

PLANES, NETS, AND CODES

STEVEN T. DOUGHERTY

1. Introduction.

Projective planes, affine planes, nets and their related structures (transversal designs, MOLS) have received a great deal of attention. An extensive literature exists using groups and combinatorial techniques to study these objects. More recently the theory of algebraic codes has been applied, see [1], [4], [7]. In this paper, we show the connection between combinatorial and geometric information in the designs and questions which arise naturally in the application of coding theory. We begin with some definitions.

Definition. A **projective plane** Π is a set of points P , a set of lines L , and an incidence relation I between them where any two points are incident with a unique line, any two lines are incident with a unique point with not all points incident with one line. It follows from the definition that, when P is a finite set with more than one point, $|P| = |L| = n^2 + n + 1$, for some n , which is called the order of the plane. An **affine plane** of order n is the design formed by removing a line and all points incident with it from a projective plane of order n , and so the affine plane $\pi = \Pi^{L_\infty} = (A, M, I)$ where $A = \{q | q \in P \text{ and } q \text{ not incident with } L_\infty\}$, $M = L - \{L_\infty\}$, and the incidence relation is the incidence relation of the projective plane restricted to these sets; it follows that $|A| = n^2$ and $|M| = n^2 + n$.

Every projective plane gives rise to $n^2 + n + 1$ affine planes – which may or may not be isomorphic. An affine plane has a unique “completion” which is a projective plane. For a fuller discussion of these points see Chapter 6 of [1]; we will adopt the notation of this book.

In a projective plane of order n there are $n + 1$ points on a line and $n + 1$ lines through a point. In an affine plane there are $n + 1$ lines through a point and n points on each line. An affine plane has $n + 1$ parallel classes each containing n lines. A projective plane of order n is said to be **desarguesian** if it is the projective plane formed in the usual way from the field F_n , n necessarily a power of a prime. An affine plane is desarguesian if its completion is a desarguesian projective plane.

Definition. A k -net of order n is an incidence structure consisting of n^2 points and nk lines satisfying the following four axioms:

- (i) every line has n points;
- (ii) parallelism is an equivalence relation on lines, where two lines are said to be parallel if they are disjoint or identical;
- (iii) there are k parallel classes each consisting of n lines;
- (iv) any two non-parallel lines meet exactly once.

From the definition it follows that an $(n+1)$ -net of order n is an affine plane. An s -transversal of a net is a set of sn points having exactly s points in common with each line of the net. A 1-transversal is known simply as a transversal.

Definition. Let S be a set of cardinality n . Let A be an $n \times n$ matrix such that each row and column of A contains each element of S exactly once. Then A is a *Latin square of order n* . Let $A = (a_{ij})$ and $B = (b_{ij})$ be Latin squares of order n ; if $\{(a_{ij}, b_{ij})\} = S \times S$ then A and B are said to be *orthogonal*. A set $\{A_1, A_2, \dots, A_s\}$ with A_i orthogonal to A_j for $1 \leq i < j \leq s$ is called a set of s *Mutually Orthogonal Latin Squares*.

It is well known that k -MOLS of order n are equivalent to a $(k+2)$ -net of order n .

A linear $[n, k]$ code C is a vector space of dimension k in F^n , where F is a field and n is the length of the code. In this paper F will always be F_p the finite field with p elements, with p a prime. The ambient space is equipped with the standard inner product, namely

$$[v, w] = \sum_{i=1}^n v_i w_i.$$

The orthogonal to the code is defined to be

$$C^\perp = \{v \in F_p^n \mid [v, w] = 0 \text{ for all } w \in C\}.$$

For any incidence structure $D = (P, B)$, we take the characteristic function of a block $b \in B$ to be:

$$v^b(q) = \begin{cases} 1, & \text{if } b \text{ is incident with } q \\ 0, & \text{if } b \text{ is not incident with } q \end{cases}$$

We define $C_p(D)$ to be the code generated by the characteristic functions of blocks over F_p , i.e. $C_p(D) = \langle v^b \mid b \text{ a block in } B \rangle$. We take p to be

a prime and for the code to be useful we take p such that p divides n . For the remainder of the paper we always assume that p divides n , the order of the design, if p divides n but p^2 does not divide n then we say that p sharply divides n .

Set $Hull_p(D) = C_p(D) \cap C_p(D)^\perp$ and let $H_p(D)$ be the code generated by differences of parallel blocks in D , i.e. $H_p(D) = \langle v^b - v^{b'} \mid b \text{ and } b' \text{ parallel blocks} \rangle$. This is a slight notational difference from the work of Assmus and Key in which $Hull_p(D)$ is often denoted by $H_p(D)$.

For the remainder of the paper we shall not distinguish between a block (in our case, a line) and its characteristic function, that is, b denotes both the line and the characteristic function depending on the context.

2. The Structure of the Codes.

In this section we shall state and prove some general structure theorems (some old and some new) about the codes of nets and planes to use later in the paper. The results in this section about planes (specifically Theorem 2.1, Theorem 2.4 and Lemma 2.2) can be found in [1] and some of those about nets (specifically Theorem 2.2, Theorem 2.3 and Lemma 2.1) are generalizations of theorems found in [4]. We include proofs for completeness.

Let Π be a projective plane of order n with $\pi = \Pi^{L\infty}$ an affine plane formed from Π . We have $N_0 \subset N_1 \subset N_2 \subset \dots \subset N_n \subset N_{n+1} = \pi$ where N_k is a k -net formed from π by removing $n + 1 - k$ parallel classes, we also consider the case where $N_0 \subset N_1 \subset N_2 \subset \dots \subset N_k$, where N_k is maximal.

Theorem 2.1. *Let Π be a projective plane of order n with p a prime dividing n . $Hull_p(\Pi)$ is of codimension 1 in $C_p(\Pi)$, and $Hull_p(\Pi) = \langle L - M \mid L, M \text{ lines of } \Pi \rangle$.*

Proof. For any two lines L and M of Π , $[L, M] = 1 \neq 0$, hence $L \notin C_p(\Pi)^\perp$, and then $L \notin Hull_p(\Pi)$, giving $Hull_p(\Pi) \neq C_p(\Pi)$. Let $G = \langle L - M \mid L \text{ and } M \text{ lines in } \Pi \rangle$. Given lines L, M , and T of Π , $[L - M, T] = 0$ giving $G \subseteq Hull_p(\Pi) \subset C_p(\Pi)$. For any line L , $\langle G, L \rangle = C_p(\Pi)$ and hence G is at most codimension 1 in $C_p(\Pi)$. Therefore $Hull_p(\Pi)$ is generated by differences of lines and is of codimension 1 in $C_p(\Pi)$. \square

For any k , $H_p(N_k) \subseteq Hull_p(N_k)$, since clearly $H_p(N_k) \subseteq C_p(N_k)$ and for parallel lines l, m we have $[l - m, t] = 0$ for any line t . Hence $H_p(N_k) \subseteq C_p(N_k) \cap C_p(N_k)^\perp = Hull_p(N_k)$.

Theorem 2.2. *If N_k has an s -transversal with p not dividing s or if $k \not\equiv 1 \pmod{p}$, then $\dim C_p(N_k) - \dim H_p(N_k) = k$.*

Proof. We know $\dim C_p(N_k) - \dim H_p(N_k) \leq k$ since $C_p(N_k) = \langle m_1, \dots, m_k, H_p(N_k) \rangle$ where $m_i \in \mathfrak{A}_i$, and \mathfrak{A}_i is the i -th parallel class. Moreover, we may assume that $k > 1$, since $\dim C_p(N_1) = n$ and $\dim H_p(N_1) = n - 1$. We shall show that m_1, m_2, \dots, m_k are linearly independent over $H_p(N_k)$. First we note that since $k > 1$, no line m is in $C_p(N_k)^\perp$ since $[l, m] \neq 0$ for l and m not parallel. Hence no line m is in $H_p(N_k)$, since $H_p(N_k) \subseteq C_p(N_k)^\perp$.

Assume $v = a_1 m_1 + a_2 m_2 + \dots + a_k m_k \in H_p(N_k) \subseteq C_p(N_k) \cap C_p(N_k)^\perp$. Since $v \in C_p(N_k)^\perp$, $[v, m] = 0$ for all lines m in N_k . Let $l_j \in \mathfrak{A}_j$; we have:

$$0 = [v, l_j] = [a_1 m_1, l_j] + \dots + [a_k m_k, l_j] = \sum_{i \neq j} a_i.$$

Therefore $0 = \left(\sum_{i=1}^k a_i\right) - a_1 = \dots = \left(\sum_{i=1}^k a_i\right) - a_k$, and so $\sum_{i=1}^k a_i = a_1 = a_2 = \dots = a_k$.

If $k \not\equiv 1 \pmod{p}$, set $a_i = a$, we have $\sum_{i \neq j} a_i = (k-1)a = 0$, if $a \neq 0$

then $(k-1) = 0$; but $k \neq rp + 1$ so $k-1 \neq 0$, and hence $a = 0$, and in this case $\{m_1, m_2, \dots, m_k\}$ are linearly independent over $H_p(N_k)$.

Now let t be an s -transversal of N_k , with p not dividing s . We know $t \in H_p(N_k)^\perp$ since $[t, m - l] = 0$ for m parallel to l . So $[v, t] = 0$, that is $[a_1 m_1, t] + \dots + [a_k m_k, t] = 0$ which implies $sa_1 + sa_2 + \dots + sa_k = s(a_1 + a_2 + \dots + a_k) = 0$, and so, again we have that $a_i = 0$ for all i , giving the result. \square

Lemma 2.1. *Let N_k be a k -net of order n , where $k \leq n$. Set $\mathfrak{A}_k = \{l_1, l_2, \dots, l_n\}$. When k is not congruent to $1 \pmod{p}$ or when N_k has an s -transversal with p not dividing s , then if $a_1 l_1 + \dots + a_n l_n \in C_p(N_{k-1})$, we have $\sum a_i = 0$ and $a_1 l_1 + \dots + a_n l_n \in H_p(N_{k-1})$ as well.*

Proof. Let $v = a_1 l_1 + \dots + a_n l_n$. If $v \in C_p(N_{k-1})$ then $v = c_1 m_1 + c_2 m_2 + \dots + c_{k-1} m_{k-1} + u$ where $u \in H_p(N_{k-1})$, $m_i \in \mathfrak{A}_i$, which implies $v + b_1 m_1 + b_2 m_2 + \dots + b_{k-1} m_{k-1} = u \in H_p(N_{k-1})$ where $b_i = -c_i$. Since u is in $H_p(N_{k-1})$, it is orthogonal to any line and any s -transversal of N_k .

We have

$$\begin{aligned} 0 &= [a_1l_1 + \dots + a_nl_n + b_1m_1 + b_2m_2 + \dots + b_{k-1}m_{k-1}, l_j] \\ &= a_j(n) + \sum b_i = \sum b_i \end{aligned}$$

and therefore $\sum b_i = 0$. If N_k has an s -transversal t , with p not dividing s , then

$$\begin{aligned} 0 &= [a_1l_1 + \dots + a_nl_n + b_1m_1 + b_2m_2 + \dots + b_{k-1}m_{k-1}, t] \\ &= s \sum a_i + s \sum b_j = s \sum a_i \end{aligned}$$

and hence $\sum a_i = 0$, since p does not divide s .

Hence

$$\begin{aligned} 0 &= \sum a_i + (\sum b_i - b_j) \\ - \sum a_i &= \sum b_i - b_j \\ - \sum a_i &= 0 - b_j \\ \sum a_i &= b_j \text{ for } 1 \leq j \leq k - 1 \end{aligned}$$

If N_k has no such s -transversal but k is not congruent to 1 mod p then we have:

$$\begin{aligned} 0 &= [a_1l_1 + \dots + a_nl_n + b_1m_1 + b_2m_2 + \dots + b_{k-1}m_{k-1}, m_j] \\ &= \sum a_i + \sum_{i \neq j} b_i \text{ for } i \leq j \leq k - 1 \end{aligned}$$

Thus all b_j are equal; set $b = b_j$. Then $\sum b = 0$ implies $(k - 1)b = 0$, but p does not divide $(k - 1)$. Therefore $b = 0$, and $\sum a_i = 0$ as desired. \square

Theorem 2.3. *Let the i -th parallel class be $\mathfrak{A}_i = \{l_j^i\}$ in a k -net N_k of order n , where N_k has a transversal or $k \not\equiv 1 \pmod p$, then if*

$$\sum a_j^1 l_j^1 + \dots + \sum a_j^k l_j^k = 0$$

then $\sum_{j=1}^n a_j^i = 0$.

Proof. Given any parallel class \mathfrak{A}_i arrange the net so that this is the k -th parallel class. Then the result follows from the previous lemma. \square

For $N_1 \subset N_2 \subset \dots \subset N_n \subset N_{n+1}$ it is clear that $C_p(N_1) \subseteq C_p(N_2) \subseteq \dots \subseteq C_p(N_n) \subseteq C_p(N_{n+1})$ and that $H_p(N_1) \subseteq H_p(N_2) \subseteq \dots \subseteq H_p(N_n) \subseteq$

$H_p(N_{n+1})$. The all one vector \mathbf{j} is in $C_p(N_k)$ for all k since the sum of any parallel class is \mathbf{j} . For any line l in the $(n+1)$ -st parallel class we have:

$$l = \mathbf{j} - \sum_{i=1}^n l_i \text{ where } l_i \text{ is incident with } q \text{ a point on } l$$

$$\text{and } l_i \in \mathfrak{A}_i \text{ for } 1 \leq i \leq n$$

This makes $C_p(N_n) = C_p(N_{n+1})$, $H_p(N_n) = H_p(N_{n+1})$, and $Hull_p(N_n) = Hull_p(N_{n+1})$. By Theorem 2.2, the codimension of $H_p(N_n)$ in $C_p(N_n)$ is n , since any line in the $(n+1)$ -st parallel class serves as a transversal, and then the codimension of $H_p(N_{n+1})$ in $C_p(N_{n+1})$ is also n .

Theorem 2.4. *Let π be an affine plane of order n and let Π be its projective completion with L_∞ the line at infinity. Then, for a prime p dividing n , $Hull_p(\pi)$ is the projection of the subcode $\{c \in Hull_p(\Pi) | c_Q = 0 \text{ for } Q \text{ on } L_\infty\}$, and $Hull_p(\pi) = \langle l - m | l \text{ and } m \text{ are parallel lines in } \pi \rangle$, and is of codimension n in $C_p(\pi)$.*

Proof. We have seen that $H_p(N_n) = H_p(N_{n+1}) = H_p(\pi) \subseteq Hull_p(\pi) \subset C_p(\pi)$ and the codimension of $H_p(N_{n+1})$ in $C_p(N_{n+1}) = C_p(\pi)$ is n . Theorem 2.2 gives that $H_p(N_n) = Hull_p(N_n)$ since the proof shows the linear independence of these n vectors over $Hull_p(N_n)$. We know $C_p(N_n) = C_p(N_{n+1})$ and so $Hull_p(N_n) = Hull_p(N_{n+1})$. Therefore $Hull_p(\pi) = H_p(\pi)$, i.e. $Hull_p(\pi) = \langle l - m | l \text{ and } m \text{ are parallel lines in } \pi \rangle$, and $Hull_p(\pi)$ has codimension n in $C_p(\pi)$.

If l and m are parallel lines in π , let L and M be their extension in Π . Then $L - M$ is 0 on the points of L_∞ . Let $G = \langle L - M | L \text{ and } M \text{ extensions of parallel lines in } \pi \rangle$. Note that all of the vectors in G have 0 on the coordinates of L_∞ . Let $l_i \in \mathfrak{A}_i$, with L_i its extension in Π . We see that $Hull_p(\Pi) = \langle G, L_1 - L_\infty, L_2 - L_\infty, \dots, L_n - L_\infty \rangle$. If a vector has 0's on the coordinates of L_∞ then it is in G and $Hull_p(\pi)$ is the projection of G . \square

Lemma 2.2. *If Π is a projective plane of order n with p a prime dividing n , and $\pi = \Pi^{L_\infty}$. Then $C_p(\pi)^\perp$ is the projection of $\{v \in C_p(\Pi)^\perp \text{ with } v_q = 0 \text{ when } q \text{ is a point on the } L_\infty\}$*

Proof. Let $Y = \{v \in C_p(\Pi)^\perp \text{ with } v_q = 0 \text{ when } q \text{ is a point on the } L_\infty\}$. It is clear that if $v \in Y$ then its projection is in $C_p(\pi)^\perp$. The dimension of Y can be found by noticing that requiring $v_q = 0$ lowers the dimension by

1 for the first n points on L_∞ giving that $\dim Y = \dim C_p(\pi)^\perp$. \square

The following is the standard definition of the direct product of nets.

Definition. Let N_k, N'_k be k -nets of order n and n' respectively. The *direct product of nets* $N_k \times N'_k$ is defined as follows:

- (1) the points $N_k \times N'_k$ are ordered pairs (q, q') with q a point of N_k and q' a point of N'_k .
- (2) the lines of $N_k \times N'_k$ are ordered pairs (m, m') with m, m' from the i -th parallel class of N_k and N'_k respectively.
- (3) the point (q, q') is incident with the line (m, m') if and only if q is incident with m and q' is incident with m' .

Let $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_k$ be the parallel classes of N_k and let $\mathfrak{A}'_1, \mathfrak{A}'_2, \dots, \mathfrak{A}'_k$ be the parallel classes of N'_k . Note that the order the parallel classes are labeled makes a difference in the direct product. Let $X_i = C_p(\mathfrak{A}_i)$ and $X'_i = C_p(\mathfrak{A}'_i)$. By the natural identification of $F_p^{P \times P'}$, where P and P' are the point sets, with $F_p^P \otimes F_p^{P'}$ we have that

$$C_p(N_k \times N'_k) = \sum_{i=1}^k X_i \otimes X'_i$$

Let $Y_i = H_p(\mathfrak{A}_i)$ and $Y'_i = H_p(\mathfrak{A}'_i)$. Let (r, r') be a point in $N_k \times N'_k$, and let (m, m') and (l, l') be lines in $N_k \times N'_k$. Then (r, r') is incident with (m, m') and (r, r') is incident with (l, l') if and only if r is incident with m and r is incident with l , and r' is incident with m' and r' is incident with l' , and hence (m, m') and (l, l') are parallel if and only if m is parallel to l or m' is parallel to l' . This gives us the following:

$$H_p(\mathfrak{A}_i \times \mathfrak{A}'_i) = (Y_i \otimes X'_i) + (Y'_i \otimes X_i) \text{ and } H_p(N_k \times N'_k) = \sum_{i=1}^k [(Y_i \otimes X'_i) + (Y'_i \otimes X_i)],$$

$$\text{and } \dim H_p(N_k \times N'_k) = \dim \left(\sum_{i=1}^k Y_i \otimes X'_i \right) + \dim \left(\sum_{i=1}^k Y'_i \otimes X_i \right) - \dim \left(\sum_{i=1}^k Y_i \otimes Y'_i \right).$$

3. s -Transversals.

A great deal of work has been done studying transversals and partial transversals to Latin squares. See chapter 8 of [3] for an explanation of what has been done. Much less attention has been given to s -transversals for $s > 1$.

One of the most interesting conjectures concerning transversals was made by Ryser. He conjectured that every Latin square of odd order has

a transversal. This remains unproven.

A similar conjecture was made by Peter Rodney, [10]. He conjectured that any Latin square has a 2-transversal, (in our definition that means that the corresponding 3-net has a 2-transversal). Later this was revised to conjecture that every Latin square can be partitioned into disjoint 2-transversals with a single transversal left over in the case of n odd.

The author and Jeanette Janssen conjectured the following:

Conjecture. *Any s -transversal to a Latin square can be partitioned into an a -transversal and a b -transversal for some a and b with $a + b = k$, if $k > 2$.*

This final conjecture implies both Ryser's and Rodney's conjectures, since a Latin square is an n -transversal of itself.

A Latin square can have a 2-transversal and not have a transversal. For example, the circulant Latin square of order 4 has no transversal, but it is easy to find a 2-transversal.

Lemma 3.1. *An affine plane of order n has no k -transversals, for $1 \leq k < n$.*

Proof. Assume a plane has a k -transversal S with $1 \leq k < n$. Pick a point q of S . Through q there are $n + 1$ lines and through q and any other point in S there is a line. If none of these lines has more than k points from S on it, then at most it can have $k - 1$ points other than q from S . This gives a total of

$$1 + (n + 1)(k - 1) = 1 + nk + k - n - 1 = nk - (n - k)$$

points on S . So if k is less than n we have a contradiction. When k is equal to n the k -transversal is simply all the points of the plane. \square

We shall show how s -transversals appear in the codes of nets.

Let v be the characteristic function of an s -transversal to a k -net N_k . We have that $v \in H_p(N_k)^\perp$ since for any two parallel lines l and m we have $[v, l - m] = [v, l] - [v, m] = s - s = 0$. If p divides s , then $v \in C_p(N_k)^\perp$ since for any line l we have $[v, l] = s$ and $s = 0$ if p divides s . For example, if we take a 3-net corresponding to a Latin square of even order, then the characteristic functions of all its s -transversals are in $Hull_2(N_3)^\perp$ and the characteristic functions of all its s -transversals for s -even are in $C_2(N_3)^\perp$ which is a subspace of $H_2(N_3)^\perp$.

We define a *constant vector* to be a vector that is a scalar multiple of a vector consisting of 1's and 0's.

Theorem 3.1. *Let θ be a constant vector of weight sn in $H_p(N_k)^\perp$ with N_k a k -net of order $n = p$, with p a prime. Then θ is the sum of s parallel lines or it is an s -transversal.*

Proof. Normalize θ so that it is made up of 0's and 1's. For l and m parallel we have:

$$0 = [\theta, l - m] = [\theta, l] - [\theta, m]$$

giving that

$$|\theta \wedge l| \equiv |\theta \wedge m| \pmod{p}$$

for l and m parallel, where $|\theta \wedge l|$ indicates the number of places where θ and l both have a 1. If $|\theta \wedge l| \equiv 0 \pmod{p}$ then $|\theta \wedge l| = 0$ or n for any line in the parallel class, giving that θ is the sum of s lines in this class. If $|\theta \wedge l| \equiv r \pmod{p}$, with $0 < r < n$ then $|\theta \wedge l| = r$ for all lines in the class since $|\theta \wedge l|$ cannot be bigger than n since there are only n points on a line. Then $rn = sn$ giving that $r = s$ and that θ is an s -transversal. \square

This allows for the number of s -transversals to be read off the complete weight enumerator of $H_p(N_k)^\perp$, when the order of the net is p .

Let l_1, l_2, \dots, l_n be any parallel class, then

$$\sum_{i=2}^n (l_1 - l_i) = -\mathbf{j}$$

where \mathbf{j} is the all one vector, hence $\mathbf{j} \in H_p(N_k)$ for any k . So if v is a constant vector in $H_p(N_k)^\perp$, then p must divide the weight of v , where the weight is the number of non-zero coordinates of the vector, since $0 = [v, \mathbf{j}] = \sum v_i$. This gives that the weight of a constant vector must be of the form rp . Note, in comparison, that for a projective plane the all one vector is not in the Hull so that the weight of a constant vector in $Hull_p(\Pi)^\perp$ need not have weight divisible by p . If $n = p$, then with the above, we have the following:

Corollary 3.1. *If v is a constant vector in $H_p(N_k)^\perp$, where N_k is a k -net of order $n = p$, then v has weight sn for some s and it is the characteristic function of an s -transversal.*

Proof. Follows from the above and Theorem 3.1. \square

The importance of this corollary is that there must be sufficient s -transversals to extend the net. Namely s transversals must be parallel to form an s -transversal (of course there are s -transversals that are not of this form). Hence, we need sufficient numbers of s -transversals for a set of transversals to be resolvable into a parallel class.

Corollary 3.2. *Let π be an affine plane of prime order p . The constant weight vectors of $Hull_p(\pi)^\perp = C_p(\pi)$ are sums of lines and have weight kn for some k .*

Proof. The fact that $Hull_p(\pi)^\perp = C_p(\pi)$ in this case follows from the order being prime, see [1]. The rest follows from Lemma 3.1 and the Corollary 3.1. \square

As an example of this corollary consider the complete weight enumerator of the affine plane of order 3, where there are $A_{a,b,c}$ vectors with a 0's, b 1's, and c 2's. Note that the number of non-trivial constant weight vectors for a given weight is $12 = 4\binom{3}{1} = 4\binom{3}{2}$.

The complete weight enumerator of the affine plane of order 3

$A_{a,b,c}$	a	b	c
1	9	0	0
12	6	3	0
12	6	0	3
54	4	4	1
108	5	2	2
54	4	1	4
12	3	6	0
108	2	5	2
168	3	3	3
12	3	0	6
108	2	2	5
54	1	4	4
12	0	6	3
1	0	9	0
12	0	3	6
1	0	0	9

Lemma 3.2. *Let n be even and set $p = 2$. Let N_k be a k -net of order n with $\mathfrak{A}_k = \{l_1^k, l_2^k, \dots, l_n^k\}$. Assume $\sum \alpha_i^k l_i^k \in C_p(N_{k-1})$; where N_{k-1} is any $(k - 1)$ -subnet of N_k , with the following relation:*

$$\sum \alpha_i^k l_i^k + \sum \alpha_i^{k-1} l_i^{k-1} + \dots + \sum \alpha_i^1 l_i^1 = 0.$$

Let α^j be the number of α_i^j which are 1, that is $\alpha^j = |\{\alpha_i^j | \alpha_i^j = 1\}|$. If α^j is odd for any j then $\dim C_p(N_k) - \dim H_p(N_k) < k$, and therefore N_k has no s -transversals for s odd, and does not extend.

Proof. Note that $C_2(N_k) = \langle l_1^1, \dots, l_1^k, H_2(N_k) \rangle$. For all α^j odd, take one line with non-zero coefficient out of the summation, and arrange it so that it is l_1^j . We have:

$$\sum_{\alpha^j \text{ odd}} l_1^j = \sum_{i=2}^n \alpha_i^k l_i^k + \dots + \sum_{i=2}^n \alpha_i^1 l_i^1$$

where all the weights in the summations are now even and so the right side is in $H_2(N_k)$. We now have a non-trivial linear combination of $\{l_1^1, \dots, l_1^k\}$ in $H_2(N_k)$ and so $\dim C_2(N_k) - \dim H_2(N_k) < k$ and hence, by Theorem 2.2, N_k does not have any s -transversals for s odd. \square

The following Lemma appears in [7] and [4].

Lemma 3.3. *Let $n \equiv 2 \pmod{4}$; then $n = 2m$ with m odd. Let N_3 be a 3-net of order n with the following parallel classes: $\mathfrak{A}_1 = \{l_1, \dots, l_n\}$, $\mathfrak{A}_2 = \{m_1, \dots, m_n\}$, and $\mathfrak{A}_3 = \{t_1, \dots, t_n\}$. If $\sum \alpha_i t_i \in C_2(N_2)$, where $N_2 = \mathfrak{A}_1 \cup \mathfrak{A}_2$, then $wt(\alpha_1, \dots, \alpha_n)$ is n , 0, or $\frac{n}{2} = m$.*

Theorem 3.2. *Let N_3 be a 3-net of order $n \equiv 2 \pmod{4}$. If $\dim C_2(N_3) - \dim C_2(N_2) < n - 1$, then N_3 does not have any s -transversals for s odd.*

Proof. We note that by Theorem 2.2, $\dim C_2(N_2) - \dim H_2(N_2) = 2$ since $k = 2$ and $2 \not\equiv 1 \pmod{2}$. Now suppose that $w = \alpha_2(t_1 + t_2) + \dots + \alpha_n(t_1 + t_n) \in H_2(N_2) \subseteq C_2(N_2)$. Write

$$w = \left(\sum_{i=2}^n \alpha_i \right) t_1 + \alpha_2 t_2 + \dots + \alpha_n t_n$$

then, when $\sum_{i=2}^n \alpha_i = 1$ an odd number of $\alpha_2, \dots, \alpha_n$ are 1 and so

$\text{wt}(\sum_{i=2}^n \alpha_i, \alpha_2, \dots, \alpha_n)$ is even and when $\sum_{i=2}^n \alpha_i = 0$ an even number of $\alpha_2, \dots, \alpha_n$ are 1 and again $\text{wt}(\sum_{i=2}^n \alpha_i, \alpha_2, \dots, \alpha_n)$ is even. Thus $\text{wt}(\sum_{i=2}^n \alpha_i, \alpha_2, \dots, \alpha_n)$ is even and by the previous lemma the weight is either 0 or n .

Since $t_1 + t_2, \dots, t_1 + t_n$ generate $H_2(N_3)$ over $H_2(N_2)$, we have shown that $\dim H_2(N_3) - \dim H_2(N_2) = n - 2$. Now, if $\dim C_2(N_3) - \dim C_2(N_2) < n - 1$ then $\dim C_2(N_3) - \dim H_2(N_3) \neq 3$ and hence by Theorem 2.2, N_3 does not even have any s -transversals for s odd. \square

Therefore if $\dim C_2(N_3) < 3n - 2$, the net does not have any s -transversals for s odd, since $\dim C_2(N_3) = n + n - 1 + \dim(C_2(N_3)) - \dim(C_2(N_2))$. For a similar result involving 1-transversals in terms of loops see Moorhouse [7].

This produces a large class of Latin squares that have no s -transversals for s odd. A subset of this class are those Latin squares which represent the multiplication table of a group of order $n \equiv 2 \pmod{4}$.

4. Linear Combinations of Lines.

Let Π be a projective plane of order n , with P representing the point set, L representing the line set and I the incidence relation between them, and let $\Sigma = (L, P, I)$ be the dual plane. Take $\pi = \Pi^{L_\infty}$ to be an affine part of Π with $N_1 \subset N_2 \subset \dots \subset N_n \subset N_{n+1} = \pi$, where N_k is formed by removing $(n + 1 - k)$ parallel classes from π . For this section the lines of the i -th parallel class \mathfrak{A}_i are denoted by $\{l_j^i\}$.

Consider a linear combination of the lines of π summing to $\mathbf{0}$, where the coefficients of the lines in the $(n + 1)$ -st parallel class are 0. That is

$$\sum a_j^i l_j^i = \mathbf{0}$$

where $a_j^i \in F$ and $a_j^{n+1} = 0$ for all j .

Let D be the space whose vectors are the coefficients of such linear combinations, i.e.

$$D = \{(a_j^i) \mid \sum a_j^i l_j^i = \mathbf{0}\}.$$

D is a linear code of length $n^2 + n$ with the coordinates corresponding to the $n^2 + n$ lines in π . The last n coordinates of any vector are 0, these coordinates correspond to the lines in \mathfrak{A}_{n+1} the $(n + 1)$ -st parallel class of π . Since these vectors correspond to linear combinations which are 0 on the coordinates of the $(n + 1)$ -st parallel class, then by Theorem 2.3 we have:

$$\sum_j a_j^i = 0$$

for all i . This implies that the sum of their extension in the projective plane Π is $\mathbf{0}$ as well, since it is 0 at all points on L_∞ , i.e.

$$\sum a_j^i L_j^i = \mathbf{0}$$

where L_j^i is the line in the projective plane Π formed from the line l_j^i in the affine plane π by completing this affine plane to the projective plane Π .

Arrange the coordinates corresponding to the points in Σ so that they are in the same order as their corresponding lines in π , with the last point being L_∞ , the line at infinity in Π . Then if $v = (a_j^i) \in D$, i.e. $\sum a_j^i L_j^i = \mathbf{0}$ then $(v, 0) \in C_p(\Sigma)^\perp$, where $(v, 0)$ is the vector v with a 0 attached as the last coordinate. Let $D' = \{(v, 0) | v \in D\}$; then we have that $D' \subseteq C_p(\Sigma)^\perp$.

The vectors in D' are all 0 on the last $(n + 1)$ points in Σ , after the points are arranged in the manner described above. These $(n + 1)$ points correspond to the n lines in the $(n + 1)$ -st parallel class of π , together with the point of Σ corresponding to L_∞ . These $(n + 1)$ points make a line in Σ . Denote this line as M_∞ and set $\sigma = \Sigma^{M_\infty}$, i.e. σ is the affine plane formed from Σ by removing M_∞ and the points incident with it.

Let B be D' projected on the n^2 coordinates corresponding to the n^2 points of σ . Note that all the vectors in D' were 0 on these coordinates. B is a code of length n^2 whose vectors represent the coefficients of linear combinations of the first n parallel classes of π summing to the zero vector.

Theorem 2.4 states that $Hull_p(\sigma) = \{w | w \in Hull_p(\Sigma) \text{ and } w_Q = 0 \text{ for } Q \text{ on } M_\infty\}$, giving $B \subseteq Hull_p(\sigma)$. Since Π and Σ are dual planes

$\dim C_p(\Pi) = \dim C_p(\Sigma)$. By Theorem 2.4 we have

$$\begin{aligned} \dim C_p(\sigma)^\perp &= n^2 - \dim C_p(\sigma) \\ &= n^2 - (\dim C_p(\Sigma) - 1) \\ &= n^2 - (\dim C_p(\Pi) - 1) \\ &= n^2 - \dim C_p(\pi) \\ &= n^2 - \dim C_p(N_n) \\ &= \dim B \end{aligned}$$

Therefore $B = C_p(\sigma)^\perp$, giving the following theorem:

Theorem 4.1. *Let π be an affine plane of order n with p a prime dividing n , the space of all vectors representing the coefficients of lines in linear combinations of lines in the first n parallel classes of π summing to the zero vector is equal to $C_p(\sigma)^\perp$ where σ is an affine part of the dual plane of the projective completion of π formed by removing the lines corresponding to the lines of the $(n+1)$ -st parallel class of π and the line at infinity of its projective completion.*

Note that only linear combinations that had 0 for coefficients of the lines in the $(n+1)$ -st parallel class were considered; this is an extremely important distinction. In a linear combination involving all $(n+1)$ parallel classes Theorem 2.3 would not apply and we would not have that the sum of the coefficients on a given parallel class was 0, which was critically used in establishing the isomorphism of the codes. The $(n+1)$ -st parallel class often serves as an anomaly in the study of nets as it does here.

If p sharply divides n , then $C_p(\Sigma)^\perp = \text{Hull}_p(\Sigma)$ and $C_p(\sigma)^\perp = \text{Hull}_p(\sigma)$. We know that $\text{Hull}_p(\sigma) = H_p(\sigma)$, that is $\text{Hull}_p(\sigma)$ is generated by differences of parallel lines in σ . It is vital then to determine what form these generators take in B , when p sharply divides n .

Let $t \in \mathfrak{A}_{n+1}$, the $(n+1)$ -st parallel class of π , and let q_1, q_2 be points on t . Then, if \mathbf{j} denotes the all one vector, we have:

$$t = \mathbf{j} - \sum l_i, \quad l_i \perp q_1, \quad l_i \in \mathfrak{A}_i$$

and

$$t = \mathbf{j} - \sum m_i, \quad m_i \perp q_2, \quad m_i \in \mathfrak{A}_i$$

This gives:

$$\mathbf{j} - \sum l_i = \mathbf{j} - \sum m_i$$

$$\sum l_i = \sum m_i$$

$$\sum l_i - \sum m_i = 0$$

This linear combination can be expressed as follows; take any two points q and r such that the line through them is in the $(n + 1)$ -st parallel class of π . Note that the lines in σ corresponding to these points are parallel since their extensions meet at M_∞ in Σ . Let v be the vector in B corresponding to this linear combination, we have for $l \in N_n$:

$$v(l) = \begin{cases} 1, & \text{if } l \text{ is incident with } q \\ -1, & \text{if } l \text{ is incident with } r \\ 0, & \text{otherwise} \end{cases}$$

We see that v has weight $2n$ and as viewed as a vector in $Hull_p(\sigma)$ is the difference of parallel lines in the first n parallel classes of σ , and is a generator for $Hull_p(\sigma)$.

Differences of parallel lines in the $(n + 1)$ -st parallel class in σ correspond to linear combinations of the form:

$$\sum s_i - \sum t_i = \mathbf{j} - \mathbf{j} = \mathbf{0}$$

where $\mathfrak{A}_\alpha = \{s_i\}$ and $\mathfrak{A}_\beta = \{t_i\}$ for $1 \leq \alpha, \beta \leq n$.

We have seen that $H_p(N_n) = H_p(N_{n+1}) = Hull_p(\sigma)$ and so it suffices to take generators of $Hull_p(\sigma)$ as differences of parallel lines in the first n classes of σ . Hence the following:

Theorem 4.2. *The space of all vectors of coefficients of lines in linear combinations of lines of the first n parallel classes of an affine plane of order n , summing to the zero vector over F_p , with p sharply dividing n , is generated by the vectors of coefficients of linear combinations of the form $\sum l_i - \sum m_i$ where $l_i, m_i \in \mathfrak{A}_i$ $1 \leq i \leq n$, and $l_i I_{q_1}, m_i I_{q_2}$ where the line through the points q_1, q_2 is in the $(n + 1)$ -st parallel class.*

Proof. Theorem 4.1 gives the space of linear combinations summing to $\mathbf{0}$ is equal to $Hull_p(\sigma)$ and Theorem 2.4 gives that $Hull_p(\sigma)$

is generated by differences of parallel lines, i.e. $Hull_p(\sigma) = \langle l - m \mid l \text{ and } m \text{ are parallel lines in } \sigma \rangle$. The above discussion gives the rest. \square

Consider the case when $n \equiv 2 \pmod{4}$. Here $Hull_2(\sigma)$ is a self-orthogonal code generated by differences of lines which have weight $2n$ and therefore are doubly-even. Hence, all the vectors in $Hull_2(\sigma)$ are doubly even giving the following:

Theorem 4.3. *Let π be an affine plane of order $n \equiv 2 \pmod{4}$. A linear combination of lines in the first n parallel classes of π summing to the zero vector must involve a doubly-even number of lines. That is, if $\sum_{j=1}^n \sum_{i=1}^n a_i^j l_i^j = \mathbf{0}$ where $\{l_1^j, l_2^j, \dots, l_n^j\}$ are the lines of the j -th parallel class, then the number of non-zero a_i^j is divisible by 4.*

This places a strong configurational constraint on any affine plane of order $n \equiv 2 \pmod{4}$. Namely, given a set of lines from all but the last parallel class, such that through any point on the plane there are evenly many lines from the set through that point, then the cardinality of the set is divisible by 4.

5. Hyperovals.

In this section we shall restrict our interest to planes of even order, and set $p = 2$.

In a projective plane Σ of even order a hyperoval is a set of $n+2$ points with no three collinear. It follows immediately that any line in Σ is either secant to the hyperoval or disjoint from the hyperoval. When the line at infinity is fixed we refer to those hyperovals that are secant to the line at infinity as hyperbolic hyperovals and to those that are disjoint from the line at infinity as elliptic hyperovals. If O is a hyperoval in Σ then $O \in C_p(\Sigma)^\perp$, and if the hyperoval is elliptic, then by Lemma 2.2, $O \in C_p(\sigma)^\perp$.

We show an example of the importance of hyperovals. Assume there is a plane of order $n = 6$ with $p = 2$ and that there is a non-trivial linear combination of the first four parallel classes:

$$\sum a_i^4 l_i^4 + \sum a_i^3 l_i^3 + \sum a_i^2 l_i^2 + \sum a_i^1 l_i^1 = \mathbf{0}$$

Let a^j denote the number of a_i^j which are non-zero. If $a^j = 6$ for any j , then by adding the all-one vector the appropriate number of times we can change this value to 0. Hence we can assume $a^j < 6$ for all j . If $a^j = 0$ then

we would have a linear combination of only three parallel classes which by Theorem 3.2 would imply that there could not be a 4-net. This gives that $0 < a^j < 6$ for all j . By adding the all-one vector the appropriate number of times it can be arranged that $a^1 = a^2 = a^3 = 2$ and a^4 is either 2 or 4. We have seen that the coefficients of this linear combinations are vectors in $C_2(\sigma)^\perp$ which, since 2 sharply divides 6, is equal to $Hull_2(\sigma)$. Thus $C_2(\sigma)^\perp$ contains only doubly-even weight vectors. Hence $a^4 \neq 4$. In [2] it is shown that there cannot be an hyperoval in a (non-existent) plane of order 6, giving that $a^4 \neq 2$ since this would correspond to a hyperoval in σ . The dimension of $C_2(\pi)$ must be $\frac{6(7)}{2} = 21$ but the above gives that it must be $6 + 5 + 5 + 5 + (\dim C_2(N_5) - \dim C_2(N_4)) + 1 > 22$. Hence there can be no plane of order 6. Of course, this can be shown many ways but it is instructive to see the role the hyperovals can play in this setting. A similar argument can show the non-existence of two mutually orthogonal Latin squares of order 6, i.e. the non-existence of a 4-net of order 6, see [5], or [11] where it is done in a different setting. See [6] for the role the non-existence of a hyperoval in a possible plane of order 10 played in the computation showing the non-existence of a plane of order 10.

A natural question arises as to whether a possible plane of order $n \equiv 2 \pmod{4}$ can have a hyperoval. This can be decided by showing that the minimum weight of $Hull_2(\sigma)$ must be greater than $n + 2$, see [1] and the next lemma which appear there.

Lemma 5.1. *Let Σ be a projective plane of even order n , σ any affine plane contained in Σ . The minimum weight of $C_2(\Sigma)^\perp$ and $C_2(\sigma)^\perp$ is at least $n+2$ and if this is the minimum weight of the codes, then the minimum weight vectors are the characteristic functions of hyperovals.*

Proof. Let v be a vector in $C_2(\Sigma)^\perp$, and let q be a point such that $v_q = 1$. There are $n + 1$ lines through q , and on each of these lines there exists a point r with $v_r = 1$ since v is orthogonal to each of these lines. It is easy to see that this configuration is an hyperoval.

The proof works for $C_2(\sigma)^\perp$ since there are $n + 1$ lines through any point in σ as well. \square

This does not guarantee the existence of hyperovals; it only states that if there are hyperovals then they are the minimum weight vectors.

This Lemma is useful in determining how many hyperovals there are in a plane when the weight enumerator of the code is known, by applying

the MacWilliams equations. For example, the weight enumerator of the unique affine and projective planes of order 2 and 4 are known. It is then a simple matter to determine that in the projective plane of order 2, there are 7 hyperovals, and 1 hyperoval in the affine plane. In the projective plane of order 4 there are 168 hyperovals, and 48 in the affine plane. Note then for these planes, given any line in the projective plane as the line at infinity the number of hyperbolic and elliptic ovals remain unchanged; that is the number of ovals which are hyperbolic and elliptic in these planes is independent of the choice of the line at infinity.

We shall denote the space generated by the hyperovals of the projective plane by $C_p(H)$ and the space generated by the hyperovals of the affine plane as $C_p(h)$.

For n even and $p = 2$ both $Hull_p(\Sigma)$ and $Hull_p(\sigma)$ are self-orthogonal codes generated by doubly-even vectors, and as a consequence have only doubly-even vectors. For $n = 0 \pmod 4$ which accounts for all known even planes except for $n = 2$ the hyperovals cannot be in the Hull, since $n + 2$ would be $2 \pmod 4$. At $n = 2$ the hyperovals are actually differences of lines in the projective plane and differences of parallel lines in the affine plane, and are in fact generators of their respective Hulls. This gives the following relationships for all known planes:

$$Hull_p(\Sigma) \subseteq C_p(H) \subseteq C_p(\Sigma)^\perp$$

$$Hull_p(\sigma) \subseteq C_p(h) \subseteq C_p(\sigma)^\perp$$

with the first inequality being strict for all $n > 2$.

In [9], Pott shows the following:

Theorem 5.1. *The characteristic vectors of hyperovals in an abelian projective plane Σ of even order generate $C_2(\Sigma)^\perp$.*

He also shows that the dual of the code of the desarguesian planes of even order is generated by the orbit of a hyperoval (coming from a conic) under a Singer cycle.

Lemma 5.2. *In a Singer cycle of a hyperoval in a desarguesian projective plane of even order there are $\frac{(n+1)(n+2)}{2}$ hyperovals that are hyperbolic and $\frac{n(n-1)}{2}$ hyperovals that are elliptic for any choice of L_∞ .*

Proof. Let O be a hyperoval in Π , where Π is the desarguesian plane

of even order. Let $\alpha^i O$ denote the image of a hyperoval in a Singer cycle with $\alpha^0 O = O$. Each point on O has $(n + 1)$ values of i such that α^i sends it to a point on L_∞ , but each of these i actually sends two points of O to L_∞ since any line is either disjoint or secant to a hyperoval.

Hence there are $\frac{(n+1)(n+2)}{2}$ hyperovals that are hyperbolic in a Singer cycle, leaving $n^2 + n + 1 - \frac{(n+1)(n+2)}{2} = \frac{n(n-1)}{2}$ hyperovals that are elliptic in a Singer cycle. \square

In comparison with Pott's theorem, there are not enough elliptic hyperovals in a Singer cycle to generate $C_2(\pi)^\perp$ with $\pi = \Pi^{L_\infty}$, nor are there enough hyperbolic hyperovals in a Singer cycle to generate $C_2(\Pi)^\perp$ alone. For example at $n = 8$ there are 28 elliptical hyperovals in a Singer cycle and $\dim C_2(\pi)^\perp = 37$. If $n = 16$ there are 153 hyperbolic hyperovals in a Singer cycle and $\dim C_2(\Pi)^\perp = 191$.

This does not say whether hyperbolic or elliptic hyperovals generate these codes but only that it does not occur within a single Singer cycle.

Let O be an hyperoval in Σ disjoint from M_∞ the line at infinity of Σ . Let q_1, q_2, \dots, q_{n+2} be the points on the hyperoval and let Q be the point in Σ corresponding to L_∞ the line at infinity of Π . We note that Q is incident with M_∞ by construction. If we take a line from Q to the points on O we see that each line is secant to O and the points of O can be arranged into $\frac{n+2}{2}$ classes with each class containing two points. By renaming we have $\{q_1, q_2\}, \{q_3, q_4\}, \dots, \{q_{n+1}, q_{n+2}\}$ where the line through the two points in a set is incident with Q .

If t_i is the line of π corresponding to q_i , then we have $\sum t_i = 0$. This is a linear combination of lines involving $\frac{n+2}{2}$ parallel classes with 2 lines from each of these classes. This hyperoval-like linear combination is the smallest possible linear combination (i.e. fewest non-zero coefficients) of lines summing to 0 by Lemma 5.1. If $v \in C_2(\sigma)^\perp$ then there exists $w \in C_2(\Sigma)^\perp$ with w having a 0 on the coordinates of L_∞ and w projecting to v on the remaining n^2 coordinates. Then we have $w = \sum h_i$ where h_i is a hyperoval of Σ . This gives the following:

Theorem 5.2. *The vector of coefficients of linear combinations of lines in a N_n contained in a plane π , where the dual of the projective completion of π is an abelian plane of even order, is of the form $\sum h_i^*$ with h_i^* the projection of h_i , a hyperoval of Σ , on the n^2 coordinates corresponding to the lines of N_n , where $\sum h_i$ is 0 on the points of line at infinity of the*

dual plane. Namely, it is the sum of elliptic hyperovals and the sum of hyperbolic hyperovals summing to 0 on the points of the line at infinity.

Proof. Lemma 2.2 gives the relationship between the vectors of linear coefficients and the orthogonal of the dual. Theorem 4.1 and Theorem 5.1 give that the dual is generated by the characteristic vectors of the hyperovals. \square

In [8] Peeters computes all possible p -ranks of desarguesian nets of order 4,8, and 16 where a desarguesian net is a net contained in a desarguesian affine plane. We also include the computation for order 2 which is trivial.

	$n = 2$	$n = 4$	$n = 8$	$n = 16$
k				
17				81
16				81
15				80
14				79
13				78
12				77
11				76
10				75
9			27	74
8			27	73
7			26	68
6			25	63
5		9	24	56; 58
4		9	23	51; 53
3	3	8	19	42
2	3	7	15	31
1	2	4	8	16

It is natural to conjecture that, for desarguesian planes of order $n = 2^r$, $\dim C_2(N_k) - \dim C_2(N_{k-1}) = 1$ for $n \geq k \geq \frac{n}{2} + 1$, i.e. that $H_2(\pi) = H_2(N_{\frac{n}{2}})$. The conjecture would be true if the sum of any two lines in the k -th parallel class, with $k > \frac{n}{2}$ were in $H_2(N_{\frac{n}{2}})$. It would appear that the structure of the hyperovals, specifically of the image of a single hyperoval under a Singer cycle, is the reason for these dimensions.

Conjecture. *If π is a desarguesian plane of even order n then*

$$H_2(N_{\frac{3}{2}}) = H_2(\pi).$$

REFERENCES

- [1] E. F. ASSMUS JR. and J. D. KEY: *Designs and their Codes*, Cambridge University Press, 1992.
- [2] A. BICHARA: *An elementary proof of the nonexistence of a projective plane of order six*, Mitt. Math. Sem. Giessen **192**(1989), 89–93.
- [3] R. A. BRUALDI and H. J. RYSER: *Combinatorial Matrix Theory*, Cambridge University Press, 1991.
- [4] S. T. DOUGHERTY: *Nets and their codes*, Designs, Codes and Cryptography **3**(1993), 315–331.
- [5] S. T. DOUGHERTY: *A Coding-Theoretic Solution to the 36 Officer Problem*, Designs, Codes and Cryptography **4**(1994), 123–128.
- [6] C. W. H. LAM, L. THIEL and S. SWIERCZ: *The non-existence of finite projective planes of order 10*, Canad. J. Math **41**(1989), 117–123.
- [7] G. E. MOORHOUSE: *Bruck nets, codes, and character of loops*, Designs, Codes and Cryptography **1**(1991), 7–29
- [8] R. PEETERS: *On the p-ranks of Net Graphs*, Designs, Codes and Cryptography **5**(1995), 139–153.
- [9] A. POTT: *On Abelian Difference Set Codes*, Designs, Codes and Cryptography **2**(1992), 263–271.
- [10] P. RODNEY: *Private communication*, Vermont Summer Conference, 1994.
- [11] D. R. STINSON: *A short proof of the non-existence of a pair of orthogonal latin squares of order six*, J. Combin. Theory **A36**(1984), 373–376.

S. T. DOUGHERTY
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SCRANTON
SCRANTON, PA 18510

(Received April 28, 1997)