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MULTIPICATIVE ELEMENTS IN MORAVA K-THEORY OF $B\mathbb{Z}/p$

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We denote the p-adic Morava K-theory by $\widehat{K(n)}^*(\ )$ so that the ring of coefficients is $\widehat{K(n)}_* = \mathbb{Z}_p[v_n, v_n^{-1}]$. Let R be a flat extension of $\mathbb{Z}_p$. We consider the ring

$$R[v_n, v_n^{-1}, t, t^{-1}]/(v_n - t^{p^n-1}) = R[t, t^{-1}],$$

where t is a variable of degree 2. This ring is flat over $\mathbb{Z}_p[v_n, v_n^{-1}]$ and we define a complex oriented cohomology theory

$$\overline{K(n)}_R^*(\ ) = \widehat{K(n)}^*(\ ) \otimes_{\mathbb{Z}_p[v_n, v_n^{-1}]} R[v_n, v_n^{-1}, t, t^{-1}]/(v_n - t^{p^n-1}).$$

Let x be an orientation class of $\overline{K(n)}_R^*(\ )$ with $\deg x = 2$. Then tx is a degree zero class. As an abuse of the notation we shell write x instead of tx. A formal group law associated with $\overline{K(n)}_R^*(\ )$ is defined over the periodic graded ring $R[t, t^{-1}]$ of period 2, but we may regard it as a non-graded formal group law $F(x, y)$ defined over R.

It is known that

$$\overline{K(n)}_R^0(B\mathbb{Z}/p) = R\,[[x]]\,/([p]x).$$

Here $[p]x = x +_F x +_F \cdots +_F x$ (p times) is the p-series of the formal group law $F(x, y)$. Since $F(x, y)$ is obtained by the ring extension of the formal group law of Morava K-theory we have that it is a Lubin-Tate formal group law. Thus we can choose such an orientation x, that

$$[p]x = px - x^{p^n}.$$

It is clear that $\overline{K(n)}_R^0(B\mathbb{Z}/p)$ is a Hopf algebra over R. We denote it by A_n,

$$A_n = \overline{K(n)}_R^0(B\mathbb{Z}/p) = R\,[[x]]\,/([p]x).$$

Since A_n is a rank p^n free R-module we can think of its algebraic dual $\Omega_R(n)$,

$$\Omega_R(n) = \hom_{R\text{-}module}(A_n, R).$$

In this paper using the calculation of an algebraic group of the Hopf algebra $\Omega_R(n)$, for certain ramified extension R of $\mathbb{Z}_p$, we get that the group of homotopy classes of ring spectra maps

$$\mathbf{B}\mathbb{Z}/p_+ \longrightarrow \overline{\mathbf{K}(n)}_R$$

is isomorphic to $(\mathbb{Z}/p)^n$. Here $\mathbf{B}\mathbb{Z}/p_+$ is the suspension spectrum of $B\mathbb{Z}/p_+$ and $\overline{\boldsymbol{K}(n)}_R$ is the ring spectrum of cohomology theory $\overline{K(n)}_R^*(\ )$. Group operation among such classes f and g is defined by the following composition

$$\mathbf{B}\mathbb{Z}/p_+ \xrightarrow{d} \mathbf{B}\mathbb{Z}/p_+ \wedge \mathbf{B}\mathbb{Z}/p_+ \xrightarrow{f\wedge g} \overline{\mathbf{K}(n)}_R \wedge \overline{\mathbf{K}(n)}_R \xrightarrow{\mu} \overline{\mathbf{K}(n)}_R.$$

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Let f be a spectra map of degree 0

$$f : \mathbf{B}\mathbb{Z}/p_+ \longrightarrow \overline{\mathbf{K}(n)}_R.$$

We can consider f as an element of cohomology ring

$$f = f(x) \in \overline{K(n)}_R^0(B\mathbb{Z}/p) = A_n$$

and thus we can consider f as a homomorphism

$$f : \Omega_R(n) \longrightarrow R.$$

Lemma 1. *The following conditions are equivalent:*
a) $f \in \hom_{R-algebra}(\Omega_R(n), R)$;
b) f *is a ring spectra map* $\boldsymbol{B}\mathbb{Z}/p_+ \longrightarrow \overline{\boldsymbol{K}(n)}_R$;
c) $f(F(x,y)) = f(x)f(y)$.

Proof. The condition a) means that the following diagram is commutative

$$\begin{array}{ccc} \Omega_R(n) \otimes_R \Omega_R(n) & \longrightarrow & \Omega_R(n) \\ \big\downarrow{\scriptstyle f\otimes f} & & \big\downarrow{\scriptstyle f} \\ R \otimes_R R & = & R. \end{array}$$

Considering the dual diagram we get

$$\begin{array}{ccc} A_n \otimes_R A_n & \xleftarrow{\Psi} & A_n \\ \big\uparrow{\scriptstyle f^*\otimes f^*} & & \big\uparrow{\scriptstyle f^*} \\ R \otimes_R R & = & R, \end{array}$$

where $f^*(1) = f(x)$ and $\Psi(x) = F(1 \otimes x, x \otimes 1)$. Thus commutativity of this diagram implies that a) $\Longleftrightarrow$ c).

The condition b) means that we have a commutative diagram

$$\begin{array}{ccc} \mathbf{B}\mathbb{Z}/p_+ \wedge \mathbf{B}\mathbb{Z}/p_+ & \xrightarrow{f\wedge f} & \overline{\mathbf{K}(n)}_R \wedge \overline{\mathbf{K}(n)}_R \\ \big\downarrow m & & \big\downarrow \mu \\ \mathbf{B}\mathbb{Z}/p_+ & \xrightarrow{f} & \overline{\mathbf{K}(n)}_R. \end{array}$$

We can consider that $f \circ m$, $\mu \circ (f \wedge f) \in \overline{K(n)}^0_R(B\mathbb{Z}/p \times B\mathbb{Z}/p)$ and by definition

$$f \circ m = m^*(f(x)) = f(F(x,y)),$$
$$\mu \circ (f \wedge f) = f(x)f(y).$$

So b) $\Longleftrightarrow$ c).

Now to find the above ring spectra maps we shall look for the structure of algebra $\Omega_R(n)$. Recall that $\overline{K(n)}^0_R(\mathbb{C}P^\infty) = R[[x]]$ is a formal Hopf algebra and $A^n = R[[x]]/([p]x)$ is its factor Hopf algebra. We introduce the notation

$$R[[x]]^* = \hom_{R-module}(R[[x]], R).$$

For $f \in R[[x]]^*$ we have

$$\{f \in \Omega_R(n)\} \Longleftrightarrow \{f(\alpha) = 0, \forall \alpha \in ([p]x)\}.$$

Let $\beta_i \in R[[x]]^*$ be dual to x^i so that $< \beta_i, \ x^j >= \delta^i_j$. For $f = \sum a_i\beta_i \in R[[x]]^*$ we have

$$\{f(\alpha) = 0, \forall \alpha \in ([p]x)\} \Longleftrightarrow$$
$$\{< \sum a_i\beta_i, \ \ x^{k-1}(px - x^{p^n}) >= 0, \quad k \geq 1\} \Longleftrightarrow$$
$$\{pa_k = a_{k+p^n-1}, \quad k \geq 1\}.$$

Thus if $pa_k = a_{k+p^n-1}$, $k \geq 1$, then $f = \sum a_i\beta_i \in \Omega_R(n)$. Therefore the following elements are in $\Omega_R(n)$:

$$\begin{aligned}
&1,\\
&\tilde{\beta}_1 = \beta_1 + p\beta_{p^n} + p^2\beta_{2p^n-1} + \cdots,\\
&\tilde{\beta}_2 = \beta_2 + p\beta_{p^n+1} + p^2\beta_{2p^n} + \cdots,\\
&\vdots\\
&\tilde{\beta}_{p^n-1} = \beta_{p^n-1} + p\beta_{2p^n-2} + p^2\beta_{3p^n-3} + \cdots .
\end{aligned}$$

The rank of $\Omega_R(n)$ is p^n. Elements $1, \tilde{\beta}_1, \dots, \tilde{\beta}_{p^n-1}$ are linearly independent in $R[[x]]^*$ and so are in $\Omega_R(n)$. Hence they can be taken as an additive bases of $\Omega_R(n)$.

Now we consider the product in $\Omega_R(n)$. Since $[p]x = px - x^{p^n}$, by easy calculations we get

$$F(x,y) = x + y + a\left[\sum_{i+1}^{p^n-1} \frac{p^n!}{i!(p^n-i)!} x^i y^{p^n-i}\right] + \sum_{q=2}^{\infty} \sum_{i+j=q(p^n-1)+1} a_{ij} x^i y^j \tag{1}$$

where $a = 1/(p - p^{p^n})$.

Let $\mathbf{L}$ be a submodule of $R[[x]]^*$ generated by the elements β_{p^n}, $\beta_{p^n+1}, \cdots$.

Lemma 2. *Let $i + j = q(p^n - 1) + k$, $0 < k < p^n$, then*

a) $\beta_i\beta_j = a_{ij}^k\beta_k + l$, where $a_{ij}^k \in \mathbb{Z}_p$ and $l \in \mathbf{L}$

b) $\tilde{\beta}_i\tilde{\beta}_j = \gamma_{ij}\tilde{\beta}_k$, where $\gamma_{ij} \in \mathbb{Z}_p$ and $\gamma_{ij} \equiv a_{ij}^k \mod p$.

Proof.

a) It is well-known that

$$\beta_i\beta_j = \sum_{r=1}^{\infty} a_{ij}^r \beta_r \tag{2}$$

where $a_{ij}^r \in \mathbb{Z}_p$ are defined from the following equation

$$(F(x,y))^r = \sum_{i,j\geq 1} a_{ij}^r x^i y^j.$$

The spearness of the formal group law (1) implies the spearness of

$(F(x,y))^r$:

$$(F(x,y))^r = \sum_{q=0}^{\infty} \sum_{i+j=q(p^n-1)+r} a_{ij}^r x^i y^j.$$

So $a_{ij}^r = 0$ if $r \not\equiv i+j \mod (p^n-1)$.

b) By the definition

$$\tilde{\beta}_i \tilde{\beta}_j = \sum_{e,q\geq 0} p^{e+q} \beta_r \beta_t,$$

where $t = q(p^n-1)+j$ and $r = e(p^n-1)+i$. Now from a) we have

$$\tilde{\beta}_i \tilde{\beta}_j = \sum_{e,q\geq 0} p^{e+q}(a_{ij}^k \beta_k + l_{e,q}),$$

where $l_{e,q} \in \mathbf{L}$. Taking $\gamma_{ij} = \sum_{e,q\geq 0} a_{ij}^k p^{e+q} \in \mathbb{Z}_p$ and $l = \sum_{e,q\geq 0} p^{e+q} l_{e,q} \in \mathbf{L}$ we get that $\gamma_{ij} = a_{ij}^k \mod p$ and

$$\tilde{\beta}_i \tilde{\beta}_j = \gamma_{ij} \beta_k + l.$$

This relation gives b) in $\Omega_R(n)$.

Lemma 3. *a) For $i < p^n - 1$ and $p \not| \, i+1$*

$$\tilde{\beta}_1 \tilde{\beta}_i = *(i+1)\tilde{\beta}_{i+1};$$

b) for $i < p-1$ and $0 < k \leq n-1$

$$\tilde{\beta}_{p^k} \tilde{\beta}_{ip^k} = *\tilde{\beta}_{(i+1)p^k};$$

c) for $0 < k < n-1$

$$\tilde{\beta}_{p^k} \tilde{\beta}_{(p-1)p^k} = *p\tilde{\beta}_{p^{k+1}};$$

d)

$$\tilde{\beta}_{p^{n-1}} \tilde{\beta}_{(p-1)p^{n-1}} = *\tilde{\beta}_1,$$

where $*$ means a multiplication by some p-adic unit of the form $1+p\nu$, $\nu \in \mathbb{Z}_p$.

Proof. The proofs of a) and b) are easy and here we give only the proofs of c) and d).

c) From (1) and (2)

$$\beta_{p^k}\beta_{(p-1)p^k} = \frac{p^{k+1}!}{p^k!\,((p-1)p^k)!}\beta_{p^{k+1}} = *p\beta_{p^{k+1}}.$$

By Lemma 2 we have

$$\tilde{\beta}_{p^k}\tilde{\beta}_{(p-1)p^k} = \gamma_{p^k,(p-1)p^k}\tilde{\beta}_{p^{k+1}}.$$

Thus we need to prove that $\gamma_{p^k,(p-1)p^k} \equiv p \mod p^2$ and so it is sufficient to show the existens of such an $l \in L$ that

$$\tilde{\beta}_{p^k}\tilde{\beta}_{(p-1)p^k} \equiv p\beta_{p^{k+1}} + l \mod p^2. \tag{3}$$

By the definition

$$\begin{aligned}\tilde{\beta}_{p^k}\tilde{\beta}_{(p-1)p^k} &\equiv \beta_{p^k}\beta_{(p-1)p^k} + p(\beta_{p^k}\beta_{p^n-1+(p-1)p^k} + \beta_{p^n-1+p^k}\beta_{(p-1)p^k}) \mod p^2\\ &\equiv p\beta_{p^{k+1}} + p(\beta_{p^k}\beta_{p^n-1+(p-1)p^k} + \beta_{p^n-1+p^k}\beta_{(p-1)p^k}) \mod p^2.\end{aligned}$$

Now Lemma 2.a) shows that for some $l \in L$ we have

$$\tilde{\beta}_{p^k}\tilde{\beta}_{(p-1)p^k} \equiv \beta_{p^{k+1}} + p(a'_k + a''_k)\beta_{p^{k+1}} + l \mod p^2,$$

where $a'_k = a^{p^{k+1}}_{p^k,\,[p^n-1+(p-1)p^k]}$ and $a''_k = a^{p^{k+1}}_{[p^n-1+p^k],\,[(p-1)p^k]}$ are the coefficients of the summands $x^{p^k}y^{p^n-1+(p-1)p^k}$ and $x^{p^n-1+p^k}y^{(p-1)p^k}$ in $(F(x,y))^{p^{k+1}}$. The degrees of this summands are not the powers of p, so their coefficients have to be divisible by p. Thus we get (3).

d) From (1) and (2) we get that for some $l \in L$

$$\beta_{p^{n-1}}\beta_{(p-1)p^{n-1}} = \frac{ap^n!}{p^{n-1}!\,((p-1)p^{n-1})!}\beta_1 + l.$$

Since $a = 1/(p - p^{p^n})$, it is easy to check that

$$\frac{ap^n!}{p^{n-1}!\,((p-1)p^{n-1})!} = 1 + p\nu,$$

for some $\nu \in \mathbb{Z}_p$.

Theorem 4. *a)* $\Omega_R(1) = R[b_1]/(b_1^p - b_1)$;

b) Let $n \geq 2$ *and let* I_n *be an ideal generated by*

$$b_p^{p^k} - a(k)b_{p^{k+1}},\ k = 1, \dots, n-2$$

and

$$b_p^{p^n} - a(n,1)b_p,$$

where $a(k) = p^{p^{k-1}+p^{k-2}+\dots+p+1}$ *and* $a(n,k) = p^{p^{n-1}+\dots+p^{k+1}+p^{k-1}+\dots+p+1}$. *Then*

$$\Omega_R(n) = R[b_p, b_{p^2}, \dots, b_{p^n-1}]/I_n.$$

Proof. From Lemma 3.a) and b) follows that $1, \tilde{\beta}_1, \tilde{\beta}_p, \dots, \tilde{\beta}_{p^n-1}$ can be taken as a multiplicative generators of $\Omega_R(n)$. Lemma 3.d) shows that $\tilde{\beta}_1$ can be omitted and Lemma 3.b), c), d) together imply that

$$\tilde{\beta}^p_{p^k} = *p\tilde{\beta}_{p^{k+1}},\ k = 1, \dots, n-2;$$
$$\tilde{\beta}^{p^2}_{p^{n-1}} = *p\tilde{\beta}_p.$$

Now it is easy to get the following equations

$$\tilde{\beta}^{p^k}_p = *a(k)\tilde{\beta}_{p^{k+1}},\ k = 1, \dots, n-2; \tag{4}$$
$$\tilde{\beta}^{p^n}_p = *a(n,1)\tilde{\beta}_p. \tag{5}$$

Let K be the quotient field of R. It is clear that $K \otimes_R \Omega_R(n)$ is multiplicatively generated by $\tilde{\beta}_p$. Therefore from the degree reason we have

$$K \otimes_R \Omega_R(n) = K[[\tilde{\beta}_p]]/(\tilde{\beta}^{p^n}_p - *a(n,1)\tilde{\beta}_p). \tag{6}$$

Hence all relations in $\Omega_R(n)$ can be reduced to the relations of $\tilde{\beta}_p, \tilde{\beta}_{p^2}, \dots, \tilde{\beta}_{p^n-1}$ to $\tilde{\beta}_p$, that is (4) and (5).

From (5) we can write $\tilde{\beta}^{p^n}_p = \mu a(n,1)\tilde{\beta}_p$, where $\mu \in \mathbb{Z}_p$, $\mu \equiv 1 \mod p$. For such μ there exists $\bar{\mu} \in \mathbb{Z}_p$ satisfying $\bar{\mu}^{p^n-1} = \mu$. Taking $b_p = \bar{\mu}\tilde{\beta}_p$, we get $b^{p^n}_p = a(n,1)b_p$. Thus multiplying by the appropriate elements of $\mathbb{Z}_p$, we can change the generators $\tilde{\beta}_{p^k}$ by b_{p^k}, $k = 1, \dots, n-1$, in such a way, that the new generators will satisfy the following relations:

$$b^{p^k}_p = a(k)b_{p^{k+1}},\ k = 1, \dots, n-2; \tag{7}$$
$$b^{p^n}_p = a(n,1)b_p.$$

Thus Theorem 4 is proved.

Corollary 5. *Let* $\mathcal{O}_n$ *be a degree* n *unramified extension of* p*-adic integers* $\mathbb{Z}_p$. *Then* $\mathcal{O}_n$*-algebra homomorphism* $f : \Omega_{\mathcal{O}_n}(n) \longrightarrow \mathcal{O}_n$ *is trivial if* $n > 1$.

Proof. b_p is a root of the following equation in $\Omega_{\mathcal{O}_n}(n)$

$$y_p^{p^n} = a(n,1)y_p. \tag{8}$$

Thus $f(b_p)$ has to be a root of the same equation in $\mathcal{O}_n$. But in $\mathcal{O}_n$ there is only trivial root of (8). So $f(b_p) = 0$ and therefore $f(b_{p^k}) = 0$ for $1 < k \leq n-1$.

Now let R be a flat extension of $\mathcal{O}_n$ containing the $(p^n - 1)$-th roots of $a(n,k)$, $k = 1, \dots, n-1$. For example such rings are:

1. $R = \mathcal{O}_{\mathbb{C}_p}$ is the ring of integers of $\mathbb{C}_p$, where $\mathbb{C}_p$ is the completion field of the algebraic closure of p-adic numbers $\mathbb{Q}_p$;

2. $R = \mathcal{O}_n[x]/(x^{p^n-1} - p)$ is rumified extension of $\mathcal{O}_n$ containing the $(p^n - 1)$-th roots of p.

Proposition 6. *For R as above, there exist exactly p^n R-algebra homomorphisms $\Omega_R(n) \longrightarrow R$.*

Proof. Using $f \to 1_K \otimes f$ we define the following injection

$$\phi : \hom_{R-algebra}(\Omega_R(n), R) \longrightarrow \hom_{K-algebra}(K \otimes_R \Omega_R(n), K).$$

Let us re-wright (6) in such a way

$$K \otimes_R \Omega_R(n) = K[[b_p]]/(b_p^{p^n} - a(n,1)b_p).$$

It is clear that the number of K-algebra homomorphisms $K \otimes_R \Omega_R(n) \longrightarrow K$ is equal to the number of different roots of (8) in K. Thus there are p^n such homomorphisms.

Now it is sufficient to prove that ϕ is surjection. Considering the inclusion

$$i : \Omega_R(n) \longrightarrow K \otimes_R \Omega_R(n)$$

we need to show that $f \circ i(\Omega_R(n)) \subset R$ if $f \in \hom_{K-algebra}(K \otimes_R \Omega_R(n), K)$. ¿From (7) we have that

$$f \circ i(b_{p^k}) = f\left(\frac{b_p^{p^{k-1}}}{a(k)}\right) = \frac{1}{a(k)}[f(b_p)]^{p^k-1},$$

where $f(b_p)$ is a root of (8). Thus

$$f(b_p) = f \circ i(b_p) = f \circ i(b_{p^2}) = \cdots = f \circ i(b_{p^{n-1}}) = 0$$

or

$$f(b_p) = \xi^{i_1}_{p^n-1} a(n,1)^{1/p^n-1},$$

where $\xi^{i_1}_{p^n-1}$ is some (p^n-1)-th root of the unity and under $p^{1/m}$ we mean the real positive m-th root of p. In this case

$$f \circ i(b_{p^k}) = \xi^{i_1}_{p^n-1} a(n,k)^{1/p^n-1}$$

and hence

$$[f \circ i(b_{p^k})]^{p^n-1} = a(n,k).$$

Now by the condition of Proposition 6 $f \circ i(b_{p^k}) \in R$, $k = 1, \dots, n-1$.

Theorem 7. *The group of homotopy classes of ring spectra maps*

$$B\mathbb{Z}/p_+ \longrightarrow \overline{K(n)}_R$$

is isomorphic to $(\mathbb{Z}/p)^n$.

Proof. Let f be such a ring spectra map. Considering the following commutative diagram

$$\begin{array}{ccccc} \mathbf{B}\mathbb{Z}/p_+ & \xrightarrow{d} & \overset{p}{\wedge}\mathbf{B}\mathbb{Z}/p_+ & \xrightarrow{\overset{p}{\wedge} f} & \overline{\mathbf{K}(n)}_R \overset{p}{\wedge} \overline{\mathbf{K}(n)}_R \\ & & \downarrow m & & \downarrow \mu \\ & & \mathbf{B}\mathbb{Z}/p_+ & \xrightarrow{f} & \overline{\mathbf{K}(n)}_R \end{array}$$

we have that $f^p = \mu \circ (\overset{p}{\wedge} f) \circ d = f \circ m \circ d$. But $m \circ d$ is induced from the trivial composition

$$\mathbb{Z}_p \longrightarrow \bigoplus_p \mathbb{Z}_p \xrightarrow{m} \mathbb{Z}_p.$$

Thus $f^p = 0$ and Theorem 7 follows from Proposition 6.

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