined above.

Define $h: S^7\times V_{7,2}\to S^6$ by $h(\alpha,(J,K))=(J\overline{\alpha})(\alpha K)$.

**Proposition 1:** For any $\alpha$ in $S^7$, $m,n$ in $\mathbb{Z}$ and for $\star=(e_1,e_2)$, the base point of $V_{7,2}$, the following relation holds

$$h(\alpha^m,H(\alpha^n))=H'(\alpha^n,h(\alpha^{m+3n},\star))$$

where $h(\beta,\star)$ is defined to be $(e_1\overline{\beta})(\beta e_2)$.

*Proof:* Recall that the principle of triality assigns to each $A$ in $SO(8)$ a pair $(B,C)$ in $SO(8)\times SO(8)$, modulo common sign, such that

$$A(xy)=B(x)C(y)\qquad\text{for all}\quad x,y\quad\text{in}\quad\mathbb{K},$$

both products being Cayley multiplications.

It was shown in [11] that if $A$ is the conjugation by a unit Cayley number $\alpha$ in $S^7$, then $B(x)=\alpha x\alpha^2$ and $C(y)=\overline{\alpha}^2y\overline{\alpha}$, i.e., $\alpha(xy)\overline{\alpha}=(\alpha x\alpha^2)(\overline{\alpha}^2y\overline{\alpha})$.

Let $x = \overline{\alpha}\xi\alpha$ and $y = \overline{\alpha}\eta\alpha$, so we have

$$\alpha[(\overline{\alpha}\xi\alpha)(\overline{\alpha}\eta\alpha)]\overline{\alpha} = (\alpha\overline{\alpha}\xi\alpha\alpha^2)(\overline{\alpha}^2\overline{\alpha}\eta\alpha\overline{\alpha}) = (\xi\alpha^3)(\overline{\alpha}^3\eta),$$

since any subalgebra of $\mathbb{K}$ generated by two elements is associative. Therefore, we also have

$$\overline{\alpha}[(\alpha x\overline{\alpha})(\alpha y\overline{\alpha})]\alpha = (x\overline{\alpha}^3)(\alpha^3 y),$$

or

$$(\mu) \qquad (\alpha x\overline{\alpha})(\alpha y\overline{\alpha}) = \alpha[(x\overline{\alpha}^3)(\alpha^3 y)]\overline{\alpha}$$

for all $\alpha$ in $S^7$, $x, y$ in $\mathbb{K}$ [8].

With the above notation we have now,

$$\begin{aligned} h(\alpha^m, H(\alpha^n)) &= h(\alpha^m, (\alpha^n e_1\overline{\alpha}^n, \alpha^n\alpha_2\overline{\alpha}^n)) = \\ &= [(\alpha^n e_1\overline{\alpha}^n)\overline{\alpha}^m][\alpha^m(\alpha^n e_2\overline{\alpha}^n)] = \\ &= [(\alpha^n(e_1\overline{\alpha}^m)\overline{\alpha}^n][\alpha^n(\alpha^m e_2)\overline{\alpha}^n] \qquad \text{(by } (\mu)) \\ &= \alpha^n[(e_1\overline{\alpha}^{m+3n})(\alpha^{m+3n}e_2)]\overline{\alpha}^n = \\ &= \alpha^n h(\alpha^{m+3n}, \star)\overline{\alpha}^n = H'(\alpha^n, h(\alpha^{m+3n}, \star)). \end{aligned}$$

**Example:** $(\alpha e_1\overline{\alpha})(\alpha e_2\overline{\alpha}) = \alpha h(\alpha^3)\overline{\alpha}$.

## §3. $S^3$-Principal Bundles over $S^7$

One could use the formula of the Example above to look for an explicitly defined section of the (trivial) $S^3$-principal bundle $E(12)$ over $S^7$ [5] or equivalently, for an explicitly written generator of $\pi_7 Sp(2) \equiv \mathbb{Z}$ ([10, p. 238]), in a sense that we describe below. We denote by $E(n)$ the pull-back of the bundle $S^3 ... Sp(2) \to S^7$ by a degree $n$ self map $f_n$ of $S^7$. For example, $f_n(\alpha) = \alpha^n$.

Consider the generator $Sp(2)$ of the $S^3$-principal bundles over $S^7$ as the pull back of $-h$ by $h$ over $S^7$ as follows:

$$Sp(2) = \left\{ A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \text{with} \quad AA^* = A^*A = I \right\}$$

where $a, b, c$ and $d$ are quaternions

$$\begin{array}{ccc} Sp(2) & \xrightarrow{2^{nd} col.} & S^7 \\ {\scriptstyle 1^{st} col.} \downarrow & & \downarrow -h \\ S^7 & \xrightarrow[h]{} & S^4 \end{array}$$

The diagram above is commutative by the definition of $Sp(2)$, i.e., because $|a|^2 - |b|^2 = -(|c|^2 - |d|^2)$ and $2a\bar{b} = -2c\bar{d}$. As $E(k)$ is the pull-back $f_k^*(Sp(2)) = f_k^*(E_1)$, where $f_k : S^7 \to S^7$ is a map of degree $k$ [5] for $k = 2, 3 \ldots$ , it follows that to write down a section of the trivial $E(12)$ is equivalent to producing a map

$$\beta : S^7 \longrightarrow S^7 \quad \text{with } -h \circ \beta(\alpha) = h(\alpha^{12})$$

and the matrix $(\alpha^{12}, \beta(\alpha))$ of $Sp(2)$ represents a generator of $\pi_7 Sp(2)$.

From the relevant part of the exact homotopy sequence of the principal fibration $S^3 ... G_2 \to V_{7,2}$ one can readily observe that the 2-primary component of $\pi_6(S^3) \cong \mathbb{Z}_{12} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_4$ is related to $\pi_7 V_{7,2} \cong \mathbb{Z}_4$. The 3-primary component is related to $\pi_6(G_2) \cong \mathbb{Z}_3$ ([7], [3]).

A strategy for obtaining a section of $E(12)$ is the following:

The principal $S^3$-bundle $S^3 \cdots G_2 \xrightarrow{p} V_{7,2}$, where $G_2$ is the exceptional Lie group of automorphisms of $\mathbb{K}$, implies that there exists a lifting to $G_2$ of the (homotopically trivial) map $\tau : S^7 \longrightarrow V_{7,2}$ defined by $\tau(\alpha) = (\alpha^4 e_1 \overline{\alpha}^4, \alpha^4 e_2 \overline{\alpha}^4)$.

Recall now ([13]) that the columns of all elements of $G_2 \subseteq SO(7)$ satisfy the same relation as the purely imaginary units of $\mathbb{K}$, i.e., $3^{rd} col. = (1^{st} col.)(2^{nd} col.)$, $5^{th} col. = (1^{st} col.)(4^{th} col.)$, $6^{th} col. = (2^{nd} col.)(4^{th} col.)$, $7^{th} col. = (3^{rd} col.)(4^{th} col.)$. As the projection $p$ above is precisely on the first two columns, it follows that to lift $\tau$ one should just determine a fourth column, i.e., an element of $S^7$ that is perpendicular, in the euclidean metric of $\mathbb{R}^8$, to $\alpha^4 e_1 \overline{\alpha}^4, \alpha^4 e_2 \overline{\alpha}^4$ as well as to their Cayley product that represents the third column and is according to the **Example** above equal $\alpha^4 h(\alpha^{12}) \overline{\alpha}^4$. As conjugation by a unitary element is $\mathbb{R}^8$, in our case by $\alpha^4$, is an isometry of the euclidean metric, one is looking for a map $\varphi : S^7 \longrightarrow S^4$, such that $\langle \varphi(\alpha), h(\alpha^{12}) \rangle = 0$, for all $\alpha$ is $S^7$. Such a $\varphi$ exists, from the above discussion. If $C(\overline{\alpha}^4)$ represents conjugation by $\overline{\alpha}^4$ in $SO(7)$ and $\psi : S^7 \longrightarrow G_2$ is a lifting of $\tau$, according to the above, then $C(\overline{\alpha}^4) \circ \psi(\alpha)$ is a matrix in $SO(5)$ and we have

**Proposition 2:** $C(\overline{\alpha}^4) \circ \psi(\alpha)$ generates $\pi_7 SO(5)$ and $\pi_7 Spin(5)$ in the sense of triality.

*Proof:* Observe that $C(\overline{\alpha}^4) \circ \psi(\alpha)$ is a $7 \times 7$ matrix with the following columns: $e_1$, $e_2$, $h(\alpha^{12})$, $\varphi(\alpha)$, $\overline{\alpha}^4[(\alpha^4 e_1 \overline{\alpha}^4)(\alpha^4 \varphi(\alpha)\overline{\alpha}^4)]\alpha^4$, $\overline{\alpha}^4[(\alpha^4 e_2 \overline{\alpha}^4)(\alpha^4 \varphi(\alpha)\overline{\alpha}^4)]\alpha^4$ and $\overline{\alpha}^4[(\alpha^4 h(\alpha^{12})\overline{\alpha}^4)(\alpha^4 \varphi(\alpha)\overline{\alpha}^4)]\alpha^4$. Call this matrix $A(\alpha)$ and we have the element $(A(\alpha), B(\alpha), C(\alpha))$ in $Spin(5) \subseteq Spin(8)$ according to triality, see also [12]. We know that the projection $Spin(5) \longrightarrow S^7$ corresponding to the first column projection $Sp(2) \longrightarrow S^7$ used above, corresponds to $B(\alpha)(1)$. If the degree of $\alpha \longmapsto B(\alpha)(1)$ as a map from $S^7$ to itself is $\pm 12$, then the map $\alpha \longmapsto (A(\alpha), B(\alpha), C(\alpha))$ represents a generator of $\pi_7 Spin(5) = \pi_7 Sp(2)$ and $\alpha \longmapsto A(\alpha)$ is a generator of $\pi_7 SO(5)$, as follows from the exact homotopy sequence of $S^3 \cdots Spin(5) \longrightarrow S^7$ and the identification $Spin(5) \cong Sp(2)$.

Let $L_\beta$, resp. $R_\beta$, denote left, resp. right, Cayley multiplication by $\beta$, then using the triality, we have

$$C(\overline{\alpha}^4) \equiv (L_{\overline{\alpha}^4} \circ R_{\alpha^4}, L_{\overline{\alpha}^4} \circ R_{\overline{\alpha}^8},\ L_{\alpha^8} \circ R_{\alpha^4}) \quad \text{in} \quad Spin(7)$$

and

$$\psi(\alpha) \equiv (\psi(\alpha), \psi(\alpha), \psi(\alpha)) \quad \text{in} \quad G_2 \subseteq Spin(7).$$

Recall that $G_2$ is characterized by its elements being of the form $(A, A, A)$ in $Spin(7)$, by its definition as the automorphism group of $\mathbb{K}$. Therefore

$$C(\overline{\alpha}^4) \circ \psi(\alpha) = (L_{\overline{\alpha}^4} \circ R_{\alpha^4} \circ \psi(\alpha), L_{\overline{\alpha}^4} \circ R_{\overline{\alpha}^8} \circ \psi(\alpha), L_{\alpha^8} \circ R_{\alpha^4} \circ \psi(\alpha))$$

and $B(\alpha)(1) = L_{\overline{\alpha}^4} \circ R_{\overline{\alpha}^8} \circ \psi(\alpha)(1) = \overline{\alpha}^4[\psi(\alpha)(1)]\overline{\alpha}^8 = \overline{\alpha}^{12}$, since $\psi(\alpha)(1) = 1$. Therefore $C(\overline{\alpha}^4) \circ \psi(\alpha)$ generates $\pi_7 Spin(5)$ in the manner described above. QED

Suppose now that we are given $\varphi$ (and $\psi$) and we want to retrieve $\beta$, a section of $E(12)$. By the above we can construct $g_1 \equiv g_1(\varphi)$ a generator of $\pi_7 Spin(5)$. By an easy argument [4], it follows that the corresponding map $g : S^7 \longrightarrow Sp(2)$ has columns $\alpha^{12}$ and $B(\alpha)(e_4)$. From the diagram below

$$\begin{array}{ccccc} S^7 & \xrightarrow{g} & Sp(2) & \xrightarrow{2^{nd}col.} & S^7 \\ & \searrow^{\lambda} & \downarrow 1^{st}col. & & \downarrow -h \\ & & S^7 & \xrightarrow[h]{} & S^4 \end{array}$$

we have $\lambda = 1^{st}col.g$, $deg(\lambda) = 12$, and $h(\lambda(\alpha)) = -h(2^{nd}col.g(\alpha)) = -h(B(\alpha)(e_4))$.

In other words, the map $S^7 \longrightarrow S^7 \times Sp(2)$ with $\alpha \longmapsto (\alpha, g(\alpha))$ is a section of $E(12)$.

Observe that the second column of the matrix $g(\alpha)$ in $Sp(2)$ has degree $-12$ too, as a map from $S^7$ to itself: As we saw $2^{nd}col.g(\alpha) = B(\alpha)(e_4) = \overline{\alpha}^4(\psi(\alpha)(e_4))\overline{\alpha}^8 = \overline{\alpha}^4(\alpha^4\varphi(\alpha)\overline{\alpha}^4)\overline{\alpha}^8 = \varphi(\alpha)\overline{\alpha}^{12}$, but $\varphi(\alpha)$ has degree zero since it lands in $S^4$.

## §4. E. Cartan's method

To construct a natural (though, fatally nullhomotopic) map from $S^7$ to $G_2$ we employ Elie Cartan's method of embedding symmetric spaces into Lie groups as totally geodesic submanifolds [2, p. 77].

In our case, we begin with the generator of $\pi_7(V_{7,2})$ described earlier, i.e., $(\alpha e_1\overline{\alpha}, \alpha e_2\overline{\alpha})$ and apply to it the section $\mathcal{X}$ to obtain the element $(\alpha e_1\overline{\alpha}, \alpha e_2\overline{\alpha}, (\alpha e_1\overline{\alpha})(\alpha e_2\overline{\alpha})) = (\alpha e_1\overline{\alpha}, \alpha e_2\overline{\alpha}, \alpha h(\alpha^3)\overline{\alpha})$ in $V_{7,3}$. Cartan's method does not apply directly in this case, since $(SO(7), SO(4))$ is not a symmetric pair ($(SO(7), SO(3) \times SO(4))$ is one). We can, however, consider the conjugate orbit of the matrix $A = \begin{pmatrix} I_3 & 0 \\ 0 & -I_4 \end{pmatrix}$, which amounts to $V_{7,2} \longrightarrow G_2/SO(4) \xrightarrow{\text{Cartan}} G_2$, since $A$ is in $G_2 \subseteq SO(7)$.

We will employ the Moufang identities to prove

**Theorem 2:** The Cartan inclusion $\Lambda$ of $G_2/SO(4)$ in $G_2$ is $\Lambda([B]) = L_{b_3} \circ L_{b_2} \circ L_{b_1}$, where $b_i$ , $i = 1, 2, 3$ are the first three columns of any matrix in the class [B] in $G_2/SO(4)$.

*Proof:* Note that $A$ is the composition of four reflections in $\mathbb{R}^7 = Im\mathbb{K}$, each one with respect to the hyperplane perpendicular to $e_4, e_5, e_6$ and $e_7$. If $v$ is in $S^6$, unitary vector in $\mathbb{R}^7$, the reflection in the hyperplane perpendicular to $v$, denoted by $R_v$ is $R_v(x) = vxv$. Therefore,

$$A(x) = e_7(e_6(e_5(e_4xe_4)e_5)e_6)e_7.$$

for the same reason $-A = \begin{pmatrix} -I_3 & 0 \\ 0 & I_4 \end{pmatrix}$, that belongs to the "negative" connected component of $O(7)$, is equal to

$$-A(x) = e_3(e_2(e_1xe_1)e_2)e_3 \quad , \quad \text{for all} \quad x \in \mathbb{R}^7,$$

and $A(x) = -e_3(e_2(e_1xe_1)e_2)e_3$.

Employing the following Mounfang identity [11], [8]

$$a(xy)a = (ax)(ya)$$

we obtain

$$\begin{aligned} A(x) &= -e_3\{e_2[(e_1x)e_1]e_2)\}e_3 \\ &= -e_3\{e_2[(e_1x)][e_1e_2]\}e_3 = \\ &= -e_3\{e_2[(e_1x)]e_3\}e_3 \\ &= -e_3\{e_2[(e_1x)]\}e_3^2 = e_3(e_2(e_1x)). \end{aligned}$$

Recall now that the columns of $[B]$ satisfy $b_3 = b_1b_2$, so that the matrix $B$ can be choosen to belong to $G_2$, as the rest of the columns beyond the third don't matter. In this case we can assume that $B$ distributes over Cayley products and we have:

$$\begin{aligned} BAB^{-1}(x) = BA(B^{-1}(x)) = B[e_3(e_2(e_1B^{-1}(x)))] = \\ = \; b_3(b_2(b_1x)) = L_{b_3} \circ L_{b_2} \circ L_{b_1}(x) \quad \text{as claimed.} \end{aligned}$$

QED

**Corollary 1:** The Cartan inclusion of $G_2/SO(4)$ in $G_2$ is represented by: $(J, K)$ in $V_{7,2}$ goes to $L_{JK} \circ L_K \circ L_J$ in $G_2$, using the projection $s$ of the fibration

$$SO(3) \dots V_{7,2} \xrightarrow{s} G_2/SO(4).$$

Let now $\Psi$ be the composition $\Lambda \circ s \circ H$, where $H$ is the generator of $\pi_7(V_{7,2})$ from §2, $s$ is the projection from $V_{7,2}$ to $G_2/SO(4)$ and $\Lambda$ is the Cartan inclusion of the symmetric space $G_2/SO(4)$ into $G_2$ described above.

**Corollary 2:** The resulting map $\Psi$ from $S^7$ to $G_2$ has columns $\Psi(\alpha)(e_i) = [\alpha h(\alpha^3)\overline{\alpha}]\{(\alpha e_2\overline{\alpha})[(\alpha e_1\overline{\alpha})e_i]\}$ for $i = 1, 2, \dots, 7$.

This map is the lifting to $G_2$ of the homotopically trivial map $\mu : S^7 \longrightarrow V_{7,2}$, with $\mu = p \circ \Psi$. In order to construct a generator of $\pi_7 Sp(2)$

one should produce a map $\Phi : S^7 \longrightarrow SO(7)$ with $[\Phi] = \pm 4$ in $\pi_7 SO(7)$, such that the first two columns of $\Phi(\alpha)$ are precisely $\mu(\alpha)$. Note that $\alpha h(\alpha^3)\overline{\alpha}$ is just $(\alpha e_1 \overline{\alpha})(\alpha e_2 \overline{\alpha})$ and that each $\alpha e_i \overline{\alpha}$ is a representative of the generator of $\pi_7(S^6)$.

*Problem:* (i) We do not know if there exists a reasonable formula describing such a $\Phi$.
(ii) Similarly, no explict formula for a map $\varphi : S^7 \longrightarrow S^4$, described above, that would leed to a lifting $\psi$ of $\tau$, is known to the authors either.

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