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A REMARK ON TORSION-FREE SUBGROUPS IN GROUP RINGS

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Introduction. Let RG be the group ring of a group G over a commutative ring R with identity. For any normal subgroup N of G we write $\Delta_R(G,N)$ for the kernel of the natural map $RG \to R(G/N)$. Also, for any ideal I of RG, we put $U(1+I) = \{u \in U(RG) \mid u-1 \in I\}$, where $U(RG)$ is the unit group of RG. Note that $U(1+I)$ forms a normal subgroup of $U(RG)$.

We shall denote by TN the set of torsion elements in a group N. The aim of this note is to prove the following theorem which generalizes [2, Theorem B]. The theorem bellow was proved in [2] under the condition that N is periodic or finitely generated.

Theorem. *Let R be an integral domain of characteristic 0 in which no rational prime is invertible. Let N be a nilpotent normal subgroup of a group G and let A be an abelian normal subgroup of G with $N \supseteq A$. Assume that N/TN is finitely generated. Then $U(1+\Delta_R(G,N)\Delta_R(G,A))$ is torsion-free.*

In what follows, unless otherwise stated, R denotes a commutative ring with identity. Also, for simplicity of notation, we omit the subscript R from $\Delta_R(G,N)$, which will be denoted by $\Delta(G,N)$.

1. Preliminaries. For the proof of our theorem we shall establish some lemmas concerning augmentation ideals. Let N be any group and let $\Delta(N)$ denote the augmentation ideal of RN. Let L be a subgroup of N, and T a right transversal for L in N with $T \ni 1$. Then each element g of N can be written uniquely in the form $g = tx$, $t \in T$, $x \in L$ and so, setting $\theta(g) = x$, we have an R-linear map $\theta : RN \to RL$. It is easy to see that θ is a right RL-homomorphism and that $\theta(\Delta(N)) = \Delta(L)$. We write $\theta = \theta(N,L,T)$. Recall that the projection map $\pi : RN \to RL$, which is defined by $\pi(\sum_{g\in N}\alpha(g)g) = \sum_{g\in L}\alpha(g)g$, is also a right RL-homomorphism.

Lemma 1.1. *Let C be a normal subgroup of N and $\mathscr{A}$ a nonempty set of normal subgroups of N which contain C. Suppose that $L_1 \cap L_2 \in \mathscr{A}$*

whenever $L_1, L_2 \in \mathscr{A}$. *Then*

$$\bigcap_{L\in\mathscr{A}}(\Delta(N,L)\Delta(N,C)) = \Delta(N,\bigcap_{L\in\mathscr{A}} L)\Delta(N,C).$$

Proof. We have only to verify that the left-hand side is contained in the right-hand side. Let $\alpha \in \bigcap_{L\in\mathscr{A}}(\Delta(N,L)\Delta(N,C))$ and set $M = \bigcap_{L\in\mathscr{A}} L$. Then, choosing a transversal T for M in N, α can be written uniquely in the form $\alpha = \sum_{i=1}^{n} t_i\alpha_i$, $\alpha_i \in RM$, $t_i \in T$. By the property of $\mathscr{A}$, we may take some $L \in \mathscr{A}$ such that $\{t_1^{-1}t_i | 1 \le i \le n\} \cap L = \{1\}$. Then, under the projection map $\pi : RN \to RL$,

$$\alpha_1 = \pi(t_1^{-1}\alpha) \in \pi(\Delta(N,L)\Delta(N,C)) = \Delta(L)\Delta(L,C).$$

Let S be a transversal for M in L with $S \ni 1$, and consider the map $\theta = \theta(L,M,S) : RL \to RM$. Then we have $\alpha_1 = \theta(\alpha_1) \in \theta(\Delta(L)\Delta(L,C)) = \Delta(M)\Delta(M,C)$. Similarly, we see that all α_i's are in $\Delta(M)\Delta(M,C)$ so that $\alpha \in \Delta(N,M)\Delta(N,C)$. This completes the proof.

Lemma 1.2. *Let* $L \supseteq A$ *be normal subgroups of* N *and let* $\mathscr{B}$ *be a nonempty set of normal subgroups of* A. *Suppose that* $\bigcap_{B\in\mathscr{B}}(\Delta(A)^2 + \Delta(A,B)) = \Delta(A)^2$. *Then*

$$\bigcap_{B\in\mathscr{B}}(\Delta(N,L)\Delta(N,A) + RN\Delta(B)) = \Delta(N,L)\Delta(N,A).$$

Proof. We have only to show that the left-hand side is contained in the right-hand side. Take $\alpha \in \bigcap_{B\in\mathscr{B}}(\Delta(N,L)\Delta(N,A) + RN\Delta(B))$ and let T be a transversal for L in N. Then α can be written uniquely as a finite sum of the form $\alpha = \sum_{t\in T} t\alpha_t$ with $\alpha_t \in RL$. Similarly, let us fix $t \in T$ and write $\alpha_t = \sum_{s\in S} s\beta_s$, $\beta_s \in RA$ where S ($\ni 1$) is a transversal for A in L. Then , under the projection map $\pi : RN \to RL$,

$$\alpha_t = \pi(t^{-1}\alpha) \in \pi(\Delta(N,L)\Delta(N,A) + RN\Delta(B)) = \Delta(L)\Delta(L,A) + RL\Delta(B)$$

for any $B \in \mathscr{B}$. Therefore, considering $\theta = \theta(L,A,S) : RL \to RA$,

$$\sum_{s\in S}\beta_s = \theta(\alpha_t) \in \theta(\Delta(L)\Delta(L,A) + RL\Delta(B)) = \Delta(A)^2 + \Delta(A,B)$$

for all $B \in \mathscr{B}$, and hence $\sum_{s\in S}\beta_s \in \Delta(A)^2$ by hypothesis. Since $\alpha_t \in \Delta(L,A)$, $\beta_s \in \Delta(A)$ for all $s \in S$. Thus $\alpha_t \in \Delta(L)\Delta(A)$ because $\alpha_t = \sum_{s\in S}(s-1)\beta_s + \sum_{s\in S}\beta_s$. This is true for any $t \in T$, so $\alpha \in \Delta(N,L)\Delta(N,A)$ and the lemma is proved.

Lemma 1.3. *Let* A *be a nontrivial elementary abelian* p*-group for* *some prime p and let $\mathscr{B}$ be the set of all maximal subgroups of A. Then*

$$\bigcap_{B\in\mathscr{B}}(\Delta(A)^2+\Delta(A,B))=\Delta(A)^2.$$

Proof. We need only to prove the following:

(*) $$\bigcap_{B\in\mathscr{B}}(\Delta(A)^2+\Delta(A,B))\subseteq\Delta(A)^2.$$

To do this, we first assume that A is finite and let $|A|=p^n$. We proceed by induction on n, the case $n=1$ being trivial. Let $n\geq 2$ and assume that (*) holds for $n-1$. Take $\alpha\in\bigcap_{B\in\mathscr{B}}(\Delta(A)^2+\Delta(A,B))$ and let $B\in\mathscr{B}$ so that $|B|=p^{n-1}$. Then clearly there is a subgroup C of A such that $A=B\times C$. Consider here the map $\theta=\theta(A,B,C):RA\to RB$. Then, for any maximal subgroup M of B, $MC\in\mathscr{B}$ so $\alpha\in\Delta(A)^2+\Delta(A,MC)$, and hence $\theta(\alpha)\in\Delta(B)^2+\Delta(B,M)$ because θ is a ring homomorphism with $Ker\ \theta=\Delta(A,C)$. Therefore $\theta(\alpha)\in\Delta(B)^2$ by induction hypothesis. On the other hand, since $\alpha\in\Delta(A)^2+\Delta(A,B)=\Delta(A)^2+\Delta(B)$, we can write $\alpha=\gamma+\beta$, $\gamma\in\Delta(A)^2$, $\beta\in\Delta(B)$, and so $\theta(\alpha)=\theta(\gamma)+\beta$. Thus we obtain $\beta\in\Delta(B)^2$ so that $\alpha\in\Delta(A)^2$ and hence (*) follows here.

Next, assume that A is infinite. Let $\alpha=\sum_{x\in A}\alpha(x)x$ be a nonzero element in $\bigcap_{B\in\mathscr{B}}(\Delta(A)^2+\Delta(A,B))$ and let A_0 be the subgroup of A generated by $\{x\in A\mid\alpha(x)\neq 0\}$. Then, $A=A_0\times C$ for some subgroup C of A. Taking here the map $\theta=\theta(A,A_0,C):RA\to RA_0$, by the same argument as above, we see that $\theta(\alpha)\in\Delta(A_0)^2+\Delta(A_0,M)$ for any maximal subgroup M of A_0. Therefore, since A_0 is finite, the previous case ensures that $\theta(\alpha)\in\Delta(A_0)^2$. Thus $\alpha=\theta(\alpha)\in\Delta(A)^2$ and (*) is established.

Remark. The above result is trivial in case $R=\mathbf{Z}$, the ring of rational integers. For, in this case, it is known that the map $f:A\to\Delta(A)/\Delta(A)^2$ defined by $f(a)=a-1+\Delta(A)^2$ is an isomorphism of abelian groups. Moreover, for any subgroup C of A, $f(C)=(\Delta(A)^2+\Delta(C))/\Delta(A)^2=(\Delta(A)^2+\Delta(A,C))/\Delta(A)^2$. Therefore, $\{\Delta(A)^2+\Delta(A,B)\mid B\in\mathscr{B}\}$ is the set of all maximal subgroups of $\Delta(A)$ which contain $\Delta(A)^2$. Since the Frattini subgroup of $\Delta(A)/\Delta(A)^2$ is 0, it follows that $\bigcap_{B\in\mathscr{B}}(\Delta(A)^2+\Delta(A,B))=\Delta(A)^2$.

Lemma 1.4. *Let N be a nilpotent group such that N/TN is finitely generated and let C be a normal subgroup of N. If C is a p-group of bounded exponent for some prime p, then*

$$\bigcap_{n=1}^{\infty}(\Delta(N)^n\Delta(N,C))\subseteq\Delta(N,TN)\Delta(N,C).$$

Proof. Since N/TN is a finitely generated torsion-free nilpotent group, it is residually 'a finite p-group' (see [3,Theorem 2.1]). Therefore, denoting by $\mathscr{A}$ the set of all normal subgroups L of N such that $L \supseteq TN$ and N/L is a finite p-group, we have $\bigcap_{L\in\mathscr{A}} L = TN$. So it follows from Lemma 1.1 that

$$\bigcap_{L\in\mathscr{A}}(\Delta(N,L)\Delta(N,C)) = \Delta(N,TN)\Delta(N,C).$$

Now, the additive group $\Delta(C)/\Delta(C)^2$ is a p-group of bounded exponent, because $R\bigotimes_{\mathbf{Z}}(C/C') \simeq \Delta(C)/\Delta(C)^2$ (see e.g. [4, p.23]). Thus $p^l\Delta(C) \subseteq \Delta(C)^2$ for some integer $l > 0$. Set $I = \bigcap_{n=1}^{\infty}(\Delta(N)^n\Delta(N,C))$. Let $L \in \mathscr{A}$ and consider the natural map $f : RN \to S(N/L)$ where $S = R/p^lR$. Then, since the augmentation ideal of $S(N/L)$ is nilpotent, $f(\Delta(N)^n) = 0$ for some integer $n > 0$. This implies that $\Delta(N)^n \subseteq \Delta(N,L) + p^lRN$ and hence

$$I \subseteq \Delta(N,L)\Delta(N,C) + p^l\Delta(N,C).$$

Furthermore, since $p^l\Delta(N,C) \subseteq \Delta(N,C)^2$, we obtain $I \subseteq \Delta(N,L)\Delta(N,C)$. This holds for any $L \in \mathscr{A}$ and so $I \subseteq \Delta(N,TN)\Delta(N,C)$ as asserted.

Remark. The above result does not hold in general without the condition that N/TN is finitely generated. For example, take the direct product $N = \mathbf{Q}/\mathbf{Z}\times\mathbf{Q}$, where $\mathbf{Q}$ is the field of rational numbers. Then N is a divisible abelian group such that N/TN is not finitely generated. Let $C = \langle c \rangle$ be a cyclic subgroup of prime order p in N and consider the case when $R = \mathbf{Z}$. Then, for any $g \in N$ there exists $x \in N$ such that $g = x^p$, and thus $g - 1 \equiv (x-1)^p \bmod p\mathbf{Z}N$. So it follows that $\Delta(N) \subseteq \Delta(N)^2 + p\Delta(N)$, and hence $\Delta(N)\Delta(N,C) \subseteq \Delta(N)^2\Delta(N,C)$, since $p\Delta(C) \subseteq \Delta(C)^2$. Consequently, $\Delta(N)\Delta(N,C) = \Delta(N)^n\Delta(N,C)$ for all $n \geq 1$. Therefore, considering an element $\alpha = (x-1)(c-1)$ with $x \in N\setminus TN$, it is sure that $\alpha \in \bigcap_{n=1}^{\infty}(\Delta(N)^n\Delta(N,C))$. However $\alpha \notin \Delta(N,TN)\Delta(N,C)$. For, if $\alpha \in \Delta(N,TN)\Delta(N,C)$, then $\alpha = \beta(c-1)$ for some $\beta \in \Delta(N,TN)$, and so we can write $x - 1 - \beta = \gamma\hat{c}$, $\gamma \in RN$, where $\hat{c} = 1 + c + \cdots + c^{p-1}$. Then under the natural map $\overline{}$: $\mathbf{Z}N \to \mathbf{Z}(N/TN)$, we have $\bar{x} - \bar{1} = p\bar{\gamma}$ and hence $\bar{x} = \bar{1}$ i.e. $x \in TN$, contrary to our choice of x.

Lemma 1.5. *Let N be a nilpotent group such that N/TN is finitely generated and let A be a central subgroup of N. Suppose that A is an elementary abelian p-group for some prime p and that p is not a zero divisor* *in R. Then, for an additive subgroup I of RN,*

$$I \subseteq \Delta(N)\Delta(N,A), \;\; pI \subseteq I^p \;\Rightarrow\; I \subseteq \Delta(N,TN)\Delta(N,A).$$

Proof. The case $A = \{1\}$ being trivial, so let $A \neq \{1\}$. In case $|A| = p$, we have $\Delta(N,A)^p = p\Delta(N,A)$ (see e.g. [1, Lemma 3.4]). Therefore, if $I \subseteq \Delta(N)^n\Delta(N,A)$ then

$$pI \subseteq I^p \subseteq \Delta(N)^{np}\Delta(N,A)^p = p\Delta(N)^{np}\Delta(N,A)$$

and so $I \subseteq \Delta(N)^{np}\Delta(N,A)$, since p is not a zero divisor in R. Thus $I \subseteq \Delta(N)^n\Delta(N,A)$ for all $n \geq 1$ and hence by Lemma 1.4, $I \subseteq \Delta(N,TN)\Delta(N,A)$.

For the general case, let $\mathscr{B}$ be the set of all maximal subgroups of A and take $B \in \mathscr{B}$. Then, under the natural map $\overline{} : RN \to R(N/B)$, $|\overline{A}| = p$ and so the previous case shows that $I \subseteq \Delta(N,TN)\Delta(N,A) + \Delta(N,B)$. Thus

$$I \subseteq \bigcap_{B\in\mathscr{B}}(\Delta(N,TN)\Delta(N,A) + \Delta(N,B)).$$

On the other hand, Lemma 1.3 and Lemma 1.2 ensure that $\bigcap_{B\in\mathscr{B}}(\Delta(N,TN)\Delta(N,A) + \Delta(N,B)) = \Delta(N,TN)\Delta(N,A)$ and hence the lemma is proved.

2. Proof of Theorem. We see that for any normal subgroup L of G,consider $\overline{G}=G/L$, and then $\overline{N}/T\overline{N}$ is finitely generated. Therefore, as in the proof of [2, Theorem B], we may harmlessly assume that A is central in N.

Now, let $u \in TU(1 + \Delta(G,N)\Delta(G,A))$. Then, to show that $u = 1$ we may assume that $u^p = 1$ for some prime p. Let $T_p(A)$ be the set of p-elements in A and consider the natural map $\overline{} : RG \to R(G/T_p(A))$. Then, since $T_p(\overline{A}) = \{1\}$, we know from [2, Lemma 2.3] that $T_p(U(1 + \Delta(\overline{G},\overline{A}))) = \{1\}$. Thus $\overline{u} = 1$ and so it follows from [2, Lemma 1.3] that

$$u - 1 \in \Delta(G,N)\Delta(G,A) \bigcap \Delta(G,T_p(A)) = \Delta(G,N)\Delta(G,T_p(A)).$$

Therefore $u - 1$ can be written in the form

$$u - 1 = \sum_{i=1}^{n} \lambda_i g_i(x_i - 1)(a_i - 1) \quad (\lambda_i \in R,\ g_i \in G,\ x_i \in N,\ a_i \in T_p(A)).$$

Set $B = \langle a_1, \cdots, a_n\rangle$ so that $u - 1 \in \Delta(G,N)\Delta(G,B^G)$, where B^G is the normal closure of B in G. Note here that B^G is of bounded exponent(see [2, Lemma 2.1]). We may therefore assume that A is a p-group of bounded exponent and hence that $A^{p^n} = \{1\}$ for some $n \geq 0$. We proceed by induction on n. The case $n = 0$ is trivial, so let $n \geq 1$ and put $C = A^{p^{n-1}}$. Then, under the natural map $\overline{}$: $RG \to R(G/C)$ we have $\overline{A}^{p^{n-1}} = \{1\}$ and hence $\overline{u} = 1$ by induction. Thus $u - 1 \in \Delta(G,N)\Delta(G,C)$ and consequently we may assmue that A is an elementary abelian p-group.

Set $I = RG(u-1)RG$ and let $\pi : RG \to RN$ be the projection map. Then, since $pI \subseteq I^p$, we obtain $p\pi(I) \subseteq \pi(I)^p$. Furthermore, $\pi(I) \subseteq \Delta(N)\Delta(N,A)$ and so, by virtue of Lemma 1.5, $\pi(I) \subseteq \Delta(N,TN)\Delta(N,A)$. Thus $I \subseteq RG\pi(I) \subseteq \Delta(G,TN)\Delta(G,A)$ and hence we see that $u \in U(1 + \Delta(G,TN)\Delta(G,A))$. So we conclude from [2, Theorem B] that $u = 1$, which completes the proof of the theorem.

REFERENCES

[1] G. H. CLIFF, S. K. SEHGAL and A. R. WEISS: Units of integral group rings of metabelian groups, J. Algebra **73**(1981), 167–185.

[2] T. FURUKAWA: Some torsion-free subgroups in group rings, Math. J. Okayama Univ. **35**(1993), 17–28.

[3] K. W. GRUENBERG: Residual properties of infinite soluble groups, Proc. London Math. Soc. (3) **7**(1957), 29–62.

[4] G. KARPILOVSKY: Unit groups of group rings, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. **47**(1989), Longman.

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