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SELF MAPS OF SUSPENSION OF SPHERE BUNDLES OVER SPHERES

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1. Introduction. Since $\Sigma(S^m \times S^n)$ is homotopy equivalent to $\Sigma S^m \vee \Sigma S^n \vee \Sigma S^{m+n}$, there exists a bijection

$$[\Sigma(S^m \vee S^n), Y] \to [\Sigma S^m \vee \Sigma S^n \vee \Sigma S^{m+n}, Y].$$

However this bijection is not always homomorphic with respect to the natural multiplication of two sets. Of course the latter is necesarily abelian for any Y. However, the first group is not always abelian. For example, let $p_n : S^m \times S^n \to S^n$ and $p_m : S^m \times S^n \to S^m$ be the projections onto each factor and let $q : S^m \times S^n \to S^{m+n}$ be the projection. For brevity, suppose i denotes canonical inclusions $\Sigma S^m \to \Sigma(S^m \times S^n)$,$\Sigma S^n \to \Sigma(S^m \times S^n)$, $\Sigma S^m \vee \Sigma S^n \to \Sigma(S^m \times S^m)$ by the same symbol. Then the commutator $< i \circ \Sigma p_n, i \circ \Sigma p_m >= \pm i_*[\iota_{n+1}, \iota_{m+1}] \circ \Sigma q \in [\Sigma(S^m \times S^n), \Sigma(S^m \times S^n)]$ is a non-trivial element(Corollary 2.2) and so $[\Sigma(S^m \times S^n), \Sigma(S^m \times S^n)]$ is non-abelian. More generally we show,

Theorem. *Let $E(\xi)$ be an S^m-bundle over S^n with its characteristic class $\xi \in \pi_{n-1}(SO(m+1))$. If $2 < m+1 < n$, then the group $[\Sigma E(\xi), \Sigma E(\xi)]$ is not abelian.*

For example the cases of $S^3 \to Sp(2) \to S^7$ and $S^3 \to SU(3) \to S^5$ are known by Ohshima.

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2.A commutator in $[\Sigma E(\xi), \Sigma E(\xi)]$ We use the notations:

$p_n : S^m \times S^n \to S^n$ and $p_m : S^m \times S^n \to S^m$ be the projections on the each of factors,respectively,

$q : S^m \times S^n \to S^{m+n} = S^m \times S^n / S^m \vee S^n$ be the projection,

$i_n : S^n \subset \Omega\Sigma(S^m \vee S^n)$ and $i_m : S^m \subset \Omega\Sigma(S^m \vee S^n)$ be the canonical inclusions respectively,

i denotes the adjoints of i_n and i_m by the same symbol,

$+, -$ denotes the loop sum operation and its inverse,

and assume $2 < m + 1 < n$.

Lemma 2.1. *In $[\Sigma(S^m \times S^n), \Sigma(S^m \vee S^n)]$, we have the following* *relation*

$$\{i \circ \Sigma p_n, i \circ \Sigma p_m\} = [\iota_{n+1}, \iota_{m+1}] \circ \Sigma q.$$

where $\{a, b\}$ *denotes the commutator* $(-a-b)+(a+b)$.

Proof. The adjoint of $\{i \circ \Sigma p_n, i \circ \Sigma p_m\}$ is represented $< i_n, i_m > \circ q$ using the Samelson product $< i_n, i_m >$ of i_n and i_m. Considering the adjoint by [3], this turns to the identity

$$\{i \circ \Sigma p_n, i \circ \Sigma p_m\} = \pm[\iota_{n+1}, \iota_{m+1}] \circ \Sigma q.$$

Applying the inclusion $\Sigma(S^m \vee S^n) \subset \Sigma(S^m \times S^n)$, the following is easy;

Corollary 2.2. $[\Sigma(S^m \times S^n), \Sigma(S^m \times S^n)]$ *is non-abelian.*

Next we consider more generally the case $[\Sigma E(\xi), \Sigma E(\xi)]$ where
$p : E(\xi) \to S^n$, an S^m-bundle over S^n with its characteristic class $\xi \in \pi_{n-1}(SO(m+1))$,
$q : E(\xi) \to E(\xi)/S^m \cup e^n = S^{m+n}$.

We assume $2 < m+1 < n$. The following lemma is an extension of [2].

Lemma 2.3. *Let X be a connected finite CW complex of dim X=n and let* $P : X \to X/X_{n-1} = \vee_k S^n$ *be the projection where* X_{n-1} *denotes the (n-1)-skelton of X. Then for* $f \in [\Sigma X, Y]$ *and* $\{\alpha_k\} \in \pi_{n+1}(Y)$, *we have the equality,*

$$f + (\vee_k \alpha_k) \circ \Sigma P = (\vee_k \alpha_k) \circ \Sigma P + f$$

in the group $[\Sigma X, Y]$.

Proof. We put $S = \Sigma(X/X_{n-1}) = \vee_k S^{n+1}$ and consider the map $\Sigma P + \Sigma id : \Sigma X \to S \vee \Sigma X$. Since the inclusion $i : S \vee \Sigma X \to S \times \Sigma X$ induces a bijection $[\Sigma X, S \vee \Sigma X] \to [\Sigma X, S \times \Sigma X]$ because of $(S \times \Sigma X)_{n+2} = S \vee \Sigma X$ we obtain $\Sigma P + \Sigma id = \Sigma id + \Sigma P$. Then the proof is completed by applying the map $(\vee_k \alpha_k) \vee f : S \vee \Sigma X \to Y$ to both sides of this equality.

Lemma 2.4. *Let* (X, μ) *be a connected CW Hopf space,* $n > m+1 > 2$, $\alpha \in \pi_n(X)$ *and* $g : E(\xi) \to X$. *Then*

$$\{\alpha \circ p, g\}_\mu = \pm < \alpha, g \circ i >_\mu \circ q,$$

where $\{\alpha, \beta\}_\mu = (-\alpha - \beta) + (\alpha + \beta)$ *is the commutator in the algebraic loop* $[E(\xi), X]$ *with respect to* μ, $< -, - >_\mu$ *is the Samelson product with* *respect to μ.*

Proof. It follows from the commutativity of the following diagram.

$$\begin{array}{ccccc} E(\xi) & \xrightarrow{q} & S^n \wedge S^m & \xrightarrow{<\alpha, g\circ i>_\mu} & X \\ \downarrow d & & \downarrow id\wedge i & & \uparrow \{\ \}_\mu \\ E(\xi)\wedge E(\xi) & \xrightarrow[(\epsilon\iota_n \circ p)\wedge id]{} & S^n \wedge E(\xi) & \xrightarrow[\alpha\wedge g]{} & X\wedge X \end{array}$$

where d denotes the diagonal map,$\{\ \}_\mu$ denotes the commutator map with respect to μ and ϵ is ± 1. The commutativity of the first square follows from the facts that $(id \wedge i)\circ q$ and $\{(\epsilon\iota_n \circ p)\wedge id\}\circ d$ have the same induced homomorphism $H^{m+n}(S^n \wedge E(\xi)) \to H^{m+n}(E(\xi))$ and the natural transformation

$$[E(\xi), S^n \wedge E(\xi)] \to Hom(H^{m+n}(S^n \wedge E(\xi)), H^{m+n}(E(\xi))).$$

is isomorphic. The commutativity of the second square follows from the definition. Thus the proof is completed.

Proposition 2.5. *If $\alpha \in [\Sigma S^n, \Sigma E(\xi)]$ and $g \in [\Sigma E(\xi), \Sigma E(\xi)]$, then we have*

$$\{\alpha\circ\Sigma p, g\}_\mu = (-1)^n\epsilon[\alpha, g\circ\Sigma i]\circ\Sigma q.$$

Proof. It follows from the adjoint isomorphism

$$\begin{array}{ccc} [\Sigma E(\xi), \Sigma E(\xi)] & \simeq & [E(\xi), \Omega\Sigma E(\xi)] \\ (\Sigma q)^* \uparrow & & \uparrow q^* \\ \pi_{m+n+1}(\Sigma E(\xi)) & \simeq & \pi_{m+n}(\Omega\Sigma E(\xi)) \end{array}$$

and Lemma 2.4.

Proof of Theorem. From the exact sequence of homotopy groups of the fiber bundle $p : E(\xi) \to S^n$, there exists an elemnet $\alpha \in \pi_{n+1}(\Sigma E(\xi))$ such that $deg(\Sigma p\circ\alpha)$ is non-zero, because $\pi_{n-1}(S^m)$ is finite. By [1], $\Sigma E(\xi)$ has the homotopy type of the mapping cone of $\Sigma\Delta\iota_n \vee J(\xi) : \Sigma S^{n-1} \vee \Sigma S^{m+n-1} \to \Sigma S^m$ where Δ denotes the boundary homomorphism$\pi_n(S^n) \to \pi_{n-1}(S^m)$ of the exact sequence of homotopy groups of the fiber bundle and J is the Hopf-Whitehead J-homomorphism. By our assumption $n > m+1$, $J(\xi) \in \pi_{n+m}(\Sigma S^m)$ has the finite order. For such an element α, it follows that $[\alpha, \Sigma i]$ has infinite order because there exists no map $S^{n+1} \times S^{m+1} \to \Sigma E(\xi)$ of type $(k\alpha, \Sigma i)$ for any integer $k \neq 0$. From the Puppe sequence

$$[\Sigma^2(S^m \cup e^n), \Sigma E(\xi)] \xrightarrow{\Sigma(\Sigma i \circ J(\xi))^*} \pi_{m+n+1}(\Sigma E(\xi)) \xrightarrow{\Sigma q^*} [\Sigma E(\xi), \Sigma E(\xi)]$$

it follows that the order of the kernel of $\Sigma q^* : \pi_{m+n+1}(\Sigma E(\xi)) \to [\Sigma E(\xi), \Sigma E(\xi)]$ is finite and so we have that the commutator $< \alpha \circ \Sigma p, id_{\Sigma E} >$ is also non-trivial by Proposition 2.5. Thus the proof is completed.

Remark. *(1) Any maps in* $[\Sigma E(\xi), \Sigma E(\xi)]$ *can be represented as the formula*

$$s\, id_{\Sigma E} + \alpha \circ \Sigma p + \beta \circ \Sigma q$$

for $s \in Z$ *(integers),* $\alpha \in \pi_{n+1}(\Sigma E(\xi))$ *and* $\beta \in \pi_{m+n}(\Sigma E(\xi))$.
(2) On the set $[\Sigma E(\xi), \Sigma E(\xi)]$ *, the iterated commutators are trivial.*
(3) For the case n=m+1, we have a counterexample for the Theorem, that is, the Hopf fibration $S^3 \to S^7 \to S^4$ *or* $S^7 \to S^{15} \to S^8$.

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