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NOTE ON HOMOTOPY CLASSES OF SELF MAPS; $[\Sigma HP^3, \Sigma HP^3]$

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1. Introduction. Let X be a CW complex and Σ denote the reduced suspension. Then the homotopy classes of self maps of ΣX $[\Sigma X, \Sigma X]$ has a group structure induced by the co-H structure of ΣX. In this paper we shall investigate $[\Sigma X, \Sigma X]$ when X is the quarternionic projective space HP^2 and HP^3. Let HP^n be the quarternionic projective space S^{4n+3}/S^3 and let $Eq(X)$ denote the group consisting of the self homotopy equivalences of X by the composition structure. Let $+$ denote the sum operation in $[\Sigma X, \Sigma X]$ and let $\circ$ denote the composition of maps. Then we have the following theorems:

Theorem 2.2. *$[\Sigma HP^2, \Sigma HP^2]$ is abelian and isomorphic to $Z + Z$.*

Theorem 2.20. *$[\Sigma HP^3, \Sigma HP^3]$ is non-abelian. We have a non trivial extension;*

$$0 \to Z\{K, P, \Xi\} \to [\Sigma HP^2, \Sigma HP^2] \to Z\{id\} \to 0.$$

The non trivial commutator is $< id, \Xi >= 24K$. The center of $[\Sigma HP^3, \Sigma HP^3] = Z\{K, P\}$.

Let

$$H_X : [X, X] \to Hom(\tilde{H}_*(X), \tilde{H}_*(X)$$
$$\pi_X : [X, X] \to Hom(\pi_*(X), \pi_*(X))$$

denote the maps defined by $H_X(f)$ =the induced homomorphism on homology groups $H_*(X)$ and $\pi_X(f)$ =the induced homomorphism on homotopy groups $\pi_*(X)$ for $f \in [X, X]$ respectively. Then we call $f \in [X, X]$ homologically trivial if $\tilde{H}_X(f)$ is the trivial homomorphism, homotopy trivial if $\pi_X(f)$ is the trivial homomorphism, homologically identity if $H_X(f)$ is the identity and homotopically identity if $\pi_X(f)$ is the identity. Then for the group of homologically identity maps of X, $H_X^{-1}(id)$, the group of homotopically identity maps of X, $\pi_X^{-1}(id)$, the group of self homotopy equivalences of X, $Eq(X)$, the homologically trivial maps of X, $H_X^{-1}(0)$ and the homotopically trivial maps of X, $\pi_X^{-1}(0)$ in the case of $X = \Sigma HP^2$ and ΣHP^3 we obtain the followings:

Theorem 5.2. *(1)* $H^{-1}_{\Sigma HP^2}(0) = \{0\} = \pi^{-1}_{\Sigma HP^2}(0)$, $H^{-1}_{\Sigma HP^2}(id) = \{id_{\Sigma HP^2}\} = \pi^{-1}_{\Sigma HP^2}(id)$ *and* $Eq(\Sigma HP^2) = \{\pm id_{\Sigma HP^2}\}$.
(2) $H^{-1}_{\Sigma HP^3}(0) = Z\{K\}, \pi^{-1}_{\Sigma HP^3}(0) = \{0\}$, $H^{-1}_{\Sigma HP^3}(id) = \{id_{\Sigma HP^3} + Z\{K\}\}$, $\pi^{-1}_{\Sigma HP^3}(id) = \{id_{\Sigma HP^3}\}$ *and* $Eq(\Sigma HP^3) = \{\pm id_{\Sigma HP^3} + Z\{K\}\}$.

We shall give the composition operations in $[\Sigma HP^2, \Sigma HP^2]$ and $[\Sigma HP^3, \Sigma HP^3]$ in the section 5.

Lemma 5.22. *The standard left distributivity law is satisfied in* $[\Sigma HP^2, \Sigma HP^2]$.

Theorem 5.25. *The left distributivity law in* $[\Sigma HP^3, \Sigma HP^3]$ *is as follows:*

$$(f+g)\circ h = f\circ h + g\circ h \text{ for the case } f,g \in Z\{K,P,\Xi\} \text{or } h \in \{id, \Xi\}$$

and exceptinal cases;

$$(m\ id)\circ K = m^2K, (a\ id + b\Xi)\circ K = a(1+24b)K$$
$$\text{and } (a\ id + b\Xi)\circ P = (a+120b)P + 4320bK.$$

Let (a,b) denote $a\ id + b\xi\circ(/S^5)$ in $[\Sigma HP^2, \Sigma HP^2]$ for abbreviation where id and $\xi\circ(/S^5)$ are generators of $[\Sigma HP^2, \Sigma HP^2]$. Then we have:

Corollary 5.26. $(a,b)\circ(c,d) = (ac, ad+bc+24bd)$.

Let (a,b,c,d) denote $a\ id + bK + cP + d\Xi$ in $[\Sigma HP^3, \Sigma HP^3]$ where id, K, P and Ξ are generators of $[\Sigma HP^3, \Sigma HP^3]$. Then we have:

Corollary 5.27. $(a,b,c,d)\circ(e,f,g,h) = (ae, a2f + af(1+24d) + be + 360bg + 4320dg + 120bh, ag + ce + 360cg + g(a+120d) + 120ch + 32dh, ah + de + 24dh)$.

2. Groups $[\Sigma HP^2, \Sigma HP^2]$ and $[\Sigma HP^3, \Sigma HP^3]$. We have the following exact sequence for the pair $(\Sigma HP^2, S^5)$:

(2.1)

$$
\begin{array}{ccccccc}
\pi_{14}(\Sigma HP^2, S^5) & \to & \pi_{13}(S^5) & \to & \pi_{13}(\Sigma HP^2) & \xrightarrow{j_*^{(1}} & \\
\| & & \| & & \| & & \\
Z_2\{[e^9, \iota_5] \circ \bar{\eta}\} & & Z_2\{\epsilon_5\} & & Z_2\{\epsilon_5\} + Z\{\kappa\} & & \\
 & & & & & & \\
\pi_{13}(\Sigma HP^2, S^5) & \to & \pi_{12}(S^5) & \to & \pi_{12}(\Sigma HP^2)^{(2} & \to & \\
\| & & \| & & & & \\
Z\{[e^9, \iota_5]\} & & Z_{30} & & & & \\
 & & & & & & \\
\pi_{12}(\Sigma HP^2, S^5) & \to & \pi_{11}(S^5) & \to & \pi_{11}(\Sigma HP^2) & \to & \\
\| & & \| & & \| & & \\
Z_{24}\{e^9 \circ \bar{\nu}\} & & Z_2\{\nu^2\} & & 0 & & \\
 & & & & & & \\
\pi_{11}(\Sigma HP^2, S^5) & \to & \pi_{10}(S^5) & \to & \pi_{10}(\Sigma HP^2) & \to & \\
\| & & \| & & \| & & \\
Z_2\{e^9 \circ \bar{\eta}^2\} & & Z_2\{\nu \circ \eta^2\} & & 0 & & \\
 & & & & & & \\
\pi_{10}(\Sigma HP^2, S^5) & \to & \pi_9(S^5) & \to & \pi_9(\Sigma HP^2) & \xrightarrow{j_*^{(3}} & \\
\| & & \| & & \| & & \\
Z_2\{e^9 \circ \bar{\eta}\} & & Z_2\{\nu \circ \eta\} & & Z\{\xi\} & & \\
 & & & & & & \\
\pi_9(\Sigma HP^2, S^5) & \to & \pi_8(S^5) & \to & \pi_8(\Sigma HP^2) & \to & \\
\| & & \| & & \| & & \\
Z\{e^9\} & & Z_{24}\{\nu\} & & 0 & &
\end{array}
$$

$;^{(1}\ j_*\kappa = [e^9, \iota_5],^{(2}\ \Sigma p \in \pi_{12}(\Sigma HP^2),^{(3}\ j_*\xi = 24e^9$

where the symbols of almost all of elements of the homotopy groups of spheres are followed by Toda[13], p is the canonical projection $S^{11} \to HP^2$ and, for example, e^9 represents the characteristic element of $\pi_9(\Sigma HP^2, S^5)$ such that $\partial e^9 = \nu$ and similarly $\bar{\eta}$ does the characteristic element of $\pi_{k+1}(e^k, S^{k-1})$ such that $\partial \bar{\eta} = \eta$.

We obtain an exact sequence

$$0 \to \pi_9(\Sigma HP^2) \to [\Sigma HP^2, \Sigma HP^2] \to \pi_5(\Sigma HP^2) \to 0$$

from the Puppe sequence associated with the cofibering $S^5 \subset \Sigma HP^2 \to S^9$ because $\nu^* : \pi_6(\Sigma HP^2) \to \pi_9(\Sigma HP^2)$ is trivial and $\pi_8(\Sigma HP^2) = 0$. Therefore $[\Sigma HP^2, \Sigma HP^2]$ is generated by id and $\xi \circ (/S^5)$ where $/S^5$:

$\Sigma HP^2 \to S^9$ denotes the collapsing map.

Theorem 2.2. *$[\Sigma HP^2, \Sigma HP^2]$ is abelian and $[\Sigma HP^2, \Sigma HP^2] = Z\{id, \xi \circ (/S^5)\}$.*

Proof. It remains to show the commutativity of id and $\xi \circ (/S^5)$. The map $-id + \xi \circ (/S^5) + id$ is represented as follows :

$$\Sigma HP^2 \xrightarrow{\nabla} \Sigma HP^2 \vee \Sigma HP^2 \vee \Sigma HP^2 \xrightarrow{\Gamma} \Sigma HP^2 \vee \Sigma HP^2 \vee \Sigma HP^2 \xrightarrow{fold} \Sigma HP^2,$$

where ∇ denotes the pinching map and $\Gamma = -id \vee \xi \circ (/S^5) \vee id$.

Since the restriction $\xi \circ (/S^5)|S^5$ is homotopy trivial, $-id + \xi \circ (/S^5) + id$ is contained in the image of $\pi_9(\Sigma HP^2)$ on the above Puppe sequence. The element $\xi \in \pi_9(\Sigma HP^2)$ is characterized by the property that the induced homomorphism ξ_* on the homology group $H_9(\Sigma HP^2)$ is $24id$ and trivial on others $\tilde{H}_*(\Sigma HP^2)$. The map $-id + \xi \circ (/S^5) + id$ has the same property and so $-id + \xi \circ (/S^5) + id = \xi \circ (/S^5)$. Hence $[\Sigma HP^2, \Sigma HP^2]$ is equal to $Z\{id, \xi \circ (/S^5)\}$ as an abelian group.

Next we shall investigate $[\Sigma HP^3, \Sigma HP^3]$. First we note

Lemma 2.3. *$(/S^4) \circ p = 2\nu$.*

Proof. Since $HP^3/S^4 = HP^3/HP^1$ is homeomorphic to the Thom space of $2[\xi]$ over HP^1 where $[\xi]$ is the canonical line bundle, $(/S^4) \circ p$ must be 2ν.

Hence it follows $j_* \Sigma p = 2e^9 \circ \nu$ where $j_* : \pi_{12}(\Sigma HP^2) \to \pi_{12}(\Sigma HP^2, S^5)$.

Lemma 2.4. *$\pi_{12}(\Sigma HP^2) = Z_{360}$ generated by Σp.*

Though the result of Lemma 2.4 is given by Morisugi[8], another proof is given in the section 4.

Lemma 2.5. *$\Sigma p \circ \eta \neq 0$ and it follows $\Sigma p \circ \eta = i_* \epsilon_5$.*

The proof of Lemma 2.5 is given in the section 3.

Thus we have the following exact sequence for the pair $(\Sigma HP^3, \Sigma HP^2)$:

$$
(2.6)\quad
\begin{array}{ccccc}
\pi_{14}(\Sigma HP^3, \Sigma HP^2) & \to & \pi_{13}(\Sigma HP^2) & \to & \pi_{13}(\Sigma HP^3) \xrightarrow{j_*} \\
\| & & \| & & \| \\
Z_2\{e^{13}\eta\} & & Z_2\{\epsilon_5 + Z\{\kappa\} & & Z\{\kappa, \rho\} \\
\\
\pi_{13}(\Sigma HP^3, \Sigma HP^2) & \to & \pi_{12}(\Sigma HP^2) & \to & \pi_{12}(\Sigma HP^3) \\
\| & & \| & & \| \\
Z\{e^{13}\} & & Z_{360}\{\Sigma p\} & & 0
\end{array}
$$

;[1] $j_*\rho = 360e^{13}$

On the other hand from the cofiber sequences

$$S^5 \subset \Sigma HP^3 \to \Sigma HP^3/S^5 = S^9 \cup_{2\nu} e^{13} \text{ and } S^9 \subset \Sigma HP^3/S^5 \to S^{13}$$

we obtain the Puppe sequences using (2.1) and (2.6):

$$
(2.7)\quad
\begin{array}{ccccc}
 & & & \pi_{10}(\Sigma HP^3) & = 0 \\
 & & & \downarrow & \\
Z_2\{\eta\} = & \pi_6(\Sigma HP^3) & & \pi_{13}(\Sigma HP^3) & = Z\{\kappa, \rho\} \\
 & \downarrow & & \downarrow (/S^9)^* & \\
 & [\Sigma HP^3/S^5, \Sigma HP^3] & = & [\Sigma HP^3/S^5, \Sigma HP^3] & \\
 & \downarrow (/S^5)^* & & \downarrow |S^9 & \\
Z\{id\} \subset & [\Sigma HP^3, \Sigma HP^3] & & \pi_9(\Sigma HP^3) & = Z\{\xi\} \\
 & \downarrow |S^5 & & \downarrow (2\nu)^* & \\
Z\{\iota\} = & \pi_5(\Sigma HP^3) & & \pi_{12}(\Sigma HP^3) & = 0 \\
 & \downarrow & & & \\
 & 0 & & &
\end{array}
$$

Here we note that $[\Sigma HP^3/S^5, \Sigma HP^3]$ is abelian because $\Sigma HP^3/S^5 = S^9 \cup_{2\nu} e^{13}$ is a double suspension. Thus we obtain

Lemma 2.8. $[\Sigma HP^3/S^5, \Sigma HP^3] = Z\{(/S^9)^*\kappa, (/S^9)^*\rho, \tilde{\xi}\}$ *where* $\tilde{\xi}$ *denotes an extension of* ξ *on* $\Sigma HP^3/S^5$ *such that* $\tilde{\xi}|S^9 = \xi$.

Now the generators $K = (/\Sigma HP^2)^*\kappa, P = (/\Sigma HP^2)^*\rho$ and $\Xi = (/S^5)^*\tilde{\xi}$ of $[\Sigma HP^3, \Sigma HP^3]$ are given as follows:

(2.9)

$$\begin{array}{llllll}
\Xi : \Sigma HP^3 & \overset{/S^5}{\rightarrow} & S^9 \cup_{2\nu} e^{13} & \overset{\tilde{\xi}}{\rightarrow} & \Sigma HP^3 & \\
 & & \cup & & \cup & \\
 & & S^9 & \overset{\xi}{\rightarrow} & \Sigma HP^2 & \overset{/S^5}{\rightarrow} S^9 \\
 & & \| & & & \quad \| \\
 & & S^9 & & \overset{24\iota_9}{\rightarrow} & \quad S^9
\end{array}$$

$$\begin{array}{lllll}
P : \Sigma HP^3 & \overset{/\Sigma HP^2}{\rightarrow} & S^{13} & \overset{\rho}{\rightarrow} & \Sigma HP^3 \\
 & & & & \downarrow /HP^2 \\
 & & & & S^{13} \quad ; \quad (/HP^2)\circ\rho = 360\iota_{13}
\end{array}$$

$$K : \Sigma HP^3 \overset{/\Sigma HP^2}{\rightarrow} S^{13} \overset{\kappa}{\rightarrow} \Sigma HP^2$$

By (2.7), it follows that $[\Sigma HP^3, \Sigma HP^3]$ is generated by 4 elements: id, K, P, Ξ.

Recall the Hilton-Milnor theorem (followed by G.W.Whitehead[14]). Let $J(X)$be the infinite reduced product spaces of James and let $J_n(X)$ be the image of X^n. Let $i : X \rightarrow J(X)$ be the canonical inlusion. It is well known that $j : J(X) \rightarrow \Omega\Sigma X$ is a homotopy equivalence where $j(x_1 \cdot x_2 \cdots x_n) = (\cdots(i(x_1) + i(x_2)) + i(x_3) + \cdots) + i(x_n)$. We identify $J(X)$ with $\Omega\Sigma X$ by j for brevity. Let $x \in [\Sigma X, Y]$. Then let $x' \in [X, \Omega Y]$ be its adjoint and occasionally x by the same sign. To state the Hilton-Milnor theorem precisely, we need some algebraic preliminaries. Let A be the free non-associative ring with n generators $x_1, x_2, \cdots, x_n$. A has an additive basis consisting of all parenthesized monomials in x_i. We shall single out certain of these, reffering to them as basic products. Let us define the weight of a monomial to be the number of its factors. Suppose Y be a connected homotopy associative H space (grouplike space). If w is a basic product and $x_i \in [X_i, Y]$, we define

$$w(x_1, x_2, \cdots, x_n) \in [X_1^{w(1)} \wedge X_2^{w(2)} \cdots \wedge X_n^{w(n)}, Y] = [X^{\wedge w}, Y]$$

using the iterated Samelson product. If $Y = \Omega\Sigma X$, its adjoint is the iterated Whitehead product

$$[x_1, x_2, \cdots, x_n]_w \in [\Sigma(X_1^{w(1)} \wedge X_2^{w(2)} \cdots \wedge X_n^{w(n)}, \Sigma X]$$

for $x_i \in [\Sigma X_i, \Sigma X] \simeq [X_i, \Omega\Sigma X]$. Then we have

Theorem (Hilton-Milnor). *Let $X_t (t = 1, \cdots, n)$ be connected CW-complexes with vertices. Then $J(X_1 \vee X_2 \vee \cdots \vee X_n)$ is homotopy equivalent to $\prod_w J(\Sigma(X_1^{w(1)} \wedge X_2^{w(2)} \wedge \cdots \wedge X_n^{w(n)})$.*

An explicit equivalence can be constructed as follows. Let $j_t : X_t \subset X = X_1 \vee X_2 \vee \cdots \vee X_n$ be the usual inclusion $(t = 1, \dots, n)$ and let $i_t : X_t \subset J(X)$ be the composite of j_t with the inclusion $i : X \to J(X)$. For each basic product w, we can form the element (the iterated Samelson products)

$$w(i_1, i_2, \dots, i_n) : X_1^{w(1)} \wedge X_2^{w(2)} \cdots \wedge X_n^{w(n)} \to J(X).$$

This map can be extended uniquely to a homomorphism

$$\widetilde{w(i_1, i_2, \dots, i_n)} : J(X_1^{w(1)} \wedge X_2^{w(2)} \wedge \cdots \wedge X_n^{w(n)}) \to J(X).$$

Their weak products induces an equivalence:

$$\prod_w \widetilde{w(i_1, i_2, \dots, i_n)} : \prod_w J(X_1^{w(1)} \wedge X_2^{w(2)} \wedge \cdots \wedge X_n^{w(n)}) \to J(X_1 \vee X_2 \vee \cdots \vee X_n).$$

Consider the case $n = 2$ and $X_1 = X_2 = X$. Let $i_1 = id_1$ and $i_2 = id_2$ denote the inclusions of ΣX into $\Sigma X \vee \Sigma X$ so that $\nabla = i_1 + i_2$. Then the formula of the Hilton-Milnor theorem for $n = 2$ is as folllows:

Corollary. *The iterated Samelson products $< id_1, id_2 >_w \in [X^w, \Omega\Sigma(X \vee X)]$ for basic products w induces a homotopy equivalence (not necessarily H space equivalent unless X is a co-H space)*

$$\psi = \prod_w \widetilde{< id_1, id_2 >_w} : \prod_w \Omega\Sigma(X^{\wedge w}) \simeq \Omega\Sigma(X \vee X)$$

where $\widetilde{< id_1, id_2 >_w} : \Omega\Sigma(X^{\wedge w}) \to \Omega\Sigma(X \vee X)$ is the unique homomorphic extension of $< id_1, id_2 >_w$. Especially $\widetilde{< id_1, id_2 >_{w}}_* : \pi_*(\Omega\Sigma(X^{\wedge w})) \to \pi_*(\Omega\Sigma(X \vee X))$ is an embedding to direct summands for each w.

It follows for any finite dimensional CW-complex A that

$$[\Sigma A, \Sigma(X \vee X)] \simeq \prod_w [A, \Omega\Sigma(X^{\wedge w})] \simeq \prod_w [\Sigma A, \Sigma(X^{\wedge w})].$$

Let $pr_w : \prod_w \Omega\Sigma(X^{\wedge w}) \to \Omega\Sigma(X^{\wedge w})$ be the projection. Especially for $A = X$, the co-Hopf structure map $\nabla \in [\Sigma X, \Sigma(X \vee X)] \simeq id_1 + id_2 \in [X, \Omega\Sigma(X \vee X)]$ corresponds to $\prod_w h_w$ where $h_w = pr_w \circ \psi^{-1} \circ \nabla$. For

$\alpha \in [\Sigma A, \Sigma X]$, define $h_w(\alpha)$ =the adjoint of $h_{w*}\alpha'$ called the Hopf-Hilton invariants of α so that $\nabla\alpha \simeq \Pi_w h_w(\alpha)$ by the above correspondence. That is,

$$\nabla\alpha = \sum_w [id_1, id_2]_w \circ h_w(\alpha)$$

and its adjoint is

$$\nabla'\alpha' = \prod_w < \widetilde{id_1, id_2} >_w \circ h_{w*}\alpha'$$

where $\nabla' = (\widetilde{id_1 + id_2})$ denotes the canonical extension. Then the generalized Hopf invariants $H_n : [\Sigma A, \Sigma X] \to [\Sigma A, \Sigma(X^{\wedge n})]$ are defined. Especially we need the evalution of H_2.

Our $X = HP^2, HP^3$ are not co-H spaces but for our requirement, it is sufficient to investigate a few invariants $H_2()$. On our cases $\pi_8(\Omega\Sigma HP^2)$ and $\pi_{12}(\Omega\Sigma HP^3)$, we have $< \widetilde{id_1, id_2} >_w =< id_1, id_2 >_w$ because

$$\pi_8(HP^2 \wedge HP^2) \simeq \pi_8(\Omega\Sigma(HP^2 \wedge HP^2)),$$
$$\pi_{12}(HP^3 \wedge HP^3) \subset \pi_{12}(\Omega\Sigma(HP^3 \wedge HP^3))$$

and

$$\pi_{12}(HP^3 \wedge HP^3 \wedge HP^3) \simeq \pi_{12}(\Omega\Sigma(HP^3 \wedge HP^3 \wedge HP^3))$$

for w with weight 2 or 3. First we consider

$$\nabla\xi = i_1 \circ \xi + i_2 \circ \xi + [id_1, id_2] \circ \{a(\xi)\iota_9\} = i_1 \circ \xi + i_2 \circ \xi + a(\xi)[\iota_5^{(1)}, \iota_5^{(2)}].$$

Let $\xi' \in \pi_8(\Omega\Sigma HP^2)$ be the adjoint of ξ. Note that $\Omega\Sigma HP^2 = S^4 \cup_\nu e^8 \cup_{[\iota,\iota]} e^8 \cup e^{12} \cup \cdots$. From the following sequence

$$\text{(2.10)} \quad \begin{array}{ccccccc} \pi_8(S^4) & \to & \pi_8(\Omega\Sigma HP^2) & \overset{j_*^{(1}}{\to} & \pi_8(\Omega\Sigma HP^2, S^4) & \to & \pi_7(S^4) \\ \| & & \| & & \| & & \| \\ \Sigma\pi_7(S^3) & & Z\{\xi'\} & & Z\{e_\nu^8, e_{[\iota,\iota]}^8\} & & Z\{\nu\} + Z_{12}\{\Sigma\omega\} \end{array}$$

;(1 $j_*\xi' = 24e_\nu^8 - 12e^8[\iota, \iota]$

we obtain

$$
(2.11)\quad
\begin{array}{ccccccc}
S^8 & \overset{\xi'}{\to} & \Omega\Sigma P & \overset{H_2}{\to} & \Omega\Sigma(P\wedge P) & \overset{\widetilde{<id_1,id_2>}}{\to} & \Omega\Sigma(P\vee P)\\
\| & & \cup & & \cup & & \uparrow < id_1, id_2 >\\
S^8 & \to & J_2(P) & \overset{/P}{\to} & J_2(P)/P & = & P\wedge P\\
\downarrow \delta & & \downarrow /S^4 & & \uparrow 0\vee id & & \\
S^8\vee S^8 & \subset & J_2(P)/S^4 & = & S^8\vee(P\wedge P) & & \uparrow 0\vee id\\
\| & & & & & & \\
S^8\vee S^8 & & & = & & & S^8\vee S^8
\end{array}
$$

where P denotes HP^2, $\delta = 24\iota_8 \vee -12\iota_8$ in the diagram and $P\wedge P$ has a CW-decomposition $S^8 \cup_\nu e^{12} \cup_\nu e^{12} \cup e^{16}$.

Hence we obtain

$$
\begin{aligned}
a(\xi)[\iota_5^{(1)}, \iota_5^{(2)}] &= \text{the adjoint of } \widetilde{< id_1, id_2 >}_* H_{2*}\xi' \\
&= \text{the adjoint of } \widetilde{< id_1, id_2 >}_*(-12\iota_8) = -12[\iota_5^{(1)}, \iota_5^{(2)}].
\end{aligned}
$$

Theorem 2.12. $\nabla\xi = i_1\circ\xi + i_1\circ\xi - 12[\iota_5^{(1)}, \iota_5^{(2)}]$.

We have stated previously that Theorem 2.2 follows from this equation and $\Phi_*[\iota_5^{(1)}, \iota_5^{(2)}] = \nu\circ\eta = 0$ in $\pi_9(\Sigma HP^2)$.
The reduced diagonal map

$$
\bar{\Delta} : HP^3 \to HP^3\wedge HP^3 = S^8 \vee (S^8\cup_\nu e^{12}\cup_\nu e^{12}\cup e^{16})\cup\cdots
$$

is decomposed as follows:

$$
(2.13)\quad
\begin{array}{ccccc}
HP^3 & \overset{/S^4}{\to} & HP^3/S^4 & \to & HP^3\wedge HP^3\\
 & & \| & & \cup\\
 & & S^8\cup_{2\nu} e^{12} & \overset{\hat{\nabla}}{\to} & S^8\cup_\nu e^{12}\cup_\nu e^{12}
\end{array}
$$

The map $\hat{\nabla} : S^8\cup_{2\nu} e^{12} \to S^8\cup_\nu e^{12}\cup_\nu e^{12}$ is the one gotten by the construction of the following cofiber sequences:

$$
(2.14)\quad
\begin{array}{ccccccc}
S^{11} & \overset{2\nu}{\to} & S^8 & \subset & S^8\cup_{2\nu} e^{12} & \overset{/S^8}{\to} & S^{12}\\
\downarrow\nabla & & \| & & \downarrow\hat{\nabla} & & \downarrow\nabla\\
S^{11}\vee S^{11} & \overset{\nu\vee\nu}{\to} & S^8 & \subset & S^8\cup_\nu e^{12}\cup_\nu e^{12} & \to & S^{12}\vee S^{12}
\end{array}
$$

Let ξ_2 be the generator of $\pi_8(S^4\cup_{2\nu} e^8)$ such that $j_*\xi_2 = 12e^8$ where $j* : S^4\cup_{2\nu} e^8 \to (S^4\cup_{2\nu} e^8, S^4)$ and for abbreviation, let ξ_2 denote its suspended generator of $\pi_{8+t}(\Sigma^t(S^4\cup_{2\nu} e^8))$, too. Let $S_0^{11} = e_+^{12}\cap e_-^{12}$ be the equator of

$S^{12} = e_+^{12} \cup e_-^{12}$ and then $S^8 \cup_\nu e^{12} \cup_\nu e^{12}$ is considered as the partial mapping cylinder of $\nu : S_0^{11} \to S^8$. Let $\hat{\iota}$ be the map $S^{12} = e_+^{12} \cup e_-^{12} \subset S^8 \cup_{S_0} S^{12}$.

Lemma 2.15 $\pi_{12}(S^8 \cup_\nu e^{12} \cup_\nu e^{12}) = Z\{\hat{\iota}, \hat{\nabla} \circ \xi_2\}$.

Proof. Consider the exact sequence of the pair $(S^8 \cup_\nu e^{12} \cup_\nu e^{12}, S^8)$:

$$\begin{array}{ccc}
\pi_{12}(S^8) & = & 0 \\
\downarrow & & \\
\pi_{12}(S^8 \cup_\nu e^{12} \cup_\nu e^{12}) & & \\
\downarrow j_* & & \\
\pi_{12}(S^8 \cup_\nu e^{12} \cup_\nu e^{12}, S^8) & = & Z\{e_+^{12}, e_-^{12}\} \\
\downarrow & & \\
\pi_{11}(S^8) & = & Z_{24}\{\nu\} \\
\downarrow & & \\
\pi_{11}(S^8 \cup_\nu e^{12} \cup_\nu e^{12}) & = & 0
\end{array}$$

Then we have $\partial^{-1}(0) = Z\{e_+^{12} + e_-^{12}, 12(e_+^{12} - e_-^{12})\}$. From the definitions of $\hat{\iota}$ and $\hat{\nabla}\xi_2$, we have

$$j_*\hat{\iota} = e_+^{12} + e_-^{12}, j_*(\hat{\nabla}\xi_2) = 12(e_+^{12} - e_-^{12}).$$

Considering the adjoints of this lemma, we obtain the following diagram:

$$\begin{array}{ccccc}
0 & & Z\{\Sigma\hat{\iota}, \Sigma(\hat{\nabla} \circ \xi_2)\} & & Z\{e_+^{13}, e_-^{13}\} \\
\| & & \| & & \| \\
\pi_{13}(S^9) & \to & \pi_{13}(S^9 \cup_\nu e^{13} \cup_\nu e^{13}) & \xrightarrow{j_*} & \pi_{13}(S^9 \cup_\nu e^{13} \cup_\nu e^{13}, S^9) \\
\downarrow [\iota_5^1, \iota_5^2]_* & & \downarrow [id_1, id_2]_* & & \downarrow [id_1, id_2]_* \\
\pi_{13}(S^5 \vee S^5)^{(1} & \to & \pi_{13}(\Omega\Sigma(P \vee P)) & \xrightarrow{j_*} & \pi_{13}(\Omega\Sigma(P \vee P, S^5 \vee S^5)
\end{array}$$

$$\begin{array}{ccccc}
 & & Z_{24}\{\nu\} & & 0 \\
 & & \| & & \| \\
\to & & \pi_{12}(S^9) & \to & \pi_{12}(S^9 \cup_\nu e^{13} \cup_\nu e^{13}) \\
 & & \downarrow [\iota_5^1, \iota_5^2]_* & & \downarrow \\
\to & & \pi_{13}(\Omega\Sigma(S^5 \vee S^5))^{(2} & \to & \pi_{13}(\Omega\Sigma(P \vee P)) \\
 & & & & \| \\
 & & & & \pi_{12}(\Sigma P)^2
\end{array}$$

; $^{(1}\pi_{13}(S^5 \vee S^5) = \pi_{13}(S^5)^2 + Z\{[[\iota_5^{(1)}, \iota_5^{(2)}], \iota_5^{(1)}], [[\iota_5^{(1)}, \iota_5^{(2)}], \iota_5^{(2)}]\}$
$^{(2}\pi_{12}(S^5)^2 + Z_{24}\{[\iota_5^{(1)}, \iota_5^{(2)}] \circ \nu\}$

(2.16)

where P denotes HP^2 in the diagram and we have

$$\partial[e^{9(1)}, \iota_5^{(2)}] = [\nu^{(1)}, \iota_5^{(2)}] = [\iota_5^{(1)}, \iota_5^{(2)}] \circ \nu = [\iota_5^{(1)}, \nu^{(2)}] = -\partial[e^{9(2)}, \iota_5^{(1)}].$$

It follows that

$$\pi_{13}(\Sigma(HP^2 \vee HP^2, S^5 \vee S^5) =$$
$$i_1 \circ \pi_{13}(\Sigma HP^2, S^5) + i_2 \circ \pi_{13}(\Sigma HP^2, S^5) + Z\{[e^{9(1)}, \iota_5^{(2)}], [e^{9(2)}, \iota_5^{(1)}]\}.$$

Let λ and μ be the adjoints of $< \widetilde{id_1, id_2} >_* \hat{\iota}$ and $< \widetilde{id_1, id_2} >_* \hat{\nabla} \circ \xi_2$ in $\pi_{12}(\Omega\Sigma(HP^2 \vee HP^2))$ respectively. Then we have the following relations

$$\text{(2.7)}\quad \begin{array}{ll} \lambda = [id_1, id_2]_* \Sigma\hat{\iota} & \mu = [id_1, id_2]_* \Sigma(\hat{\nabla} \circ \xi_2) \\ j_*\lambda = [e^{9(1)}, \iota_5^{(2)}] + [e^{9(2)}, \iota_5^{(1)}] & j_*\mu = 12([e^{9(1)}, \iota_5^{(2)}] - [e^{9(2)}, \iota_5^{(1)}]) \\ [id_1, id_2]_* e_+^{13} = [e^{9(1)}, \iota_5^{(2)}] & [id_1, id_2]_* e_-^{13} = [e^{9(2)}, \iota_5^{(1)}] \\ \Phi_*\lambda = 2\kappa & \Phi_*\mu = 0 \end{array}$$

It follows that

$$\pi_{13}(\Sigma(HP^2 \vee HP^2)) = i_1 \circ \pi_{13}(\Sigma HP^2) + i_2 \circ \pi_{13}(\Sigma HP^2)$$
$$+ Z\{\lambda, \mu\} + Z\{[[\iota_5^{(1)}, \iota_5^{(2)}], [\iota_5^{(1)}], [[[\iota_5^{(1)}, \iota_5^{(2)}], \iota_5^{(2)}].$$

Similarly we have

$$\pi_{13}(\Sigma(HP^3 \vee HP^3)) = i_1 \circ \pi_{13}(\Sigma HP^3) + i_2 \circ \pi_{13}(\Sigma HP^3)$$
$$+ Z\{\lambda, \mu\} + Z\{[[\iota_5^{(1)}, \iota_5^{(2)}], [\iota_5^{(1)}], [[\iota_5^{(1)}, \iota_5^{(2)}], \iota_5^{(2)}]\}.$$
$$\pi_{13}(\Sigma(HP^3 \vee HP^3), S^5 \vee S^5) = i_1 \circ \pi_{13}(\Sigma HP^3, S^5) + i_2 \circ \pi_{13}(\Sigma HP^3, S^5)$$
$$+ Z\{[e^{9(1)}, \iota_5^{(2)}], [e^{9(2)}, \iota_5^{(1)}]\}.$$

Suppose X is a connnected finite complex. Let $id_j : X \to \Omega\Sigma(X \vee X)$ be the canonical inclusion into j-th factor and also its adjoint by the same sign. Then the commutator for $f, g \in [\Sigma X, \Sigma X]$ is given by

$$-f - g + f + g = \Phi_*[f \circ id_1, g \circ id_2] \circ \Sigma\bar{\Delta} :$$
$$\Sigma X \to \Sigma(X \wedge X) \to \Sigma(X \vee X) \to \Sigma X$$

where $[f \circ id_1, g \circ id_2] : \Sigma(X \wedge X) \to \Sigma(X \vee X)$ denotes the Whitehead product of $f \circ id_1$ and $g \circ id_2$ and $\Phi : \Sigma(X \vee X) \to \Sigma X$ denotes the folding map. In our case $X = HP^3$, we have the following commutative diagram:

$$
\begin{array}{lccccccc}
 & & & \Sigma P & \overset{/S^5}{\to} & S^9 \cup_{2\nu} e^{13} \overset{/S^9}{\to} S^{13} & & \\
 & & & \downarrow \Sigma\bar{\Delta} & & \downarrow \Sigma\,\hat{\nabla} & & \\
 & & & \Sigma(P \wedge P) & \supset & K & & \\
 & & & \downarrow [id_1, id_2] & & & & \\
 & \Sigma(P \vee P) & = & \Sigma(P \vee P) & & \downarrow /S^9 & & \| \\
 & id \vee /\Sigma HP^2 \downarrow & & \downarrow \Sigma(id \vee /S^4) & & & & \\
(2.18) & \Sigma(P \vee S^{13}) & \overset{id\vee/S^9}{\leftarrow} & \Sigma P \vee (P/S^5) & \overset{0\vee-[\iota_5^{(1)},\iota_9]}{\leftarrow} & S^{13} \vee S^{13} & & \\
 & id \vee \kappa \downarrow & & & & & & \\
 & id \vee \lambda \downarrow & & \downarrow id \vee \tilde{\xi} & & \| & & \\
 & \Sigma(P \vee P) & & \Sigma(P \vee P) & \overset{0\vee-[\iota_5^{(1)},\xi^{(2)}]}{\leftarrow} & S^{13} \vee S^{13} & \overset{\nabla}{\leftarrow} & S^{13} \\
 & \Phi \downarrow & & \downarrow \Phi & & \| & & \\
 & \Sigma P & & \Sigma P & \overset{[\xi,\iota_5]}{\leftarrow} & S^{13} \vee S^{13} & &
\end{array}
$$

where P denotes HP^3 and $K = S^9 \cup_\nu e^{13} \cup_\nu e^{13}$.

From this diagram and $[\xi, \iota_5] = 24\kappa$ (see Theorem 7.1), we obtain

Lemma 2.19. *The commutators in* $[\Sigma HP^3, \Sigma HP^3]$ *are*

$$< id, K >=< id, P >= 0 \text{ and } < id, \Xi >= 24K.$$

Theorem 2.20 $[\Sigma HP^3, \Sigma HP^3]$ *is non-abelian. We have a non trivial extension:*

$$0 \to Z\{K, P, \Xi\} \to [\Sigma HP^3, \Sigma HP^3] \to Z\{id\} \to 0.$$

The non trivial commutator is $< id, \Xi >= 24K$. *The center of* $[\Sigma HP^3, \Sigma HP^3] = Z\{K, P\}$.

Proof. The extension diagram follows from (2.7). The commutativities of K, P and Ξ is easily verified because these are contained in the image of the abelian group $[\Sigma HP^3/S^5, \Sigma HP^3]$. By Lemma 2.19, it follows that the non trivial commutator is $< id, \Xi >= 24K$.

3. Proof of Lemma 2.5. Let F be the homotopy fibre of the collapsing map $/S^4 : HP^2 \to S^8$. Then it is easily verified that the 10-skelton of F is S^4. Let i be the inclusion $S^4 \subset F$. The exact sequence of the fibration $F \to HP^2 \to S^8$ must be as follows:

$$
\begin{array}{ccc}
\pi_{13}(S^8) = 0 & \overset{c_*}{\leftarrow} & \pi_{13}(HP^2, S^4) = Z_2\{[e^8, \iota_4] \circ \bar{\eta}^2]\} \\
\downarrow \partial & & \downarrow \partial \\
\pi_{12}(F) = Z_2\{\gamma\} + Z_2\{\iota_* \epsilon_4\} & \overset{\iota_*}{\leftarrow} & \pi_{12}(S^4) = Z_2\{\epsilon_4\} \\
\downarrow & & \downarrow i_* \\
\pi_{12}(HP^2) = Z_2\{p \circ \eta\} + Z_2\{i_* \epsilon_4\} & = & \pi_{12}(HP^2) \\
\downarrow (/S^4)_* & & \downarrow j_*^{(1} \\
\pi_{12}(S^8) = 0 & \leftarrow & \pi_{12}(HP^2), S^4) = Z_2\{[e^8, \iota_4] \circ \ bar\eta]\} \\
\downarrow & & \downarrow \\
\pi_{11}(F) = Z\{\beta\} + \iota_* \Sigma \pi_{10}(S^3) & \leftarrow & \pi_{11}(S^4) = Z_{15} \\
\downarrow i_*^{(2} & & \downarrow \\
\pi_{11}(HP^2) = Z\{p\} + i_* \Sigma \pi_{10}(S^3) & = & \pi_{11}(HP^2) \\
\downarrow (/S^4)_*^{(3} & & \downarrow j_*^{(4} \\
\pi_{11}(S^8) = Z_{24}\{\nu\} & \leftarrow & \begin{array}{l} \pi_{11}(HP^2, S^4) = \\ \qquad Z_2\{[e^8, \iota_4]\} + Z_{24}\{e^8 \circ \bar{\nu}\} \end{array} \\
\downarrow \partial^{(5} & & \downarrow \partial^{(6} \\
\pi_{10}(F) = Z_2\{\alpha\} + \iota_* \Sigma \pi_9(S^3) & \overset{\iota_*^{(7}}{\leftarrow} & \pi_{10}(S^4) = Z_{24}\{\nu^2\} + \Sigma \pi_9(S^3) \\
\downarrow & & \downarrow i_* \\
\pi_{10}(HP^2) = i_* \Sigma \pi_9(S^3) = Z_3 & = & \pi_{10}(HP^2) \\
\downarrow & & \downarrow j_* = 0 \\
\pi_{10}(S^8) = Z_2\{\eta^2\} & \leftarrow & \pi_{10}(HP^2, S^4) = Z_2\{e^8 \circ \bar{\nu}^2\}
\end{array}
$$

;$^{(1}$ $j_* p \circ \bar{\eta} = [e^8, \iota_4] \circ \bar{\eta}$, $^{(2}$ $i_* \beta = 12p$, $^{(3}$ $(/S^4)_* p = 2\nu$, $^{(4}$ $j_* p = [e^8, \iota_4] + 2e^8 \circ \bar{\nu}$, $c_*[e^8, \iota_4] = 0$,$c_* e^8 \circ \bar{\nu} = \nu$, $^{(5}$ $\partial \nu = \alpha$, $^{(6}$ $\partial [e^8, \iota_4] = [\nu, \iota_4]$, $\partial e^8 \circ \bar{\nu} = \nu^2$ $^{(7}$ $\iota_* \nu^2 = \alpha$,

Since $(/S^4)_* p = 2\nu$, there exists an element $\alpha \in \pi_{10}(F)$ such that $\pi_{10}(F) = Z_2\{\alpha\} + i_* \Sigma \pi_9(S^3)$ and $\iota_* \nu^2 = \alpha$. On the E^2-term of the homology spectral sequence of the fibration $F \to HP^2 \to S^8$, the first non-trivial differential on $H_8(S^8) \otimes H_4(F)$ just hits $H_{11}(F) \simeq Z$. Let $K = S^4 \cup e^{11}$ be the mapping cone of $2\nu^2$. Then we may regard K as the 11-skelton of F. Let β be the generator of Z-summand of $\pi_{11}(F)$ such that $i_* \beta = 12p$ and let γ be the element of $\pi_{12}(F)$ so that γ corresponds to $p \circ \eta$. Then we obtain the following exact sequence of homotopy groups of the pair (F, S^4):

$$\begin{array}{rcl}
Z_2\{\gamma\}+i_*\Sigma\pi_{11}(S^3) = & \pi_{12}(F) & \overset{i_*^{(1}}{\to} \pi_{12}(HP^2)=Z\{p\}+\iota_*\Sigma\pi_{10}(S^3) \\
 & \downarrow j_* & \\
Z_2\{e^{11}\circ\bar{\eta}\} & =\pi_{12}(F,S^4) & \\
 & \downarrow & \\
\Sigma\pi_{10}(S^3) = & \pi_{11}(S^4) & \qquad\qquad \pi_{11}(S^4) \\
 & \downarrow & \qquad\qquad \downarrow \iota_*^{(3} \\
Z\{\beta\}+i_*\Sigma\pi_{10}(S^3) = & \pi_{11}(F) & \overset{i_*^{(2}}{\to} \pi_{11}(HP^2)=Z\{p\}+\iota_*\Sigma\pi_{10}(S^3) \\
 & \downarrow j_*^{(3} & \\
Z\{e^{11}\} & =\pi_{11}(F,S^4) & \\
 & \downarrow \partial^{(4} & \\
Z_{24}\{\nu^2\}+\Sigma\pi_9(S^3) = & \pi_{10}(S^4) &
\end{array}$$

$^{(1}$ $i_*\gamma=p\circ\eta$, $^{(2}$ $i_*\beta=12p$, $^{(3}$ $j_*\beta=12e^{11}$, $^{(4}$ $\partial e^{11}=2\nu^2$

Consider the exact ladder of homotopy groups of the pairs (K,S^4) and (F,S^4):

$$\begin{array}{ccccccc}
\pi_{12}(S^4)\to & \pi_{12}(K) & \to\pi_{12}(K,S^4) & \to\pi_{11}(S^4) & \to \\
\| & \downarrow & \downarrow & \| & \\
\pi_{12}(S^4)\to & \pi_{12}(F) & \to\pi_{12}(F,S^4) & \to\pi_{11}(S^4) & \to
\end{array}$$

$$\begin{array}{cccccc}
\pi_{11}(K) & \to\pi_{11}(K,S^4)\to & \pi_{10}(S^4) & \to\pi_{10}(K) & \to \\
\downarrow & \downarrow & \| & \downarrow & \\
\pi_{11}(F) & \to\pi_{11}(F,S^4)\to & \pi_{10}(S^4) & \to\pi_{10}(F) & \to.
\end{array}$$

This ladder is equivalent to

$$\begin{array}{cccccc}
Z_2 & \to Z_2+Z_2\to & Z_2 & \to & Z_{15} & \to \\
\| & \downarrow & \downarrow & & \| & \\
Z_2 & \to Z_2+Z_2\to & Z_2 & \to & Z_{15} & \to
\end{array}$$

$$\begin{array}{ccccc}
Z+Z_{15} & \to Z\{e^{11}\} & \to Z_{24}+Z_3 & \to Z_2+Z_3 & \to \\
\downarrow (12,0) & \downarrow & \| & \| & \\
Z+Z_{\{}\beta\} & \to Z\{e^{11}\} & \to Z_{24}+Z_3 & \to Z_2+Z_3 & \to.
\end{array}$$

Thus we may regard K as the 12-skelton of F and so $\pi_{12}(F,K)=0$. Then the 13-skelton of ΣF is of the form $\Sigma K = S^5\vee S^{12}$. Suppose c denotes the collapsing maps; $K\to K/S^4=S^{11}$, $S^5\vee S^{12}\to S^{12}$ in the following diagram.

$$\begin{array}{ccccccc}
Z_2\{\epsilon_4\}+Z_{15} & & & & & & Z_2\{\eta\} \\
\| & & & & & & \| \\
\pi_{12}(S^4) & \to & \pi_{12}(K) & \to & \pi_{12}(K,S^4) & \xrightarrow{c_*} & \pi_{12}(S^{11}) \\
\downarrow\Sigma & & \downarrow\Sigma & & & & \downarrow\Sigma \\
\pi_{13}(S^5) & \to & \pi_{13}(S^5\vee S^{12}) & = & \pi_{13}(S^5)+\pi_{13}(S^{12}) & \xrightarrow{c_*} & \pi_{13}(S^{12}). \\
\| & & & & & & \| \\
0 & & & & & & Z_2\{\eta\}
\end{array}$$

The left suspension and the right one are isomorphic and so is the central suspension. Hence $\Sigma p \circ \eta = \Sigma(p \circ \eta) \neq 0$ and it follows $\Sigma p \circ \eta = i_* \epsilon_5$ for $i : S^5 \subset \Sigma HP^2$. Then we know the following groups are isomorphic to each other.

$$\pi_{12}(HP^2) \leftarrow \pi_{12}(F) \leftarrow \pi_{12}(K) \xrightarrow{\Sigma} \pi_{13}(S^5 \vee S^{12})$$

4. Proof of Lemma 2.4. The Steenrod operations on the cohomology groups of HP^3,$Sq^4(p=2)$ on H^4, $\wp^1(p=3)$ on H^4 and H^8,$\wp^2$ on H^4 and $\wp^1(p=5)$ on H^4 are non trivial. Hence Σp cannot be divisible by 2,3,5. The generator of Z_5-summand of $\pi_{10}(S^3) = Z_{15}$ can be detected by the property that $\wp^1(p=5)$ on H^3 is non trivial. It follows that the Z_5-component of Σp in $\pi_{12}(E(\hat{\nu})) \supset \pi_{12}(S^5) = \Sigma^2\pi_{10}(S^3) + Z_2\{\sigma'''\}$ contains the generator of the Z_5-summand. For the generator of Z_3-summand of $\pi_{10}(S^3) = Z_{15}$, $\wp^1(p=3)$ on $H^3(S^3 \cup e^{11})$ is trivial since the mod 3 Hopf invariant is trivial. Since $(/S^5)_*\Sigma\, p = 2\nu$, it follows that the order of Σp is divisible by 60. Recall that the homotopy groups of $S^5 \cup_\nu e^9$ are $\pi_5 = Z$, $\pi_6 = \pi_7 = Z_2$,$\pi_8 = 0$, $\pi_9 = Z$, $\pi_{10} = \pi_{11} = 0$ and

$$0 \to Z_{30} \to \pi_{12}(S^5 \cup_\nu e^9) \to Z_{12} \to 0$$

is an exact sequence. Hence it is enough for us to investigate the mod 3 and mod 2 extension problems for the group structure of $\pi_{12}(S^5 \cup_\nu e^9)$. Then we obtain the followings:

Lemma 4.1. *$\pi_{12}(S^5 \cup_\nu e^9)$ has Z_8 as the 2-component.*

Lemma 4.2. *$\pi_{12}(S^5 \cup_\nu e^9)$ has Z_9 as the 3-component.*

First for our calculation of the 2-component of $\pi_{12}(S^5 \cup_\nu e^9)$, consider the following the tower of fiber spaces (the connective fiber spaces of $(S^5 \cup_\nu e^9)$):

$$\begin{array}{lccl}
K(Z,8) \to & X_4 & & \\
 & \downarrow & & \\
K(Z_2,6) \to & X_3 & \overset{\xi}{\to} & K(Z,9) \\
 & \downarrow & & \\
K(Z_2,5) \to & X_2 & \overset{\eta^2}{\to} & K(Z_2,7) \\
 & \downarrow & & \\
K(Z,4) \to & X_1 & \overset{\eta}{\to} & K(Z_2,6) \\
 & \downarrow & & \\
 & (S^5 \cup_\nu e^9) & \overset{\iota}{\to} & K(Z,5)
\end{array}$$

$$;\eta^*\iota = a,\ \eta^{2*}\iota = a',\ \xi^*\iota = a''$$

Proof of Lemma 4.1. The mod 2 cohomology algebra of $K(Z_2,n)$is the polynomial algebra over Z_2 with the generators $Sq^I\iota$ where I runs through all admissible sequences of excess less than n and the one of $K(Z,n)$ is the polynomial algebra over Z_2 with the generators $Sq^I\iota$ where I runs through all admissible sequences of excess less than n and of the form $Sq^I \neq Sq^J Sq^1$. We know $H^*(S^5 \cup_\nu e^9; Z_2) = Z_2\{\iota\} + Z_2\{Sq^4\iota\}$. Consider the cohomology spectral sequences of the tower of fiber spaces (figure 1-4), where the following above condensed statesments mean the full statesments below:

$$Sq^2\iota \quad \swarrow \quad a$$

;a hits $Sq^2\iota$ by the differential d_*,i.e., $d_*(a) = Sq^2\iota$

$$\begin{array}{ccccc}
 & \swarrow & Sq^5\iota & \nwarrow & \\
Sq^4Sq^2a' & & & & Sq^2a'' \\
 & \nwarrow & Sq^4Sq^1\iota & \swarrow &
\end{array}$$

; $d_*(Sq^5\iota) = d_*(Sq^4Sq^1\iota) = Sq^4Sq^2a'$, and $d_*(Sq^2a'') = Sq^5\iota + Sq^4Sq^1\iota$,

$$Sq^1a \quad \leftarrow \quad Sq^1\iota$$

;the horizontal arrow means the element $Sq^1\iota$ of base space survives the correspondent Sq^1a of total space.

(figure 1) $X_1 \subset S^5 \cup_\nu e^9 \to K(Z,5)$

$H^*(K(Z,5);Z_2)$		$H^*(X_1;Z_2)$			
ι					
$Sq^2\iota$	$\swarrow$	a			
$Sq^3\iota$	$\swarrow$	$Sq^1 a$			
$Sq^4\iota$		(1			
ι^2	$\swarrow$	Sq^2Sq^1a	(2		
$Sq^4Sq^2\iota$	$\swarrow$	Sq^4a	(3		
$Sq^5Sq^2\iota$	$\swarrow$	$Sq^5a = Sq^4Sq^1a$			
$\iota \cdot Sq^2\iota$	$\swarrow$	$\iota \otimes a$	(4		
$Sq^6Sq^2\iota$	$\swarrow$	$Sq^6a = a^2$	(5		
$\iota \cdot Sq^3\iota$	$\swarrow$	$\iota \otimes Sq^1a$			
$Sq^6Sq^3\iota$	$\swarrow$	Sq^6Sq^1a	(6		
$\iota \cdot Sq^4\iota$	$\swarrow$	b	(7		
$(Sq^2\iota)^2$	$\swarrow$	$Sq^2\iota \otimes a$	(8		
$Sq^7Sq^3\iota$	$\swarrow$	$(Sq^1a)^2$	(9		
$Sq^2\iota \cdot Sq^3\iota$	$\swarrow$	$Sq^3\iota \otimes a$		$\nwarrow$	$a \cdot Sq^1a$
	$\nwarrow$	$Sq^2\iota \otimes Sq^1a$		$\swarrow$	
ι^3	$\swarrow$	$\iota \otimes Sq^2Sq^1a\iota$	(10		
$Sq^2\iota \cdot Sq^4\iota$	$\swarrow$	$Sq^4\iota \otimes a$	(11		
$(sq^3\iota)^2$	$\swarrow$	$Sq^3\iota \otimes Sq^1a\iota$	(12		

;$^{(1}Sq^2 = 0$ by Adem relation $Sq^2Sq^2 = Sq^3Sq^1$, $^{(2}Sq^2Sq^3\iota = (Sq^5 + Sq^4Sq^1)\iota = \iota^2$, $^{(3}Sq^3Sq^1a = Sq^2Sq^2a = 0$, $^{(4}Sq^4Sq^2a = 0$, $^{(5}Sq^5Sq^1a = Sq^2Sq^3Sq^1a = 0$, $^{(6}Sq^5Sq^2a = 0$, $^{(7}Sq^6Sq^2a = 0$, $^{(8}Sq^4Sq^2Sq^1a = 0$, $^{(9}Sq^6Sq^3a = Sq^6Sq^1Sq^2a = 0$, $^{(10}Sq^1b = 0, Sq^5Sq^2Sq^1a = 0$, $^{(11}Sq^2b = 0$ by $Sq^2Sq^4 = Sq^6 + Sq^5Sq^1$, $^{(12}Sq^3b = 0, Sq^6Sq^2Sq^1a = Sq^7Sq^2Sq^1a = 0$

(figure 2) $X_2 \subset X_1 \to K(Z_2,6)$

$H^*(X_1;Z_2)$		$H^*(K(Z_2,6);Z_2)$		$H^*(X_2;Z_2)$
a	$\leftarrow$	ι		
Sq^1a	$\leftarrow$	$Sq^1\iota$		
$0=Sq^2a$		$Sq^2\iota$	$\swarrow$	a'
$0=Sq^3a$		$Sq^3\iota$	$\swarrow$	Sq^1a'
Sq^2Sq^1a	$\leftarrow$	$Sq^2Sq^1\iota$		
Sq^4a	$\leftarrow$	$Sq^4\iota$		
$0=Sq^3sq^1a$		$Sq^3Sq^1\iota=Sq^2Sq^2\iota$	$\swarrow$	Sq^2a'
$Sq^5a=Sq^4Sq^1a$	$\swarrow$	$Sq^5\iota$	$\nwarrow$	Sq^2Sq^1a' :(1
	$\nwarrow$	$Sq^4Sq^1\iota$	$\swarrow$	
$Sq^6a=a^2$	$\leftarrow$	ι^2		$Sq^3a'=0$
$0=Sq^5Sq^1a$		$Sq^5Sq^1\iota$	$\swarrow$	$Sq^2Sq^2a'=Sq^3Sq^1a'$
$0=Sq^4Sq^2a$		$0=Sq^4Sq^2\iota$	$\swarrow$	Sq^4a'
$0=Sq^6Sq^1a$	$\leftarrow$	$Sq^6Sq^1\iota$		
$0=Sq^5Sq^2a$		$Sq^5Sq^2\iota$	$\swarrow$	$Sq^5a'=Sq^4Sq^1a'$:(2
$0=Sq^4Sq^2Sq^1a$		$Sq^4Sq^2Sq^1\iota$	$\swarrow$	c
$a\cdot Sq^1a$	$\leftarrow$	$\iota\cdot Sq^1\iota$		
b				b
$(Sq^1a)^2$	$\leftarrow$	$(Sq^1\iota)^2$		
$0=Sq^6Sq^2a$		$Sq^6Sq^2\iota$	$\swarrow$	Sq^6a'
$0=Sq^5Sq^2Sq^1a$		$0=Sq^4Sq^2Sq^1\iota$	$\swarrow$	$Sq^4Sq^2a'=Sq^1c$:(3
		$\iota\cdot Sq^2\iota$	$\swarrow$	$\iota\otimes a'$
		$\iota\cdot Sq^3\iota$	$\swarrow$	$\iota\otimes Sq^1a'$
		$Sq^1\iota\cdot Sq^2\iota$	$\swarrow$	$Sq^1\iota\otimes a'$
$0=Sq^7Sq^2a$		$Sq^7Sq^2\iota$	$\swarrow$	a'^2
$0=Sq^6Sq^3a$		$Sq^6Sq^3\iota$	$\swarrow$	Sq^6Sq^1a'
$0=Sq^6Sq^3Sq^1a$		$Sq^6Sq^2Sq^1\iota$	$\swarrow$	$Sq^4Sq^2Sq^1a'+a'^2$:(4

;(1 $Sq^2Sq^3 = Sq^5 + Sq^4Sq^1$, (2 $Sq^5Sq^1a' = 0$, (3 $Sq^4Sq^3 = Sq^5Sq^2$, (4 $Sq^5Sq^2a'=0, Sq^2c=0$

(figure 3) $X_3 \subset X_2 \to K(Z_2, 7)$

$H^*(X_2;Z_2)$		$H^*(K(Z_2,7);Z_2)$		$H^*(X_3;Z_2)$
a'	$\leftarrow$	ι		
Sq^1a'	$\leftarrow$	$Sq^1\iota$		
Sq^2a'	$\leftarrow$	$Sq^2\iota$		
$0 = Sq^3a'$		$Sq^3\iota\iota$	$\swarrow$	a''
Sq^2Sq^1a'	$\leftarrow$	$Sq^2Sq^1\iota$		$Sq^1a'' = 0$
Sq^4a'	$\leftarrow$	$Sq^4\iota$		
Sq^3Sq^1a'	$\leftarrow$	$Sq^3Sq^1\iota$		
$Sq^5a' = Sq^4Sq^1a'$	$\swarrow$	$Sq^5\iota$	$\nwarrow$	Sq^2a'
	$\nwarrow$	$Sq^4Sq^1\iota$	$\swarrow$	
c				c :(1
Sq^6a'	$\leftarrow$	$Sq^6\iota$		
$0 = Sq^5Sq^1a'$		$Sq^5Sq^1\iota$	$\swarrow$	Sq^3a''
$Sq^4Sq^2a' = Sq^1c$	$\leftarrow$	$Sq^4Sq^2\iota$		(2
b				b
a'^2	$\leftarrow$	ι^2		
Sq^6Sq^1a'	$\leftarrow$	$Sq^6Sq^1\iota$		
$0 = Sq^5Sq^2a'$		$Sq^5Sq^2\iota$	$\swarrow$	Sq^4a'' :(3
$Sq^4Sq^2Sq^1a'$	$\leftarrow$	$Sq^4Sq^2Sq^1\iota$		(4

;(1 $Sq^1c = 0$, (2 $Sq^2Sq^1a'' = 0$, (3 $Sq^4Sq^3 = Sq^5Sq^2$, (4 $Sq^5a'' = 0, Sq^2c = 0$

(figure 4) $X_4 \subset X_3 \to K(Z, 9)$

$H^*(X_3;Z_2)$		$H^*(K(Z,9);Z_2)$		$H^*(X_4;Z_2)$	
a''	$\leftarrow$	ι			
Sq^2a''	$\leftarrow$	$Sq^2\iota$			
Sq^3a''	$\leftarrow$	$Sq^3\iota$			
c				c	$Sq^1c = 0$
Sq^4a''	$\leftarrow$	$Sq^4\iota$			
b				b	$Sq^1b = 0$
$0 = Sq^5a''$		$Sq^5\iota$	$\swarrow$	d	

Therefore we obtain $\pi_{12}(S^5 \cup_\nu e^9) \otimes Z_2 = Z_2$. To know the 2-order of c, we must investigate the Bockstein spectral sequence. Let d_r be the r-th Bockstein operator and let τ be the transgression. Then we have the Bockstein relations:

Lemma 4.3. $d_rb = 0$ *for any* r, *i.e.,* b *is a* mod 2 *reduction of a class* *in* $H^{13}(X_i;Z)(i=1,2,3,4)$ *with infinite order.*

Proof. Consider the mapping cone of κ, $E(\hat{\nu})=S^5\cup_\nu e^9\cup_\kappa e^{14}$ which is the S^5-bundle over S^9 corresponding to the generator $\hat{\nu}$ of $\pi_8(SO(6))=Z_{24}\{\hat{\nu}\}$ such that $\pi_*\hat{\nu}=\nu$ because $\pi_*:\pi_8(SO(6))=Z_{24}\{\hat{\nu}\}\to\pi_8(S^5)=Z_{24}\{\hat{\nu}\}$ is an isomorphism induced by the projection $\pi:SO(6)\to S^5$. On the fiber sequence $K(Z,13)\to F\to E(\hat{\nu})\to K(Z,14)$ induced by the orientation class $[E]\in H^{14}(E(\hat{\nu});Z)$, the homotopy fibre F of $[E]$ is $S^5\cup_\nu e^9$ up to 13 dimension. The class $b\in H^{13}(X_i;Z_2)$ is the one correspondent to $\kappa\in\pi_{13}(S^5\cup_\nu e^9)$. Thus b is a mod 2 reduction of a class in $H^{13}(X_i;Z)$ with infinite order and so $d_rb=0$ for any r.

The next lemma necessary for our calculations is known as the "Bockstein lemma" (for example, see[9]).

Lemma 4.4 [Bockstein lemma]. *Let* $F\overset{i}{\subset}E\overset{p}{\to}B$ *be a fiber space. Let the class* $u\in H^n(F;Z_2)$ *be transgressive, and suppose* $d_rv=\tau(u)$ *for some positive integer* r *and for some class* $v\in H^n(B;Z_2)$. *Then* $d_{r+1}p^*v$ *is defined, and moreover* $i^*d_{r+1}p^*v=d_1u$.

Corollary 4.5. *(1) For* $K(Z_2,6)\subset X_3\to X_2$, $d_2c=Sq^4a''$ *in* $H^{13}(X_3;Z_2)$.
(2) For $K(Z,8)\subset X_4\to X_3$, *there exists* $d\in H^n(X_4;Z_2)$ *such that* $d_3c=d$ *and* $\tau(d)=Sq^5\iota$ *for the fiber space* $X_4\subset X_3\to K(Z,9)$.

Proof. (1) $d_1c=Sq^1c=Sq^4Sq^2a'=\tau(Sq^4Sq^2\iota)$. Hence $i^*d_2c=Sq^5Sq^2\iota$ and so $d_2c=Sq^4a''$.
(2) $d_2c=Sq^4a''=\tau(Sq^4\iota)$. Hence $i^*d_3c=Sq^5\iota$. Since $Sq^5a''=0$, there exists $d\in H^n(X_4;Z_2)$ such that $d_3c=d$ and $\tau(d)=Sq^5\iota$ for the fiber space $X_4\subset X_3\to K(Z,9)$.

Theorem 4.6. $H^{12}(X_4;Z_2)=Z_2\{c\}$ *and hence* $\pi_{12}(S^5\cup_\nu e^9)\cong\pi_{12}(X_4)\cong Z_{360}$.

From the structure of $\pi_{12}(S^5\cup_\nu e^9)=\pi_{12}(X_4)$ as a group, it is compatible that c is the mod 2 reduction class of a Z_8-class in $H^{12}(X_4;Z_8)$.

Remark 4.7. *In addition, construct the connective fiber space* $X_5\subset X_4\to K(Z_8,12)$ *killing* c *by the fundamental class* $\iota\in H^{12}(K(Z_8,12);Z_2)$.

(figure 5)

$H^*(X_4;Z_2)$		$H^*(K(Z_8);Z_2)$		$H^*(X_5;Z_2)$
c	$\leftarrow$	ι		
$d = d_3c$	$\leftarrow$	$d_3\iota$	$\swarrow$	e(correspondent to ϵ_5)
b				b(correspondent to κ)
$0 = Sq^1c$				
$0 = Sq^1b$		$Sq^3\iota$	$\swarrow$	$0 = Sq^1e$

Then we obtain the 2-component of $\pi_{13}(S^5 \cup_\nu e^9) = \pi_{13}(X_5)$ *is equal to* $Z_2\{\epsilon_5\} + Z\{\kappa\}$.

Similarly for our calculation of the 3-component of $\pi_{12}(S^5 \cup_\nu e^9) = \pi_{12}(E(\hat{\nu}))$, consider the following the tower of fiber spaces (the connective fiber spaces of $S^5 \cup_\nu e^9$) :

$$\begin{array}{cccc} K(Z,8) \rightarrow & X_2 & & \\ & p\downarrow & & \\ K(Z,4) \rightarrow & X_1 & \xrightarrow{\xi} & K(Z,9) \\ & \downarrow & & \\ & S^5 \cup_\nu e^9 & \xrightarrow{\iota} & K(Z,5) \end{array}$$

$$1^*\iota = a, \xi^*\iota = b,$$

Proof of Lemma 4.2. The mod 3 cohomology algebra of $K(Z,n)$ is the free commutative algebra over Z_3 with the generators $\{\wp^I\iota\}$ where I runs through all admissible sequences of excess less than n and of the form $\wp^I \neq \wp^J\beta$ (β denotes the Bockstein operation). We know $H^*(S^5 \cup_\nu e^9; Z_3) = Z_3\{\iota\} + Z_3\{\wp^1\iota\}$.
Consider the cohomology spectral sequences of the tower of fiber spaces

(figure 6) $X_1 \subset S^5 \cup_\nu^9 \rightarrow K(Z,5)$

$H^*(K(Z,5);Z_3)$		$H^*(X_1;Z_3)$	
ι			
$\wp^1\iota$			
$\beta\wp^1\iota$	$\swarrow$	a	$\beta a = 0$
$\wp^2\iota$	$\swarrow$	b	
$\iota\cdot\wp^1\iota$	$\swarrow$	c	$\beta c = 0$
$\beta\wp^2\iota$	$\swarrow$	$\wp^1 a = \beta b$	$\wp^1\beta\wp^1 = \beta\wp^2 + \wp^2\beta$
$\iota\cdot\beta\wp^1\iota$	$\swarrow$	$\iota\otimes a$	

(figure 7) $X_2 \subset X_1 \to K(Z,9)$

$H^*(X_1;Z_3)$		$H^*((K(Z,9);Z_3)$		$H^*(X_2;Z_3)$	
a	$\leftarrow$	ι			
$0=\beta a$					
b				b	$\beta b = 0$
c				b	$\beta c = 0$
b				b	$\beta b = 0$
$\wp^1 a = \beta b$	$\leftarrow$	$\wp^1 \iota$			
$0 = \beta\wp^1 a$		$\beta\wp^1\iota$	$\swarrow$	$d = d_2 b$	

It follows $H^{12}(X_2;Z_3) = Z_3\{b\}$. Apply (4.3) (use the prime number 3 instead of 2) for the fibration $K(Z,8) \subset X_2 \to X_1$. Since $d_1 b = \beta b \wp^1 a$ in $H^{13}(X_1;Z_3)$, we obtain the relation $\imath^* d_2 b = \beta\wp^1\iota$ and so $d_2 b = d$ in $H^{13}(X_2;Z_3)$. Therefore b is the mod 3 reduction class of a class in $H^{12}(X_2;Z_9)$ with order 9. Hence $\pi_{12}(S^5 \cup_\nu e^9) \otimes Z_3 = \pi_{12}(X_2) \otimes Z_3 \simeq H^{12}(X_2;Z_3) = Z_3$ and so it follows that $\pi_{12}(S^5 \cup_\nu e^9)$ has Z_9 as the 3-component.

Remark 4.8. *We can also know the class $c \in H^{13}(X_2;Z_3)$ is the one correspondent to κ from the similar argument as (4.3).*

5. Left distributive law and composition law. For abbreviation, let P^n denote the quorternionic projective space HP^n in this section. First we shall investigate the composition structure in $[\Sigma P^2, \Sigma P^2] = Z\{id, \xi \circ (/S^5)\}$. We have

Theorem 5.1. $(\xi \circ (/S^5)) \circ (\xi \circ (/S^5)) = 24\xi \circ (/S^5)$.

Proof. $(\xi \circ (/S^5) \circ (\xi \circ (/S^5) = \xi \circ (/S^5) \circ \xi \circ (/S^5) = \xi \circ (24\iota_9) \circ (/S^5) = 24\xi \circ (/S^5)$.

Recall the diagram (2.7), the following diagram

$$\begin{array}{ccc}
 & & \pi_{10}(\Sigma P^3) = 0 \\
\downarrow & & \downarrow 2\nu^* \\
Z_2\{\eta\} = \pi_6(\Sigma P^3) & & \pi_{13}(\Sigma P^3) = Z\{\rho, \kappa\} \\
\downarrow & & \downarrow \\
[\Sigma P^3/S^5, \Sigma P^3] & = [S^9 \cup_{2\nu} e^{13}, \Sigma P^3] = & Z\{(/S^9)^*\kappa, (/S^9)^*\rho, \tilde{\xi}_5\} \\
\downarrow (/S^5)^* & & \downarrow \\
Z\{id\} \subset [\Sigma P^3, \Sigma P^3] & & \pi_9(\Sigma P^3) = Z\{\xi\} \\
\downarrow |S^5 & & \downarrow 2\nu^* \\
Z\{\iota_5\} = \pi_5(\Sigma P^3) & & \pi_{12}(\Sigma P^3) = 0 \\
\downarrow (/S^5)^* & & \downarrow \\
0 & &
\end{array}$$

and the facts that the extension $\tilde{\xi} \in [S^9 \cup_{2\nu} e^{13}, \Sigma P^3]$ of ξ satisfies the condition that $(/\Sigma P^2) \circ \tilde{\xi}$ is $120(/S^9)$ and ρ satisfies the condition that $(/\Sigma P^2) \circ \rho$ is $360\iota_{13}$. We collect the behaviors of the induced homomorphisms on homology groups of Ξ, P, K; the non-trivial cases are as follows:

$$\Xi_* \text{ on } H_9(\Sigma P^3, Z) = 24id_*, \Xi_* \text{ on } H_{13}(\Sigma P^3, Z) = 120id_*$$
$$\text{and } P_* \text{ on } H_{13}(\Sigma P^3, Z) = 360id_*, \text{especially } K_* \text{is trivial on } \tilde{H}_*(\Sigma P^3, Z)$$

For the group of homologically identity maps of X, $H_X^{-1}(id)$, the group of homotopically identity maps of X,$\pi_X^{-1}(id)$, the group of self homotopy equivalences of X,$Eq(X)$, the homologically trivial maps of X, $H_X^{-1}(0)$ and the homotopically trivial maps of X,$\pi_X^{-1}(0)$ for $X = \Sigma P^2$ and ΣP^3 we obtain the following.

Theorem 5.2. *(1)* $H^{-1}_{\Sigma P^2}(0) = \{0\} = \pi^{-1}_{\Sigma P^2}(0)$, $H^{-1}_{\Sigma P^2}(id) = \{id\} = \pi^{-1}_{\Sigma P^2}(id)$, *and* $Eq(\Sigma P^2) = \{\pm id\}$.
(2) $H^{-1}_{\Sigma P^3}(0) = Z\{K\}$, $\pi^{-1}_{\Sigma P^3}(0) = \{0\}$, $H^{-1}_{\Sigma P^3}(id) = Z\{id + Z\{K\}\}$, $\pi^{-1}_{\Sigma P^3}(id) = \{id\}$ *and* $Eq(\Sigma P^3) = \pm id + Z\{K\}\}$.

Proof. Since $K_*\rho = 360\kappa$ on $\pi_{13}(\Sigma P^3)$, it is easily verified that $\pi^{-1}_{\Sigma P^3}(id)$ and $\pi^{-1}_{\Sigma P^3}(0)$ are singletons.

We have

Theorem 5.3. $\xi \circ \nu = 60\Sigma p$.

Consequently there exists an extension $\tilde{\xi}$ of ξ as follows:

$$\begin{array}{ccccccc} S^{12} & \stackrel{2\nu_*}{\to} & S^9 & \to S^9\cup_{2\nu} e^{13} \to & & S^{13} \\ 120\iota_{12}\downarrow & & \downarrow\xi & \downarrow\tilde{\xi} & & \downarrow 120\iota_{12} \\ S^{12} & \stackrel{\Sigma p}{\to} & S^5\cup_\nu e^9 \to & \Sigma P^3 & \to & S^{13} \end{array}$$

The proof of Theorem 5.3 is given in the section 6.

Next we shall investigate the composition structures of $[\Sigma P^3, \Sigma P^3]$. In our choices of ρ and $\tilde{\xi}$ there exists an ambiguity. We must choose the appropriate generators of $[\Sigma P^3, \Sigma P^3]$ to know the composition structures of $[\Sigma P^3, \Sigma P^3]$. Here we shall give the precise definition of them. The homotopy fibre of $\Sigma P^3 \vee (\Sigma P^3/S^5) \to \Sigma P^3 \times (\Sigma P^3/S^5)$ is S^{13} up to dim 15 and the inclusion $S^{13} \subset \Sigma P^3 \vee (\Sigma P^3/S^5)$ is $[\iota_9, \iota_5]$, and so we define $\phi : [\Sigma P^3/S^5, \Sigma P^3] \to Z$ as follows;

(5.4) $$((id \vee /S^5)\nabla)_*(f) = f + (/S^5)\circ f + \phi(f)[\iota_9, \iota_5]\circ(/S^9)$$
for $f \in [\Sigma P^3/S^5, \Sigma P^3]$.
$$\Sigma P^3/S^5 \xrightarrow{f} \Sigma P^3 \xrightarrow{\nabla} \Sigma P^3 \vee \Sigma P^3 \xrightarrow{id\vee/S^5} \Sigma P^3 \vee (\Sigma P^3/S^5)$$

It is easily verified that f is a homomorphism. And moreover we have

Lemma 5.5. $\phi(\kappa\circ(/S^5)) = 1$.

Proof. It is enough to show that $((id \vee /S^5)\nabla)_*(\kappa) = \kappa + [\iota_9, \iota_5]$ in $\pi_{13}(\Sigma P^3 \vee (\Sigma P^3/S^5))$. Consider the homotopy groups of pairs:

$$\begin{array}{ccccc} & & \pi_{13}(\Sigma P^2) & \to & \pi_{13}(\Sigma P^2, S^5) \\ & & \downarrow (id\vee/S^5)\nabla & & \downarrow (id\vee/S^5)\nabla \\ \pi_{13}(S^5) = Z_2\{\epsilon\} & \to & \pi_{13}(\Sigma P^2\vee S^9) & \stackrel{j_*}{\to} & \pi_{13}(\Sigma P^2\vee S^9, S^5\vee *). \end{array}$$

From the equalities $j_*\kappa = [e^9, \iota_5]$, $((id \vee /S^5)\nabla)_* e^9 = e^9 + j_*\iota_9$, we obtain

$$j_*(id \vee /S^5)\nabla)_*\kappa = ((id \vee /S^5)\nabla)_*\kappa = j_*(\kappa + [e^9, \iota_5]).$$

Hence $((id \vee /S^5)\nabla)_*\kappa = \kappa + [e^9, \iota_5]$ or $k + [e^9, \iota_5] + \epsilon$, however $i_*\epsilon = 0$ where $i_* : \pi_{13}(\Sigma P^2) \to \pi_{13}(\Sigma P^3)$, and so we have $((id\vee /S^5)\nabla)_*\kappa = [e^9, \iota_5]$ in $\pi_{13}(\Sigma P^3 \vee (\Sigma P^3/S^5))$.

$J(X)$ has the filtration $J(X) = \cup_n J_n(X)$ where $J_n(X)$ is the image of n-fold product X^n. It is well known $J_2(S^4) = S^4 \cup_{[\iota,\iota]} e^8$. $J_3(S^4)$ has a CW-decomposition as follows:

$$J_3(S^4) = S^4 \cup_{[\iota,\iota]} e^8 \cup_r e^{12} = S^4 \times S^4 \times S^4/(\text{some relations})$$

where e^8 is attached by $[\iota_4, \iota_4]$ and r denotes the attaching map of e^{12}. Then $j_*r = 3[e^8, \iota_4]$ by the cohomological computations on $S^4 \times S^4 \times S^4 \to J_3(S^4)$ where $j_* : \pi_{11}(S^4 \cup_{[\iota,\iota]} e^8) \to \pi_{11}(S^4 \cup_{[\iota,\iota]} e^8, S^4)$. Let q be the attaching map of 12-cell of

$$P^2 \times S^4/(S^4 \vee S^4 \equiv S^4) = P^2 \cup_{[\iota,\iota]} e^8 \cup_q e^{12}$$

where the identification map is the folding map $\Phi : S^4 \vee S^4 \to S^4$. Let $e^8_{[\iota,\iota]}, e^8_\nu \in \pi_8(P^2 \cup_{[\iota,\iota]} e^8, S^4)$ be the characterisitic elements such that $\partial e^8_{[\iota,\iota]} = [\iota_4, \iota_4]$, $\partial e^8_\nu = \nu$ respectively. Then we have

$$j_*q = [e^8_{[\iota,\iota]}, \iota_4] + [e^8_\nu, \iota_4] + e^8_\nu \circ \bar{\nu}$$

by the cohomological computations on $P^2 \times S^4 \to P^2 \cup_{[\iota,\iota]} e^8 \cup_q e^{12}$ where $j_* : \pi_{11}(P^2 \cup_{[\iota,\iota]} e^8) \to \pi_{11}(P^2 \cup_{[\iota,\iota]} e^8, S^4)$. The 15-skeltons of $J_2(S^4)$, $J_2(P^2)$, $J_2(P^3)$, $J_3(S^4)$, $J_3(P^2)$ and $J_3(P^3)$ have the following CW-decompositions:

$$\begin{array}{ccccccc}
S^4 \cup_{[\iota,\iota]} e^8 = & J_2(S^4) & \subset & J_3(S^4) & = S^4 \cup_{[\iota,\iota]} e^8 \cup_r e^{12} \\
 & \cap & & \cap & \\
K_1 \subset & J_2(P^2) & \subset & J_3(P^2) & \supset K_3 \\
 & \cap & & \cap & \\
K_2 \subset & J_2(P^3) & \subset & J_3(P^3) & \supset K_4
\end{array}$$

$;K_1 = P^2 \cup_{[\iota,\iota]} e^8 \cup_q e^{12} \cup_q e^{12}$ $\quad K_2 = P^3 \cup_{[\iota,\iota]} e^8 \cup_q e^{12} \cup_q e^{12}$
$K_3 = P^2 \cup_{[\iota,\iota]} e^8 \cup_q e^{12} \cup_q e^{12} \cup_r e^{12}$, $K_4 = P^3 \cup_{[\iota,\iota]} e^8 \cup_q e^{12} \cup_q e^{12} \cup_r e^{12}$

(5.6)

Then we have diagrams of the exact sequences of homotopy groups of some pairs in the above CW-decompositions, (5.7),(5.8),(5.9), (5.10),(5.11) and (5.12):

$$(5.7)\quad \begin{array}{ccccc}
\pi_{11}(S^4) & \to & \pi_{11}(P^2) & \to & \pi_{11}(P^2, S^4) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_{11}(S^4 \cup_{[\iota,\iota]} e^8) & \to & \pi_{11}(P^2 \cup_{[\iota,\iota]} e^8) & \to & \pi_{11}(P^2 \cup_{[\iota,\iota]} e^8, S^4) \\
\downarrow & & \downarrow & & \\
\pi_{11}(S^4 \cup_{[\iota,\iota]} e^8, S^4) & \to & \pi_{11}(P^2 \cup_{[\iota,\iota]} e^8, P^2) & & \\
\downarrow & & \downarrow & & \\
\pi_{10}(S^4) & \to & \pi_{10}(P^2) & \to & \pi_{10}(P^2, S^4) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_{10}(S^4 \cup_{[\iota,\iota]} e^8) & \to & \pi_{10}(P^2 \cup_{[\iota,\iota]} e^8) & \to & \pi_{11}(P^2 \cup_{[\iota,\iota]} e^8, S^4)
\end{array}$$

Generators of these groups and their homomorphic images in (5.7) are enumerated as follows:

$$
\begin{array}{ccccc}
Z_{15}\{\alpha\} & \rightarrow & Z\{p\}+Z_{15}\{\alpha\} & \rightarrow & \begin{array}{c} Z\{[e^8_{\nu},\iota_4]\} \\ +Z_{24}\{e^8_{\nu}\circ\bar{\nu}\} \end{array} \\
\downarrow & & \downarrow & & \downarrow \\
\begin{array}{c} Z\{r\}+Z_{15}\{\alpha\} \\ +Z_2\{\sigma'''\} \end{array} & \overset{i_*^{(1}}{\rightarrow} & \begin{array}{c} Z\{p,q\} \\ +Z_{120}\{\hat{\alpha}\} \end{array} & \overset{j_*^{(2}}{\rightarrow} & \begin{array}{c} Z\{[e^8_{[\iota,\iota]},\iota_4],[e^8_{\nu},\iota_4]\} \\ +Z_{24}\{[e^8_{[\iota,\iota]}\circ\bar{\nu},e^8_{\nu}\circ\bar{\nu}\} \end{array} \\
\downarrow j_*^{(3} & & \downarrow & & \\
\begin{array}{c} Z\{[e^8_{[\iota,\iota]},\iota_4]\} \\ +Z_{24}\{e^8_{[\iota,\iota]}\circ\bar{\nu}\} \end{array} & \rightarrow & \begin{array}{c} Z\{[e^8_{[\iota,\iota]},\iota_4]\} \\ +Z_{24}\{e^8_{[\iota,\iota]}\circ\bar{\nu}\} \end{array} & & \\
\downarrow \partial^{(4} & & \downarrow & & \\
\begin{array}{c} Z_{24}\{\nu^2\} \\ +Z_3\{\Sigma\omega\circ\nu\} \end{array} & \rightarrow & Z_3\{\Sigma\omega\circ\nu\} & \rightarrow & Z_2\{[e^8_{[\iota,\iota]}\circ\bar{\eta}^2\} \\
\downarrow i_*^{(5} & & \downarrow & & \downarrow \\
Z_6\{\nu^2\} & \rightarrow & 0 & \rightarrow & Z_2\{e^8_{[\iota,\iota]}\circ\bar{\eta}^2, e^8_{\nu}\circ\bar{\eta}^2\}
\end{array}
$$

(1 $i_*r=\hat{\alpha}+3q-3p, i_*\alpha=8\hat{\alpha}, i_*\sigma'''=60\hat{\alpha}$,
(2 $j_*\hat{\alpha}=3e^8_{\nu}\circ\bar{\nu}$,
(3 $j_*r=3[e^8_{[\iota,\iota]},\iota_4], j_*\sigma'''=12e^8_{[\iota,\iota]}\circ\bar{\nu}$,
(4 $\partial e^8_{[\iota,\iota]}=[[\iota_4,\iota_4],\iota]=\Sigma\omega\circ\nu, \partial e^8_{[\iota,\iota]}\circ\bar{\nu}=2\nu^2-\Sigma\omega\circ\nu$,
(5 $2i_*\nu^2=2\nu^2=\Sigma\omega\circ\nu$

(5.7)'

And let $L_1=HP^2\cup_{[\iota,\iota]}e^8\subset L_2=HP^3\cup_{[\iota,\iota]}e^8$,then we have

$$
\text{(5.8)}\qquad
\begin{array}{ccccc}
\pi_{12}(L_1) & \rightarrow & \pi_{12}(J_2(P^2)) & \rightarrow & \pi_{12}(J_2(P^2),L_1) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_{12}(L_2) & \rightarrow & \pi_{12}(J_2(P^3)) & \rightarrow & \pi_{12}(J_2(P^3),L_2) \\
\downarrow & & \downarrow & & \\
\pi_{12}(L_2,L_1) & \rightarrow & \pi_{12}(J_2(P^3),J_2(P^2)) & & \\
\downarrow & & \downarrow & & \\
\pi_{11}(L_1) & \rightarrow & \pi_{11}(J_2(P^2)) & \rightarrow & \pi_{11}(J_2(P^2),L_1) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_{11}(L_2) & \rightarrow & \pi_{11}(J_2(P^3)) & \rightarrow & \pi_{11}(J_2(P^3),L_2).
\end{array}
$$

Generators of these groups and their homomorphic images in (5.8) are enumerated as follows:

(5.8)'

$$\begin{array}{ccccc}
Z_2\{\epsilon_4, p\circ\eta, q\circ\eta\} & \to & Z_2\{\epsilon_4\}+Z\{\kappa'\} & \to & Z\{e_q^{12(1)}, e_q^{12(2)}\} \\
\downarrow & & \downarrow & & \downarrow \\
Z_2\{\epsilon_4, q\circ\eta\} & \to & Z\{\kappa'\} & \to & Z\{e_q^{12(1)}, e_q^{12(2)}\} \\
\downarrow & & \downarrow & & \\
Z\{e_p^{12}\} & \to & Z\{e_p^{12}\} & & \\
\downarrow & & \downarrow & & \\
Z\{p,q\}+Z_{120}\{\hat\alpha\} & \to & Z\{p\}+Z_{120}\{\hat\alpha\} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \\
Z\{q\}+Z_{120}\{\hat\alpha\} & \to & Z_{120}\{\hat\alpha\} & \to & 0
\end{array}$$

And we have

(5.9)

$$\begin{array}{ccccc}
\pi_{12}(J_2(P^2)) & \to & \pi_{12}(J_3(P^2)) & \to & \pi_{12}(J_3(P^2), J_2(P^2)) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_{12}(J_2(P^3)) & \to & \pi_{12}(J_3(P^3)) & \to & \pi_{12}((J_3(P^3), J_2(P^2)) \\
\downarrow & & \downarrow & & \\
\pi_{12}(J_2(P^3), J_2(P^2)) & \to & \pi_{12}(J_3(P^3), J_2(P^2)) & & \\
\downarrow & & \downarrow & & \\
\pi_{11}(J_2(P^2)) & \to & \pi_{11}(J_3(P^2)) & \to & \pi_{11}(J_3(P^2), J_2(P^2)) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_{11}(J_2(P^3)) & \to & \pi_{11}(J_3(P^3)) & \to & \pi_{11}((J_3(P^3), J_2(P^2))
\end{array}$$

Generators of these groups and their homomorphic images in (5.9) are enumerated as follows:

(5.9)'

$$\begin{array}{ccccc}
Z_2\{\epsilon_4\}+Z\{\kappa'\} & \to & Z\{\kappa'\} & \to & Z\{e_p^{12}\} \\
\downarrow & & \downarrow & & \downarrow \\
Z_2\{\epsilon_4\}+Z\{\kappa'\} & \to & Z\{\kappa', \rho'\} & \to & Z\{e_p^{12}\} \\
\downarrow & & \downarrow j_*^{(1} & & \\
Z\{e_r^{12}\} & \to & Z\{e_r^{12}\} & & \\
\downarrow \partial^{(2} & & \downarrow \partial^{(3} & & \\
Z\{p\}+Z_{120}\{\hat\alpha\} & \xrightarrow{6} & Z_{120}\{\hat\alpha\} & \to & 0 \\
\downarrow i_*^{(4} & & \downarrow & & \downarrow \\
Z_{360}\{\Sigma p'\} & \to & 0 & \to & 0
\end{array}$$

$^{(1}$ $j_*\rho' = 120e_r^{12}$, $^{(2}$ $\partial e_r^{12} = \hat\alpha - 3p$, $^{(3}$ $\partial e_r^{12} = \hat\alpha$, $i_*p = 0$, $^{(4}$ $i_*p = \Sigma p'$,$i_*\hat\alpha = 3\Sigma p'$,

And we have

$$\begin{array}{ccc}
Z_2\{\epsilon_4, p\circ\eta\} = \pi_{12}(P^2) & = & \pi_{12}(P^2) = Z_2\{\epsilon_4, p\circ\eta\} \\
\downarrow i_* & & \downarrow i_*^{(1} \\
Z_2\{\epsilon'_5\} + Z\{\kappa'\} = \pi_{12}(J_2(P^2)) & & \pi_{12}(J_3(P^2)) = Z_2\{\epsilon'_5\} + Z\{\kappa'\} \\
\downarrow & & \downarrow j_* \\
Z\{j_*\kappa'\} = \pi_{12}(J_2(P^2), P^2) & \simeq & \pi_{12}(J_3(P^2), P^2) = Z\{j_*\kappa', x\} \\
\downarrow & & \downarrow \partial^{(2} \\
Z\{p\} + Z_{15}\{\alpha\} = \pi_{11}(P^2) & = & \pi_{11}(P^2) = Z\{p\} + Z_{15}\{\alpha\} \\
\downarrow i_*^{(3} & & \downarrow i_*^{(4} \\
Z\{p\} + Z_{120}\{\hat{\alpha}\} = \pi_{11}(J_2(P^2) & \to & \pi_{11}(J_3(P^2)) = Z_{360}\{\Sigma p'\} \\
\downarrow j_*^{(5} & & \downarrow j_* \\
Z_{24}\{e^8_{[\iota,\iota]} \circ \bar{\nu}\} \circ \bar{\nu}\} = \pi_{12}(J_2(P^2), P^2) \to & & \pi_{12}(J_3(P^2), P^2) = 0
\end{array}$$

$^{(1}$ $i_* p\circ\eta = i_*\epsilon_4 = \epsilon'_5$, $^{(2}$ $\partial x = \alpha - 24p$, $^{(3}$ $i_*\alpha = 8\hat{\alpha}$, $i_* p = 0$, $^{(4}$ $i_*\alpha = 24\Sigma p'$, $^{(5}$ $i_*\hat{\alpha} = 3e^8_{[\iota,\iota]} \circ \bar{\nu}$,

(5.10)

And we have

(5.11)
$$\begin{array}{lcl}
\pi_{12}(J_2(P^2)) & = & Z_2\{\epsilon'_5\} + Z\{\kappa'\} \\
\downarrow & & \\
\pi_{12}(J_3(P^2)) & = & Z_2\{\epsilon'_5\} + Z\{\kappa'\} \\
\downarrow & & \\
\pi_{12}(J_3(P^2), J_2(P^2)) & = & Z\{e^{12}_\tau\} \\
\downarrow \partial^{(1} & & \\
\pi_{11}(J_2(P^2)) & = & Z\{p\} + Z_{120}\{\hat{\alpha}\} \\
\downarrow i_*^{(2} & & \\
\pi_{11}(J_3(P^2)) & = & Z_{360}\{\Sigma p'\}
\end{array}$$

$^{(1}$ $\partial p\circ\eta = i_*\epsilon_4 = \epsilon'_5$, $^{(2}$ $i_*\hat{\alpha} = 3\Sigma p'$,

and moreover,

$$\begin{array}{ccc}
Z_2\{\epsilon_4\} = \pi_{12}(HP^3)) & \to & \pi_{12}(P^3 \cup e^8) = Z_2\{\epsilon_4, q \circ \eta\} \\
\downarrow & & \downarrow \\
Z\{\kappa', \rho'\} & \simeq & \pi_{12}(J_3(P^3)) = Z\{\kappa', \rho'\} \\
\downarrow j_*^{(1} & & \downarrow j_*^{(2} \\
Z\{j_*\kappa', x\} = \pi_{12}(J_3(P^3), P^3) & \stackrel{\hat{j}_*}{\to} & \pi_{12}(J_3(P^3), P^3 \cup e^8) = Z\{e_q^{12(1)}, e_q^{12(2)}, e_r^{12}\} \\
\downarrow \partial^{(3} & & \downarrow \partial^{(3} \\
Z_{15}\{\alpha\} = \pi_{11}(P^3) & \to & \pi_{11}(P^3 \cup e^8) = Z\{q\} + Z_{120}\{\hat{\alpha}\} \\
\downarrow & & \downarrow \\
0 = \pi_{11}(J_3(P^2)) & = & \pi_{11}(J_3(P^2)) = 0
\end{array}$$

$^{(1}$ $j_*\rho' \equiv 15x \ mod \ Z\{j_*\kappa'\}$
$^{(2}$ $j_*\kappa' = e_q^{12(1)} - e_q^{12(2)}$,
$^{(3}$ $\partial x = \alpha$
$^{(4}$ $\partial e_q^{12(j)} = q, \partial e_r^{12} = 3q + \hat{\alpha}$

(5.12)

The choices of $\rho' \in \pi_{12}(J_3(P^3))$ and $x \in \pi_{12}(J_3(P^2), P^2)$ have an ambiguity of mod $Z\{\kappa'\}$ and mod $Z\{j_*\kappa'\}$ respectively. They must satisfy the conditions $j_*\rho' = 120e_r^{12} - (ae_q^{12(1)} + be_q^{12(2)})$, $a + b = 360$ and $j_*x = 8e_r^{12} - (ce_q^{12(1)} + de_q^{12(2)})$, $c + d = 24$. Thus we may choose ρ' and x so that

$$j_*\rho' = 120e_r^{12} - 180(e_q^{12(1)} + e_q^{12(2)})$$

and

$$\hat{j}_*x = 8e_q^{12} - 12(e_q^{12(1)} + e_q^{12(2)}),$$

because of $j_*\kappa' = e_q^{12(1)} - e_q^{12(2)}$. Then we have $j_*\rho' = 15x$. Let ρ be the adjoint of ρ'.

We are in a position to calculate the Hopf invariants of κ and ρ.

Lemma 5.13. *We have the following commutative diagram:*

$$\begin{array}{ccccccc}
S^{12} & \stackrel{\kappa'}{\to} & \Omega\Sigma P^2 & \stackrel{H_2}{\to} & \Omega\Sigma(P^2 \vee P^2) & \stackrel{<id_1 \tilde{,} id_2>}{\to} & \Omega\Sigma(P^2 \vee P^2) \\
\| & & \cup & & \cup & & \uparrow < id_1, id_2 > \\
S^{12} & \to & J_2(P^2) & \to & J_2(P^2)/P^2 & = & P^2 \wedge P^2 \\
\downarrow 0 \vee \hat{\imath} & & \downarrow /S^4 & & & & \uparrow 0 \vee id \\
S^8 \vee K & \subset & J_2(P^2)/S^4 & & = & & S^8 \vee P^2 \vee P^2
\end{array}$$

and $P^2 \wedge P^2$ *has a CW-decomposition* $K \cup e^{16} = S^8 \cup_\nu e^{12} \cup_\nu e^{12} \cup e^{16}$.

Proof. The map $(/P^2) \circ \kappa' = \hat{\iota}$ is followed by the definition of $\hat{\iota}$ and $j_*\kappa' = e_+^{12} + e_-^{12}$.

Lemma 5.14. *We have the following commutative diagram:*

$$\begin{array}{ccccccc} S^{12} & \xrightarrow{\rho'} & \Omega\Sigma P^3 & \xrightarrow{H_2} & \Omega\Sigma(P^3 \wedge P^3) & \xrightarrow{<id_1, id_2>} & \Omega\Sigma(P^3 \vee P^3) \\ \| & & \cup & & \cup & & \uparrow < id_1, id_2 > \\ S^{12} & \to & J_3(P^3) & \to & P^3 \vee P^3 & = & P^3 \vee P^3 \\ \downarrow f & & \downarrow /S^4 & & & & \uparrow g \\ Q \vee K \vee S^{12} & \subset & J_3(P^3)/S^4 & & = & & Q \vee K \vee M, \end{array}$$

where $Q = S^8 \cup_{2\nu} e^{12}$, $K = S^8 \cup_\nu e^{12} \cup_\nu e^{12}, M = S^{12} \cup e^{16} \cup \cdots, P^3 \wedge P^3 = K \cup e^{16} \cup \cdots$ *and* $f = 30\xi_2 \vee -15\hat{\nabla} \circ \xi_2 \vee 120\iota_{12}$, $g = 0 \vee inclusion \vee 0 \cup \cdots$.

Proof. The equality $\hat{j}_* x = 8e_r^{12} - 12(e_q^{12(1)} + e_q^{12(2)})$ in (5.12) shows $(/P^3)_* x = -\hat{\nabla} \circ \xi_2 \vee 8\iota_{12}$ in $\pi_{12}(\Omega\Sigma P^3/P^3)$. It follows the lemma by $j_*\rho' = 15x$.

By the equalities (2.17), we obtain

Theorem 5.15. $\nabla\kappa = i_1 \circ \kappa + i_2 \circ \kappa + \lambda$

$\nabla\rho = i_1 \circ \rho + i_2 \circ \rho - 15\mu \bmod Z\{[[\iota_5^{(1)}, \iota_5^{(2)}], \iota_5^{(1)}], [[\iota_5^{(1)}, \iota_5^{(2)}], \iota_5^{(2)}]\}$.

Corollary 5.16. $\phi((/S^9)^*\rho) = 180$, *i.e.*,

$$((id \vee /S^5)\nabla)_*\rho = \rho + (/S^5) \circ \rho + 180[\iota_9, \iota_5] = \rho + 30\xi_2 + 180[\iota_9, \iota_5].$$

Remark 5.17. *The first equation in Theorem 5.15 also shows that* $\phi(\kappa \circ (/S^5)) = 1$ *because* $(id \vee /S^5)_*\lambda = [\iota_9, \iota_5]$.

Remark 5.18. *Provided we take the suitable choice of the generator* τ *of* $\pi_{13}(\Sigma P^3, S^5) = Z\{\tau, [\iota_9, \iota_5]\}$,*we obtain*

$$\nabla\tau = i_1 \circ \tau + i_2 \circ \tau + 6([\iota_9^{(1)}, \iota_5^{(2)}] - [\iota_9^{(2)}, \iota_5^{(1)}])$$

and this relation is compatible with

$$\nabla\sigma''' = i_1 \circ \sigma''' + i_2 \circ \sigma''' + 12[\iota_9^{(1)}, \iota_5^{(2)}] \circ \nu$$

and $\pi_{12}(S^5) = Z_{30}\{\partial\tau = \alpha + \sigma'''\}$.

We note $\phi(/S^9)^* : \pi_{13}(\Sigma P^3) \to [\Sigma P^3/S^5, \Sigma P^3] \to Z$ is an ontohomomorphism and $180\kappa - \rho$ is an generator of $(\phi(/S^9)^*)^{-1}(0)$. And so we can define $\tilde{\xi}$ so that $\tilde{\xi}$ and $(/S^9)^*(180\kappa - \rho)$ are generators of $\phi^{-1}(0) \simeq Z + Z \subset [\Sigma P^3/S^5, \Sigma P^3]$. Let $P = (/\Sigma P^2)^*\rho, \Xi = (/S^5)^*\tilde{\xi}$. Since κ is homologically trivial, the homological degrees are independent to the choices of ρ and $\tilde{\xi}$.

Lemma 5.19. $(/S^5) \circ \tilde{\xi} = 24id + 8\xi_2 \circ (/S^9)$ *where id denotes the identity map on* $S^9 \cup_{2\nu} e^{13}$.

Proof. The Puppe sequence obtained from the cofiber sequence $S^9 \subset S^9 \cup_{2\nu} e^{13} \to S^{13}$ induces that self maps of $S^9 \cup_{2\nu} e^{13}$ are characterized by the homological degrees of the induced homomorphisms on $H^*(S^9 \cup_{2\nu} e^{13}; Z)$. It follows $(/S^5) \circ \xi_2 = 24id + 8\xi_2 \circ (/S^9)$.

$$\begin{array}{ccccc}
\pi_{10}(S^9 \cup_{2\nu} e^{13}) & \overset{(2\nu)^*}{\to} & \pi_{13}(S^9 \cup_{2\nu} e^{13}) & \overset{(/S^9)^*}{\to} & [S^9 \cup_{2\nu} e^{13}, S^9 \cup_{2\nu} e^{13}] \\
\| & & \| & & \| \\
Z_4 & & Z\{\xi_2\} & & Z\{id, \xi_2 \circ (/S^9)\}
\end{array}$$

$$\begin{array}{cccc}
\to & \pi_9(S^9 \cup_{2\nu} e^{13}) & \overset{(2\nu)^*}{\to} & \pi_{12}(S^9 \cup_{2\nu} e^{13}) \\
& \| & & \| \\
& Z\{\iota_9\} & & Z_2\{\nu\}
\end{array}$$

Lemma 5.20. $\tilde{\xi} \circ \xi_2 = 4\rho$.

Proof. Considering the homological degrees, we have $\tilde{\xi} \circ \xi_2 = 4\rho + u\kappa$ for some integer u. Apply $/S^9$ to this equation and so we obtain $(\tilde{\xi} \circ \xi_2) \circ (/S^9) = 4\rho \circ (/S^9) + u\kappa \circ (/S^9)$ because $/S^9$ is an suspension. On the other hand we have $\phi(f \circ \Sigma g) = \phi(f)\ degH_{13}(\Sigma g)$ for $f \in [\Sigma P^3/S^5, \Sigma P^3]$ and $g \in [P^3/S^4, P^3/S^4]$. Hence $u = \phi((\tilde{\xi} \circ \xi_2) \circ (/S^9)) - 4\phi(\rho \circ (/S^9)) = \phi(\tilde{\xi})\ degH_{13}((\tilde{\xi} \circ (/S^9)) = 0$.

Theorem 5.21. *Some relations of the compositions of the generators of* $[\Sigma P^3, \Sigma P^3]$ *are as follows:*

$$K \circ K = P \circ K = \Xi \circ K = 0, P \circ P = 360P, K \circ P = 360K, \Xi \circ P = 120P,$$
$$\Xi \circ \Xi = 24\Xi + 32P, P \circ \Xi = 120P, K \circ \Xi = 120K.$$

Proof. By the definition (2.9) and $(/S^5) \circ \kappa = 0$, the first three equations are easily verified. $(/\Sigma P^2)_*\rho = 360\iota_{13}$ implies $P \circ P = 360P$ and $K \circ P = 360K$. The equations $(\Sigma P^2)_*\tilde{\xi} = 120(/S^9)$ and Lemma 5.5 imply $P \circ \Xi =$

$120P$ and $K \circ \Xi = 120K$. Recall that the map $\Xi \circ P$ is represented as the following composition:

$$\begin{array}{lccccc} \Xi \circ P : \Sigma P^3 \overset{/P^2}{\to} & S^{13} & \overset{\rho}{\to} & \Sigma P^3 & \overset{/S^5}{\to} S^9 \cup_{2\nu} e^{13} \overset{\tilde{\xi}}{\to} \Sigma P^3 \\ & \| & & \downarrow /\Sigma P^2 & \downarrow /S^9 \\ & S^{13} & \overset{360\iota_{13}}{\to} & S^{13} & = \quad S^{13}. \end{array}$$

The relation $(/S^5)_* \rho = 30\xi_2$ is easily verified. By Remark 5.18, it follows $\Xi \circ P = 120P$. Finally, since the following maps are suspensions except $\tilde{\xi}$, it holds that

$$\begin{aligned} \Xi \circ \Xi &= \tilde{\xi} \circ (/S^5) \circ \tilde{\xi} \circ (/S^5) = \tilde{\xi} \circ \{24id + 8\xi_2 \circ (/S^9)\} \circ (/S^5) \\ &= \tilde{\xi} \circ \{24(/S^5) + 8\xi_2 \circ (/\Sigma P^2)\} = 24\Xi + 32P. \end{aligned}$$

We have the following left distributive law related with the composition and the sum by co-H structure.

Lemma 5.22. *For any $f, g, h \in [\Sigma P^2, \Sigma P^2]$, $(f+g) \circ h = f \circ h + g \circ h$.*

Proof. It is sufficient to prove for $h = \xi \circ (/S^5)$. Recall Theorem 2.12;

$$\nabla \xi = i_1 \circ \xi + i_2 \circ \xi - 12([\iota_5^{(1)}, \iota_5^{(2)}] \text{ and } \Phi_*[\iota_5^{(1)}, \iota_5^{(2)}] = [\iota_5, \iota_5] = \nu \circ \eta = 0,$$

where $\iota_5^{(j)} (j = 1, 2)$ donotes the inclusion $S^5 \subset \Sigma P^2 \vee \Sigma P^2$ to the j-th factor and $\Phi : \Sigma P^2 \vee \Sigma P^2 \to \Sigma P^2$ be the folding map. And so it follows that the out-side square of the following diagram is commutative ($P = P^2$).

$$\begin{array}{ccccccc} \Sigma P & \overset{/S^5}{\to} & S^9 & \overset{\xi}{\to} & \Sigma P & \overset{f+g}{\to} & \Sigma P \\ \downarrow \nabla & & \downarrow \nabla & & \downarrow \nabla & & \| \\ \Sigma P \vee \Sigma P & \overset{/S^5 \vee /S^5}{\to} & S^9 \vee S^9 & \overset{\xi \vee \xi}{\to} & \Sigma P \vee \Sigma P \overset{f \vee g}{\to} \Sigma P \vee \Sigma P & \overset{\Phi}{\to} & \Sigma P \end{array}$$

For $[\Sigma P^3, \Sigma P^3]$, we have

Lemma 5.23. *Let $f \in [\Sigma P^3, \Sigma P^3]$ and $g, h \in [\Sigma P^3/S^5, \Sigma P^3]$. Then we have a left distributive law in $[\Sigma P^3, \Sigma P^3]$ as follows:*

$$\begin{aligned} & f + g \circ (/S^5) \circ (h \circ (/S^5)) \\ &= f \circ h \circ (/S^5) + g \circ (/S^5) \circ h \circ (/S^5) + \phi(h)[g_* \iota_9, f_* \iota_5] \circ (/\Sigma P^2). \end{aligned}$$

Proof. Apply $f \vee g$ and $/S^5$ to the relation

$$(id + /S^5) \circ h = h + (/S^5) \circ h + \phi(h)[\iota_9, \iota_5] \circ (/S^9)$$

and we obtain

$$(f \vee g) \circ (id + /S^5) \circ h \circ (/S^5) = \\ f \circ h \circ (/S^5) + g \circ (/S^5) \circ h \circ (/S^5) + \phi(h)[g_* \iota_9, f_* \iota_5] \circ (/\Sigma P^2).$$

$$\Sigma P^3 \overset{/S^5}{\to} \Sigma P^3/S^5 \overset{h}{\to} \Sigma P^3 \overset{id+/S^5}{\to} \Sigma P^3 \vee (\Sigma P^3/S^5) \overset{f\vee g}{\to} \Sigma P^3.$$

Lemma 5.24. *$(m\ id_{\Sigma P^3}) \circ \kappa = m^2\kappa$, $(m\ id_{\Sigma P^3}) \circ \rho = m\rho$ in $\pi_{13}(\Sigma P^3)$ and $(m\ id_{\Sigma P^3}) \circ \tilde{\xi} = m\tilde{\xi}$ in $[\Sigma P^3/S^5, \Sigma P^3]$.*

Proof. Let $\nabla^m : \Sigma P^n \to \Sigma P^n \vee \Sigma P^n \vee \cdots \vee \Sigma P^n$ be the m-fold pinching map and let $\Phi : \Sigma P^n \vee \Sigma P^n \vee \cdots \vee \Sigma P^n \to \Sigma P^n$ be the folding map. Then our primary ∇ is represented ∇^2 by the new notation. Then we have

$$j_* \nabla \kappa = \nabla j_* \kappa = \nabla [e^9, \iota_5] = [\nabla e^9, \nabla \iota_5] = [e_9^{(1)} + e_9^{(2)}, \iota_5^{(1)} + \iota_5^{(2)}] \\ = j_*(\kappa + \kappa) + [e_9^{(1)}, \iota_5^{(2)}] + [e_9^{(2)}, \iota_5^{(1)}]$$

$$\begin{array}{ccccc} \pi_{13}(\Sigma P^2) & \overset{\nabla}{\to} & \pi_{13}(\Sigma P^2 \vee \Sigma P^2) & \overset{\Phi_*}{\to} & \pi_{13}(\Sigma P^2) \\ \downarrow j_* & & \downarrow j_* & & \downarrow j_* \\ \pi_{13}(\Sigma P^2, S^5) & \overset{\nabla}{\to} & \pi_{13}(\Sigma P^2 \vee \Sigma P^2, S^5 \vee S^5) & \overset{\Phi_*}{\to} & \pi_{13}(\Sigma P^2, S^5) \\ \| & & \| & & \\ Z\{[e^9, \iota_5]\} & & Z\{[e^{9(1)}, \iota_5^{(1)}], [e^{9(2)}, \iota_5^{(1)}], & & \\ & & [e^{9(1)}, \iota_5^{(2)}], [e^{9(2)}, \iota_5^{(2)}]\} & & \end{array}$$

and

$$\Phi_*[e^{9(1)}, \iota_5^{(2)}] = \Phi_*[e^{9(2)}, \iota_5^{(1)}] = [e^9, \iota_5] = j_*\kappa.$$

Hence

$$j_* \Phi_* \nabla_* \kappa = 4[e^9, \iota_5] = 4 j_* \kappa$$

and so we obtain

$$(id_{\Sigma P^3} + id_{\Sigma P^3}) \circ \kappa = \Phi_* \nabla_* \kappa = 4\kappa$$

in $\pi_{13}(\Sigma P^3)$. Also we obtain

$$j_* \Phi_* \nabla_*^m \kappa = m^2 [e^9, \iota_5] = m^2 j_* \kappa$$

and

$$(m\ id_{\Sigma P^3}) \circ \kappa = \Phi_* \nabla_*^m \kappa = m^2 \kappa$$

in $\pi_{13}(\Sigma P^3)$. By Theorem 5.15, we obtain

$$\nabla_*\rho = i_1 \circ \rho + i_2 \circ \rho - 15\mu \bmod Z\{[[\iota_5^{(1)}, \iota_5^{(2)}], \iota_5^{(1)}], [[\iota_5^{(1)}, \iota_5^{(2)}], \iota_5^{(2)}]$$

Since $\Phi_*\mu = \Phi_*[[\iota_5^{(1)}, \iota_5^{(2)}], \iota_5^{(1)}] = \Phi_*[[\iota_5^{(1)}, \iota_5^{(2)}], \iota_5^{(2)}] = 0$, we obtain $(id_{\Sigma P^3} + id_{\Sigma P^3}) \circ \rho = 2\rho$ and similarly $(m\ id_{\Sigma P^3}) \circ \rho = \Phi_*\nabla_*^m \rho = m\rho$ for any integer m. Considering the homological degrees, we have

$$(m\ id_{\Sigma HP^3}) \circ \tilde{\xi} = m\tilde{\xi} + u\kappa \circ (/S^9)$$

for some integer u. Composing the suspension element ξ_2, we have

$$\begin{aligned}((m\ id_{\Sigma HP^3}) \circ \tilde{\xi}) \circ \xi_2 &= (m\ id_{\Sigma HP^3}) \circ (4\rho) = 4(m\ id_{\Sigma HP^3}) \circ \rho = 4m\rho \\ &= (m\tilde{\xi} + u\kappa \circ (/S^9)) \circ \xi_2 = m\tilde{\xi} \circ \xi_2 + u\kappa \circ (/S^9) \circ \xi_2 = 4m\rho + 12u\kappa.\end{aligned}$$

Since κ has the infinite order, we obtain $u = 0$,i.e., $(m\ id_{\Sigma HP^3}) \circ \tilde{\xi} = m\tilde{\xi}$.

Thus summing up Lemmas 5.23 and 5.24, we obtain

Theorem 5.25. *The left distributivity law in* $[\Sigma P^3, \Sigma P^3]$ *is as follows:*

$(f+g) \circ h = f \circ h + g \circ h$ for the case $f, g \in Z\{K, P, \Xi\}$or $h \in Z\{id, \Xi\}$

and exceptinal cases;

$$\begin{gathered}(m\ id_{\Sigma HP^3}) \circ K = m^2 K, (a\ id_{\Sigma HP^3} + b\Xi) \circ K = a(1+24b)K \\ \text{and } (a\ id_{\Sigma HP^3} + b\Xi) \circ P = (a+120b)P + 4320bK.\end{gathered}$$

Let (a, b) denote $a\ id + b\xi \circ (/S^5)$ in $[\Sigma P^2, \Sigma P^2]$ for abbreviation. Then we have

Corollary 5.26. $(a, b) \circ (c, d) = (ac, ad + bc + 24bd)$.

Let (a, b, c, d) denote $a\ id + bK + cP + d\Xi$ in $[\Sigma P^3, \Sigma P^3]$ for abbreviation. Then we have

Corollary 5.27. $(a, b, c, d) \circ (e, f, g, h) = (ae, a^2 f + af(1+24d) + be + 360bg + 4320dg + 120bh, ag + ce + 360cg + g(a+120d) + 120ch + 32dh, ah + de + 24dh)$.

For example, the composition law is simple and plain in the group $H_{\Sigma HP^3}^{-1}(id)$ of homologically identity maps of ΣP^3; $(id + aK) \circ (id + bK) = id + (a+b)K$.

6. Proof of Theorem 5.4. We shall give the order of $\xi \circ \nu$ in this section. We know $(k\xi) \circ \nu = k\xi \circ \nu$ because ν is a suspension. Since

$j_*\xi \circ \nu = 0$ and $\xi \circ \nu$ is contained in the image of $\pi_{12}(S^5) = Z_{30}$ where $j_* : \pi_{12}(\Sigma HP^2) \to \pi_{12}(\Sigma HP^2, S^5)$, $30\xi \circ \nu = 0$. On the other hand we have $24\xi \circ \nu = \xi \circ (24\nu) = 0$. Hence $6\xi \circ \nu = 0$ is easily verified. In fact

Theorem 6.1. *The element $\xi \circ \nu$ has order 6.*

Proof. Consider the mapping cone of κ, $E(\hat{\nu}) = S^5 \cup_\nu e^9 \cup_\kappa e^{14}$ which is the S^5-bundle over S^9 corresponding to the generator $\hat{\nu}$ of $\pi_8(SO(6)) = Z_{24}\{\hat{\nu}\}$ such that $\pi_*\hat{\nu} = \nu$ because $p_* : \pi_8(SO(6)) = Z_{24}\{\hat{\nu}\} \to \pi_8(S^5) = Z_{24}\{\nu\}$ is an isomorphism induced by the projection $p : SO(6) \to S^5$. We know the Hopf-Whitehead J-homomorphism $J : \pi_8(SO) = Z_2 \to \pi_8^s$ is an embedding to a direct summand and so the following diagram implies $J(\hat{\nu})$ is a genertor of Z_{24}-summand. It follows $\Sigma\kappa = J(\hat{\nu})$.

$$\begin{array}{ccccc}
Z_{24} & & Z_2 & & 0 \\
\| & & \| & & \| \\
\pi_8(SO(6)) & \to & \pi_8(SO(10)) & \to & \pi_8(SO(10)/SO(6)) \\
\downarrow J & & \downarrow J & & \\
\pi_{14}(S^6) & \xrightarrow{\Sigma^4} & \pi_{18}(S^{10}) & & \\
\| & & \| & & \\
Z_{24}\{\bar{\nu}_6\} & & Z_2\{\bar{\nu}_{10}\} + Z_2\{\epsilon\} & &
\end{array}$$

By Lemma 2.4, the exact sequence of homotopy groups of the fibration $S^5 \to E(\hat{\nu}) \to S^9$ is as follows:

$$\begin{array}{ccccccc}
\to \pi_{12}(S^5) & \to & \pi_{12}(E(\hat{\nu})) & \xrightarrow{\pi_*} & \pi_{12}(S^9) & \xrightarrow{\Delta^{(1}} & \pi_{11}(S^5) \\
\| & & \| & & \| & & \| \\
0 \to Z_{30} & \to & Z_{360}\{\Sigma p\} & \to & Z_{24}\{\nu\} & \to & Z_2\{\nu^2\}
\end{array}$$

$^{(1}$ $\Delta\nu = \nu^2$

where π is the bundle projection. Let τ_{S^6} be the tangent sphere bundle of S^6 and let $E(\tau_{S^6})$ denote its total space. Then the pull back diagram

$$\begin{array}{ccc}
E((12\iota_6)^*\tau_{S^6}) & \to & E(\tau_{S^6}) \\
\downarrow & & \downarrow \\
S^6 & \xrightarrow{12\iota_6} & S^6
\end{array}$$

induces the following commutative diagram:

$$\begin{array}{ccccccc}
 & & S^5 & \overset{24\iota_5}{\rightarrow} & S^5 & & \\
 & & \cap & & \| & & \\
S^9 & \overset{\xi}{\rightarrow} & S^5 \cup_\nu e^9 & \rightarrow & S^5 & = & S^5 \\
\| & & \downarrow S^5 & & \cap & & \cap \\
S^9 & \overset{24\iota_9}{\rightarrow} & S^9 & \overset{\nu'}{\rightarrow} & S^5 \cup_{24} e^6 & \subset & S^5 \cup_{24} e^6 \cup e^{11} = E((12\iota_6)^*\tau_{S^6}) \\
 & & \| & & \downarrow /S^5 & & \downarrow \\
 & & S^9 & \overset{\nu}{\rightarrow} & S^6 & = & S^6
\end{array}$$

(figure 8)

There exists an elements ν' which is a coextension of $\nu \in \pi_9(S^6)$ and we have

$$\pi_9(S^5 \cup_{24} e^6) = \pi_9(E((12\iota_6)^*\tau_{S^6})) = Z_{24}\{\nu'\} + Z_2\{\nu \circ \eta\}$$

since $\pi_9(E(\tau_{S^6})) = Z_2 + Z_2\{\nu \circ \eta\}$. Recall the definition of $\{\nu_5, 24\iota_8, \nu_8\}$ represented by the Toda braket.

$$\begin{array}{ccccccc}
S^{11} & \overset{\nu}{\rightarrow} & S^8 & = & S^8 & & \\
 & & \cap & & \downarrow 24\iota_8 & & \\
S^{12} & \overset{\Sigma^3\xi}{\rightarrow} & S^8 \cup_\nu e^{12} & \rightarrow & S^8 & = & S^8 \\
\downarrow 24\iota_{12} & & \downarrow S^8 & & \cap & & \downarrow \nu \\
S^{12} & = & S^{12} & \overset{\Sigma^3\nu'}{\rightarrow} & S^8 \cup_{24} e^9 & \overset{\tilde{\nu}}{\rightarrow} & S^5 \\
 & & \downarrow \nu & & \downarrow /S^8 & & \downarrow i \\
 & & S^9 & = & S^9 & \rightarrow & S^5 \cup_\nu e^9 \\
 & & & & \downarrow 24\iota_9 & & \downarrow /S^5 \\
 & & & & S^9 & = & S^9
\end{array}$$

(figure 9)

The element $\{\nu_5, 24\iota_8, \nu_8\}$ is the composition of $\Sigma^3\nu'$ and $\tilde{\nu}$ where $\Sigma^3\tilde{\nu}$ is a coextension of $\nu \in \pi_{12}(S^9)$ and $\tilde{\nu}$ is an extension of $\nu \in \pi_8(S^5)$ to $S^8 \cup_{24} e^9$. The indeterminacy on this case is $\nu_5 \circ \pi_{12}(S^8) + \pi_9(S^5) \circ \nu_9 = 0$. From the above diagram we obtain $\xi \circ \nu = i_*\{\nu_5, 24\iota_8, \nu_8\}$. On the other hand, by Adams[1] 7.17 p45-6 and Examples of 11.1 p53, $\{j_3, 24\iota, j_3\} = 40j_7$ where $j_r(r = 3, 7)$ denotes the image of the generator of $\pi_r(SO)$ under the stable J homomorphism:$\pi_r(SO) \rightarrow \pi_r^s$. We may consider $j_3 = \nu$ and $j_7 = \sigma$ which are the Hopf maps. Since $\pi_7^s = Z_{240}\{\sigma\}$,$\{j_3, 24\iota, j_3\}$ has the order 6. Since $i_* : \pi_{12}(S^5) \rightarrow \pi_{12}(S^5 \cup_\nu e^9)$ is injective, it follows the order of $\xi \circ \nu$ must be also 6.

7. Whitehead product. Our κ is related with the Whitehead product in $\pi_*(\Sigma HP^3)$. In fact

Theorem 7.1. $[\xi, \iota_5] = 24\kappa$ *in* $\pi_{13}(\Sigma HP^3)$.

Proof. Let $\pi_k : E_k \to S^9$ be the induced sphere bundle of $E(\tilde{\nu})$ by the map $k\iota_9 : S^9 \to S^9$. The pull back diagram

$$\begin{array}{ccc} E_k & \to & E(\hat{\nu}) \\ \pi_k \downarrow & & \downarrow \pi \\ S^9 & \xrightarrow{k\iota_9} & S^9 \end{array}$$

induces

$$\begin{array}{ccccccccc} 0 \to \pi_{12}(S^5) & \to & \pi_{12}(E_k) & \xrightarrow{\pi_{k*}} & \pi_{12}(S^9) & \to & \pi_{11}(S^5) \to \\ \| & & \downarrow & & \downarrow k\iota_{9*} & & \| \\ \to \pi_{12}(S^5) & \to & \pi_{12}(E(\hat{\nu})) & \xrightarrow{\pi_*} & \pi_{12}(S^9) & \to & \pi_{11}(S^5) \to \\ \| & & \| & & \| & & \| \\ 0 \to \quad Z_{30} & \to & Z_{360}\{\Sigma p\} & \to & Z_{24}\{\nu\} & \to & Z_2\{\nu^2\} \to 0. \end{array}$$

Especially E_{24} is homotopy equivalent to $S^5 \times S^9 = (S^9 \vee S^5) \cup e^{14}$ attached by $[\iota_9, \iota_5]$, since $\pi_8(SO(5)) = 0$. It follows $\pi_{12}(E_{24}) = \pi_{12}(S^9) + \pi_{12}(S^5) = Z_{24} + Z_{30}$. The following diagram

$$\begin{array}{ccc} Z\{e^{14}\} = \pi_{14}(E_{24}, S^9 \vee S^5) & \xrightarrow{\times 24} & \pi_{14}(E(\hat{\nu}), S^5 \cup_\nu e^9) = Z\{e^{14}\} \\ \partial^{(1} \downarrow & & \downarrow \partial^{(2} \\ \pi_{13}(S^9 \vee S^5) & \xrightarrow{\xi \vee \iota_5} & \pi_{13}(S^5 \cup_\nu e^9) \end{array}$$

$^{(1}\ \partial e^{14} = [\iota_9, \iota_5]$, $^{(2}\ \partial e^{14} = \kappa$

implies $(\xi \vee \iota_5)_*[\iota_9, \iota_5] = [\xi, \iota_5] = 24\kappa$.

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