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THETA FUNCTIONS, II

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0 In this paper, we define four kinds of the theta functions following the definition of Jacobi's theta functions. We also define the $\wp$-function $P(u)$ of Weierstrass by our theta functions. We calculate the values $e_p = P(\tau_p)$ and others concretely. We also define the Jacobi elliptic functions newly. The main references are [2] and *Theta functions*, I [4]. Note that we quote from the appendix of [2] simply by [A17], for example.

1 In *Theorie der Elliptischen Funktionen, aus den Eigenshaften der Thetareihen abgeleitet* [1] Jacobi defined four kinds of the theta functions as

$$(1.1)\qquad \begin{cases} \theta(x) = \sum (-1)^\nu q^{\nu^2} e^{2\nu x i} \\ \theta_1(x) = -\sum i^{2\nu+1} q^{\frac{1}{4}(2\nu+1)^2} e^{(2\nu+1)xi} \\ \theta_2(x) = \sum q^{\frac{1}{4}(2\nu+1)^2} e^{(2\nu+1)xi} \\ \theta_3(x) = \sum q^{\nu^2} e^{2\nu x i} \end{cases}$$

where, in the summation, ν runs over the integers set Z([1], p. 501).Here the sign - in the definition of $\theta_1(x)$ was added, so as the function $\theta_1(x)$ to be positive for a real value x in the interval $[0, \pi] \subset$ R, when $q = e^{-\pi t}$ with t real and positive.

We define the four kinds of theta functions $\Theta_p(u) = \Theta_p(u, \tau)$ $(p \in \{1,2,3,4\})$, as follows, with slight changes in two variables x and q.

$$(1.2)\begin{cases} \Theta_1(u) = -\sum i^{2\nu+1} q^{\frac{1}{4}(2\nu+1)^2} e^{(\nu+\frac{1}{2})\pi i u} = -i\sum (-1)^\nu x^{\nu+\frac{1}{2}} q^{\nu^2+\nu+\frac{1}{4}} \\ \Theta_2(u) = \sum q^{\frac{1}{4}(2\nu+1)^2} e^{(\nu+\frac{1}{2})\pi i u} = \sum x^{\nu+\frac{1}{2}} q^{\nu^2+\nu+\frac{1}{4}} \\ \Theta_3(u) = \sum q^{\nu^2} e^{\nu\pi i u} = \sum x^\nu q^{\nu^2} \\ \Theta_4(u) = \sum (-1)^\nu q^{\nu^2} e^{\nu \pi i u} = \sum (-1)^\nu x^\nu q^{\nu^2} \end{cases}$$

where τ is a complex number with $\mathrm{Im}(\tau) > 0$, and $q = e^{\pi i \tau}$ and

$$(1.3)\qquad x = e^{\pi i u} \quad for\ u \in \mathrm{C}$$

Now, from the definition of the function $\theta(x,q)$:

$$\theta(x,q)=\sum_{n\in Z}x^nq^{n^2} \tag{1.4}$$

we can rewrite the definition of the function $\Theta_p(u,\tau)=\Theta_p(u)$ as follows.

$$\begin{cases}\Theta_1(u)=-i\varphi(u)\theta(-xq,q)\\ \Theta_2(u)=\varphi(u)\theta(xq,q)\\ \Theta_3(u)=\theta(x,q)\\ \Theta_4(u)=\theta(-x,q)\end{cases} \tag{1.5}$$

where

$$\varphi(u)=x^{\frac{1}{2}}q^{\frac{1}{4}}=e^{\frac{\pi iu}{2}}e^{\frac{\pi i\tau}{4}} \tag{1.6}$$

From the triple product theorem for the function $\theta(x,q)$(A17), we can derive the followings.

$$\begin{cases}\Theta_1(u)=2q^{\frac{1}{4}}I\sin\frac{\pi u}{2}\ \prod_{\nu=1}^{\infty}\{(1-xq^{2\nu})(1-x^{-1}q^{2\nu})\}\\ \Theta_2(u)=2q^{\frac{1}{4}}I\cos\frac{\pi u}{2}\ \prod_{\nu=1}^{\infty}\{(1+xq^{2\nu})(1+x^{-1}q^{2\nu})\}\\ \Theta_3(u)=I\ \prod_{\nu=1}^{\infty}\{(1+xq^{2\nu-1})(1+x^{-1}q^{2\nu-1})\}\\ \Theta_4(u)=I\ \prod_{\nu=1}^{\infty}\{(1-xq^{2\nu-1})(1-x^{-1}q^{2\nu-1})\}\end{cases} \tag{1.7}$$

where $I=\prod_{\nu=1}^{\infty}(1-q^{2\nu})$ is the factor which depends only on the variable τ. Note that we use the formula; $-i(x^{\frac{1}{2}}-x^{\frac{-1}{2}})=2\sin\frac{\pi u}{2}$ and $x^{\frac{1}{2}}+x^{\frac{-1}{2}}=2\cos\frac{\pi u}{2}$.

From the definition (1.2) or the formula (1.7), it follows that

Theorem 1.1. *The theta function $\Theta_p(u,\tau)$ satisfies the following heat equation.*

$$\frac{\partial\Theta_p(u,\tau)}{\partial\tau}=\pi i\frac{\partial^2\Theta_p(u,\tau)}{\partial u^2} \tag{1.8}$$

Theorem 1.2. *The function $\Theta_1(u)$ is an odd function in the variable u, and $\Theta_p(u)$ ($p\in\{2,3,4\}$) are even functions in the variable u.*

We use the notations

$$\theta_p^{(\nu)}=\theta_p^{(\nu)}(\tau)=\frac{\partial^\nu\Theta_p(u,\tau)}{\partial u^\nu}|_{u=0} \tag{1.9}$$

for $\nu\in N$. If $\nu=0$ or 1, then we denote $\theta_p=\theta_p(\tau)$ or $\theta_p'=\theta_p'(\tau)$. From

Theorem 1.2 it follows that

$$\theta_p^{(2\nu+1)}(\tau) = 0 \quad \text{for } p \in \{2,3,4\} \tag{1.10}$$

and that

$$\theta_1^{(2\nu)}(\tau) = 0. \tag{1.11}$$

2 From the elementary formulae (A.9∼11) we can deduce

$$\begin{cases} \Theta_1(u+1) = \Theta_2(u) \\ \Theta_2(u+1) = -\Theta_1(u) \\ \Theta_3(u+1) = \Theta_4(u) \\ \Theta_4(u+1) = \Theta_3(u) \end{cases} \tag{2.1}$$

$$\begin{cases} \Theta_1(u+\tau) = \frac{i}{\varphi(u)}\Theta_4(u) \\ \Theta_2(u+\tau) = \frac{1}{\varphi(u)}\Theta_3(u) \\ \Theta_3(u+\tau) = \frac{1}{\varphi(u)}\Theta_2(u) \\ \Theta_4(u+\tau) = \frac{i}{\varphi(u)}\Theta_1(u) \end{cases} \tag{2.2}$$

so that

$$\begin{cases} \Theta_1(u+\tau+1) = \frac{1}{\varphi(u)}\Theta_3(u) \\ \Theta_2(u+\tau+1) = \frac{-i}{\varphi(u)}\Theta_4(u) \\ \Theta_3(u+\tau+1) = \frac{i}{\varphi(u)}\Theta_1(u) \\ \Theta_4(u+\tau+1) = \frac{1}{\varphi(u)}\Theta_2(u) \end{cases} \tag{2.3}$$

and that

$$\begin{cases} \Theta_1(u+2) = -\Theta_1(u) \\ \Theta_2(u+2) = -\Theta_2(u) \\ \Theta_3(u+2) = \Theta_3(u) \\ \Theta_4(u+2) = \Theta_4(u) \end{cases} \tag{2.4}$$

$$\begin{cases} \Theta_1(u+2\tau) = -(xq)^{-1}\Theta_1(u) \\ \Theta_2(u+2\tau) = (xq)^{-1}\Theta_2(u) \\ \Theta_3(u+2\tau) = (xq)^{-1}\Theta_3(u) \\ \Theta_4(u+2\tau) = -(xq)^{-1}\Theta_4(u) \end{cases} \tag{2.5}$$

The formulae (2.4) and (2.5) show the *quasi* double periodicity of the function $\Theta_p(u)=\Theta_p(u,\tau)$ for the parallelotope $[2,\ 2\tau]=\{\alpha+\tau\beta; 0 \le \alpha,\beta \le 2\}$. Here we construct the table of the special values of our functions $\Theta_p(u)$:

This table and the refined table are fundamental in our discussion. From (1.7) we see that there exists only one zero point for each $\Theta_p(u)$ in the parallelotope $[2,\ 2\tau]$.

$\Theta\setminus u$	0	1	$\tau+1$	τ
Θ_1	0	θ_2	$\varepsilon\theta_3$	$i\varepsilon\theta_4$
Θ_2	θ_2	0	$-i\varepsilon\theta_4$	$\varepsilon\theta_3$
Θ_3	θ_3	θ_4	0	$\varepsilon\theta_2$
Θ_4	θ_4	θ_3	$\varepsilon\theta_2$	0

Table 1: (with $\varepsilon = q^{-\frac{1}{4}}$)

3 In the formula (1.7), letting $u = 0$ (so $x = 1$), we have

$$(3.1)\qquad \begin{cases} \theta_1 = \theta_1(\tau) = 0 \\ \theta_2 = q^{\frac{1}{4}}\ \theta(q,q) = 2q^{\frac{1}{4}} I \prod_{\nu=1}^{\infty}(1+q^{2\nu})^2 \\ \theta_3 = \theta(1,q) \quad = I\prod_{\nu=1}^{\infty}(1+q^{2\nu-1})^2 \\ \theta_4 = \theta(-1,q) \quad = I\prod_{\nu=1}^{\infty}(1-q^{2\nu-1})^2 \end{cases}$$

As $\prod\{(1-q^{2\nu})(1+q^{2\nu})(1+q^{2\nu-1})(1-q^{2\nu-1})\} = \prod(1-q^{2\nu})$, we have

$$(3.2)\qquad \theta_2\theta_3\theta_4 = 2q^{\frac{1}{4}}\prod_{\nu=1}^{\infty}(1-q^{2\nu})^3 = 2\eta(\tau)^3$$

Now we consider the limit $\lim_{u\to 0}\Theta_1(u)/u$ which is equal to $\Theta_1'(0)=\theta_1'=\theta_1'(\tau)$, so

$$(3.3)\qquad \theta_1' = \pi q^{\frac{1}{4}} I^3 = \pi\eta(\tau)^3$$

Thus $\theta_1'(\tau) \neq 0$ for any $\tau \in \mathrm{C}$ with $\mathrm{Im}(\tau) > 0$, and

$$(3.4)\qquad \theta_1'(\tau) = \frac{\pi}{2}\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)$$

By the logarithmic differentiation with regard to τ, noting (1.8), we have

$$(3.5)\qquad \frac{\theta_1'''}{\theta_1'} = \frac{\theta_2''}{\theta_2} + \frac{\theta_3''}{\theta_3} + \frac{\theta_4''}{\theta_4}$$

Also by the logarithmic differentiation of (3.3) with regard to τ, we have

$$(3.6)\qquad \frac{\theta_1'''}{\theta_1'} = 3\pi i\frac{\dot\eta}{\eta}$$

where we denote $d\eta/d\tau$ by $\dot\eta = \dot\eta(\tau)$. The logarithmic differentiation of

$$\eta(\tau) = e^{\frac{\pi i\tau}{12}}\prod_{\nu=1}^{\infty}(1-e^{2\pi i n\tau})$$

gives

$$\frac{\dot{\eta}(\tau)}{\eta(\tau)} = \frac{\pi i}{12} + \sum_{n=1}^{\infty} \frac{-2\pi i n q^{2n}}{1-q^{2n}} = \frac{\pi i}{12}(1 - 24\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}}) = \frac{\pi i}{12}(1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^{2n})$$

where $\sigma_1(n) = \sum_{d|n} d$ and $\sigma_k(n) = \sum_{d|n} d^k$. Abbreviating by

$$\delta = \delta(\tau) = \frac{\theta_1'''(\tau)}{\theta_1'(\tau)} = \frac{\theta_1'''}{\theta_1'} \tag{3.7}$$

we have

$$\delta(\tau) = -\frac{\pi^2}{4}(1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^{2n}) \tag{3.8}$$

From the differentiation of the formula $\eta(\frac{-1}{\tau}) = \lambda^{-3}\sqrt{\tau}\eta(\tau)$ ([4](1.13)), it follows that

$$\delta(\frac{-1}{\tau}) = \frac{3\pi i \tau}{2} + \tau^2 \delta(\tau). \tag{3.9}$$

Note that $\lambda = e^{\frac{2\pi i}{24}}$ is the 24-th root of unity.

4 We consider the Taylor expansion of $\Theta_p(u)$ at $u = 0$. By (1.10) and (1.11), we have

$$\Theta_1(u) = \theta_1' u + \frac{1}{6}\theta_1''' u^3 + \cdots + \frac{1}{(2\nu-1)!}\theta_1^{(2\nu-1)} u^{2\nu-1} + \cdots \tag{4.1}$$

$$\Theta_p(u) = \theta_p + \frac{1}{2}\theta_p'' u^2 + \cdots + \frac{1}{(2\nu)!}\theta_p^{(2\nu)} u^{2\nu} + \cdots, \tag{4.2}$$

where $p \in \{2,3,4\}$. We define the functions $f_p(u)$ for $p \in \{2,3,4\}$ by putting

$$f_p(u) = \frac{\theta_1'}{\theta_p}\frac{\Theta_p(u)}{\Theta_1(u)} \tag{4.3}$$

These functions are odd functions and meromorphic in the whole complex plane, with simple poles at the zero points of $\Theta_1(u)$, and zero at the zero points of $\Theta_p(u)$. The Laurent expansions of $f_p(u)$ at $u = 0$, have the following forms

$$f_p(u) = \frac{1}{u} + \frac{A_p}{2}u + \frac{B_p}{12}u^3 + \cdots \tag{4.4}$$

where

$$A_p = \frac{\theta_p''}{\theta_p} - \frac{1}{3}\delta \tag{4.5}$$

$$B_p = \frac{1}{2}\frac{\theta_p^{(4)}}{\theta_p} - A_p\delta - \frac{1}{10}\frac{\theta_1^{(5)}}{\theta_1'} \tag{4.6}$$

Note that by (3.5),

$$A_2 + A_3 + A_4 = 0 \tag{4.7}$$

From (2.4) and (2.5), it follows that

$$\left\{\begin{array}{l} f_2(u+2) = \ \ f_2(u) \\ f_3(u+2) = -f_3(u) \\ f_4(u+2) = -f_4(u) \end{array}\right. \tag{4.8}$$

$$\left\{\begin{array}{l} f_2(u+2\tau) = -f_2(u) \\ f_3(u+2\tau) = -f_3(u) \\ f_4(u+2\tau) = \ \ f_4(u) \end{array}\right. \tag{4.9}$$

Thus the function

$$Q(u) = f_2(u)f_3(u)f_4(u) \tag{4.10}$$

and the squares $f_p(u)^2 (p \in \{2,3,4\})$ are the elliptic functions for the fundamental parallelotope $[2, 2\tau]$. The Laurent expansions of $f_p(u)^2$ have the forms

$$f_p(u)^2 = \frac{1}{u^2} + A_p + (\frac{1}{4}{A_p}^2 + \frac{1}{6}B_p)u^2 + \cdots \tag{4.11}$$

We may define the $\wp$-function $P(u)$ of Weierstrass as

$$P(u) = \frac{1}{3}(f_2(u)^2 + f_3(u)^2 + f_4(u)^2) \tag{4.12}$$

noting the formula (4.7). The Laurent expansion of $P(u)$ has the form

$$P(u) = \frac{1}{u^2} + 0 + c_2u^2 + c_4u^4 + \cdots \tag{4.13}$$

where c_2 and c_4 are the appropriate coefficients. The function $Q(u)$ is an odd function in the variable u, and has the Laurent expansion of the following form at $u = 0$.

$$Q(u) = \frac{1}{u^3} + \frac{1}{12}(\sum B_p + 3\sum A_pA_q)u + \cdots \tag{4.14}$$

From $0 = (\sum A_p)^2 = \sum A_p{}^2 + 2\sum A_pA_q$, this can be written in the following way.

$$Q(u) = \frac{1}{u^3} + \frac{1}{24}\sum(2B_p - 3A_p{}^2)u + \cdots \tag{4.15}$$

The usual argument in the elliptic function theory shows the following theorem.

Theorem 4.1. *Between the elliptic functions $P(u)$, $Q(u)$ and $f_p(u)^2$ stand the following relations.*

$$P'(u) = -2Q(u) \tag{4.16}$$

$$f_p(u)^2 = P(u) + A_p \tag{4.17}$$

The detailed calculations will be given later.

5 From Schröter's lemma (Lemma A.2), it follows that

$$\theta(x,q)^2 = \theta(1,q^2)\theta(x^2,q^2) + q^{\frac{1}{2}}\theta(q^2,q^2)xq^{\frac{1}{2}}\theta(x^2q^2,q^2) \tag{5.1}$$

Denoting by

$$\begin{cases} \Phi_3 = \Phi_3(u) = \qquad \theta(x^2,q^2) = \Theta_3(2u,2\tau) \\ \Phi_2 = \Phi_2(u) = xq^{\frac{1}{2}}\theta(x^2q^2,q^2) = \Theta_2(2u,2\tau) \end{cases} \tag{5.2}$$

and by

$$\begin{cases} \varphi_3 = \quad \varphi_3(\tau) = \Phi_3(0) = \theta_3(2\tau) \\ \varphi_2 = \quad \varphi_2(\tau) = \Phi_2(0) = \theta_2(2\tau) \\ \varphi_4 = \theta(-1,q^2) \qquad\quad = \theta_4(2\tau) \end{cases} \tag{5.3}$$

we have

Proposition 5.1.

$$\begin{cases} {\Theta_1}^2 = \varphi_2\Phi_3 - \varphi_3\Phi_2 \\ {\Theta_2}^2 = \varphi_2\Phi_3 + \varphi_3\Phi_2 \\ {\Theta_3}^2 = \varphi_3\Phi_3 + \varphi_2\Phi_2 \\ {\Theta_4}^2 = \varphi_3\Phi_3 - \varphi_2\Phi_2 \end{cases} \tag{5.4}$$

Proof. For example, by (1.5)

$${\Theta_1}^2 = -xq^{\frac{1}{2}}\theta(-xq,q)^2 = -xq^{\frac{1}{2}}\theta(x^2q^2,q^2)\varphi_3 + \varphi_2x^2q^2\theta(x^2q^4,q^2)$$

This shows the first formula by (A.11). The others are obtained in the

similar ways.
From (5.4) it follows that

$$\Theta_1{}^4 + \Theta_3{}^4 = \Theta_2{}^4 + \Theta_4{}^4 = (\varphi_2{}^2 + \varphi_3{}^2)(\Phi_2{}^2 + \Phi_3{}^2) \tag{5.5}$$

This is equivalent to

$$\Theta_2{}^4 - \Theta_1{}^4 = \Theta_3{}^4 - \Theta_4{}^4 = 4\varphi_2\varphi_3\Phi_2\Phi_3 \tag{5.6}$$

From the formula

$$\theta(x,q)\theta(-x,q) = \theta(-1,q^2)\theta(-x^2,q^2) \tag{5.7}$$

we have

$$\begin{cases} \Theta_1\Theta_2 = \varphi_4\Phi_1 \\ \Theta_3\Theta_4 = \varphi_4\Phi_4 \end{cases} \tag{5.8}$$

where

$$\begin{cases} \Phi_1 = \Phi_1(u) = -ix^{\frac{1}{2}}\theta(-x^2q^2,q^2) = \Theta_1(2u,2\tau) \\ \Phi_4 = \Phi_4(u) = \qquad\qquad \theta(-x^2,q^2) = \Theta_4(2u,2\tau) \end{cases} \tag{5.9}$$

Lastly from the formula

$$\theta(x,q)\theta(\delta xq,q) = \theta(\delta q,q^2)(\theta(\delta x^2q,q^2) + \delta x^{-1}\theta(\delta x^{-2}q,q^2)) \tag{5.10}$$

with $\delta = \pm 1$, we have

$$\begin{cases} \Theta_2\Theta_3 = \quad \chi_0(X_0(u) + X_0(-u)) \\ \Theta_2\Theta_4 = \quad \chi_1(X_1(u) + X_1(-u)) \\ \Theta_1\Theta_4 = -i\chi_0(X_0(u) - X_0(-u)) \\ \Theta_1\Theta_3 = -i\chi_1(X_1(u) - X_1(-u)) \end{cases} \tag{5.11}$$

where

$$\begin{cases} X_0(u) = x^{\frac{1}{2}}q^{\frac{1}{8}}\theta(x^2q,q^2) \\ X_1(u) = x^{\frac{1}{2}}q^{\frac{1}{8}}\theta(-x^2q,q^2) \end{cases} \tag{5.12}$$

and

$$\begin{cases} \chi_0 = \chi_0(\tau) = X_0(0) \\ \chi_1 = \chi_1(\tau) = X_1(0) \end{cases} \tag{5.13}$$

Note that $\chi_0(\tau)$ and $\chi_1(\tau)$ are the same as in [4] (1.9). In these formulae, letting $u = 0$ (so $x = 1$), we get many familiar relations between the theta zeros. In the formulae which involve Θ_1 and Φ_1, we can use the facts

$$\lim_{u\to 0}\Theta_1(u)/u=\theta_1'(\tau)\quad\text{and}\quad\lim_{u\to 0}\Phi_1(u)/u=2\theta_1'(2\tau)$$

Here we summarise some of them for later use;

$$(5.14)\qquad \begin{cases}\theta_2{}^2=2\varphi_2\varphi_3\\ \theta_3{}^2=\varphi_3{}^2+\varphi_2{}^2\\ \theta_4{}^2=\varphi_3{}^2-\varphi_2{}^2\\ \theta_3\theta_4\ =\varphi_4{}^2\end{cases}$$

and

$$(5.15)\qquad \theta_2{}^4-\theta_3{}^4+\theta_4{}^4=0$$

6 From the formula (5.4), eliminating Φ_2 and Φ_3, we have

$$\theta_2{}^2\begin{pmatrix}\Theta_1{}^2\\ \Theta_2{}^2\end{pmatrix}=\theta_2{}^2\begin{pmatrix}-\varphi_3 & \varphi_2\\ \varphi_3 & \varphi_2\end{pmatrix}\begin{pmatrix}\Phi_2\\ \Phi_3\end{pmatrix}=\begin{pmatrix}-\varphi_3 & \varphi_2\\ \varphi_3 & \varphi_2\end{pmatrix}\begin{pmatrix}\varphi_3 & -\varphi_3\\ \varphi_2 & \varphi_2\end{pmatrix}\begin{pmatrix}\Theta_3{}^2\\ \Theta_4{}^2\end{pmatrix}$$
$$=\begin{pmatrix}-\theta_4{}^2 & \theta_3{}^2\\ \theta_3{}^2 & -\theta_4{}^2\end{pmatrix}\begin{pmatrix}\Theta_3{}^2\\ \Theta_4{}^2\end{pmatrix}$$

using (5.14). From this, eliminating each one $\Theta_p{}^2$, we have

$$(6.1)\qquad \begin{cases} \qquad\qquad\ \theta_2{}^2\Theta_2{}^2-\theta_3{}^2\Theta_3{}^2+\theta_4{}^2\Theta_4{}^2=0\\ -\theta_2{}^2\Theta_1{}^2\qquad\qquad\ -\theta_4{}^2\Theta_3{}^2+\theta_3{}^2\Theta_4{}^2=0\\ \ \theta_3{}^2\Theta_1{}^2+\theta_4{}^2\Theta_2{}^2\qquad\qquad\ -\theta_2{}^2\Theta_4{}^2=0\\ -\theta_4{}^2\Theta_1{}^2-\theta_3{}^2\Theta_2{}^2+\theta_2{}^2\Theta_3{}^2\qquad\qquad\ =0\end{cases}$$

or

$$\begin{pmatrix}0 & \theta_2{}^2 & -\theta_3{}^2 & \theta_4{}^2\\ -\theta_2{}^2 & 0 & -\theta_4{}^2 & \theta_3{}^2\\ \theta_3{}^2 & \theta_4{}^2 & 0 & -\theta_2{}^2\\ -\theta_4{}^2 & -\theta_3{}^2 & \theta_2{}^2 & 0\end{pmatrix}\begin{pmatrix}\Theta_1{}^2\\ \Theta_2{}^2\\ \Theta_3{}^2\\ \Theta_4{}^2\end{pmatrix}=0$$

This will be called *the quadratic relations between* Θ_p. It is easy to see that any two kinds of the squares of theta functions are linearly independent but any three kinds are linearly dependent, and that any linear combination of the functions $\Theta_p{}^2$ can be represented by linear combination of two kinds of the square of theta functions.

By the definition (4.12) of $P(u)$,

$$P(u)=\Gamma(u)/\Theta_1(u)^2$$

where $\Gamma(u)$ is a linear combination of $\Theta_p(u)^2$:

$$\Gamma(u) = c\sum \theta_p{}^{-2}\Theta_p{}^2 \quad \text{with} \quad c = \frac{1}{3}\theta_1'(\tau)^2 = \frac{\pi^2}{12}\theta_2{}^2\theta_3{}^2\theta_4{}^2 \tag{6.2}$$

Representing $\Gamma(u)$ by linear combination of $\Theta_1{}^2$ and $\Theta_2{}^2$, as $\Theta_3{}^2 = \theta_2{}^{-2}(\theta_4{}^2\Theta_1{}^2 + \theta_3{}^2\Theta_2{}^2)$ and $\Theta_4{}^2 = \theta_2{}^{-2}(\theta_3{}^2\Theta_1{}^2 + \theta_4{}^2\Theta_2{}^2)$, we have

$$\Gamma(u) = \frac{\pi^2}{12}(\theta_3{}^4 + \theta_4{}^4)\Theta_1{}^2 + c\cdot 3\theta_2{}^{-2}\Theta_2{}^2$$

Thus

$$P(u) = \frac{\pi^2}{12}(\theta_3{}^4 + \theta_4{}^4) + f_2(u)^2 \tag{6.3}$$

If we put $\tau_2 = 1$, then τ_2 is a zero point of $f_2(u)$. So letting $u = \tau_2$, we have

$$e_2 := P(\tau_2) = \frac{\pi^2}{12}(\theta_3{}^4 + \theta_4{}^4) \tag{6.4}$$

and in the formula (4.17), we have $A_2 = -e_2$, that is

$$\frac{1}{3}\delta(\tau) - \frac{\theta_2''}{\theta_2} = \frac{\pi^2}{12}(\theta_3{}^4 + \theta_4{}^4) \tag{6.5}$$

and that

$$f_2(u)^2 = P(u) - e_2 = P(u) - P(\tau_2) \tag{6.6}$$

In the similar way,

$$\begin{cases} P(u) = \dfrac{\pi^2}{12}(\theta_2{}^4 - \theta_4{}^4) + f_3(u)^2 \\ e_3 := P(\tau_3) = \dfrac{\pi^2}{12}(\theta_2{}^4 - \theta_4{}^4) = \dfrac{1}{3}\delta(\tau) - \dfrac{\theta_3''}{\theta_3} \end{cases} \tag{6.7}$$

where $\tau_3 = \tau + 1$ and τ_3 is a zero point of $f_3(u)$. Also we have

$$\begin{cases} P(u) = -\dfrac{\pi^2}{12}(\theta_2{}^4 + \theta_3{}^4) + f_4(u)^2 \\ e_4 := P(\tau_4) = -\dfrac{\pi^2}{12}(\theta_2{}^4 + \theta_3{}^4) = \dfrac{1}{3}\delta(\tau) - \dfrac{\theta_4''}{\theta_4} \end{cases} \tag{6.8}$$

where $\tau_4 = \tau$ and τ_4 is a zero point of $f_4(u)$. (Mnemonics: if we write $\theta_3{}^4 - \theta_4{}^4 - \theta_2{}^4 = 0$ by $X + Y + Z = 0$, then $e_2 = \frac{\pi^2}{12}(X - Y)$, $e_3 = \frac{\pi^2}{12}(Y - Z)$ and $e_4 = \frac{\pi^2}{12}(Z - X)$.)

Theorem 6.1. *For the $\wp$-function $P(u)$ to the parallelotope $[2, 2\tau]$, we have*

$$f_p(u)^2 = P(u) - P(\tau_p) = P(u) - e_p \tag{6.9}$$

for $p \in \{2,3,4\}$ where τ_p is the zero point of the function $f_p(u)$. The differential equation for $P(u)$ is given by

$$\begin{aligned} P'(u)^2 &= 4(P(u)-e_2)(P(u)-e_3)(P(u)-e_4) \\ &= 4P(u)^3 - \frac{\pi^4}{12}E(\tau)P(u) - \frac{\pi^6}{216}F(\tau) \end{aligned} \tag{6.10}$$

where

$$E(\tau) = \frac{1}{2}({\theta_2}^8 + {\theta_3}^8 + {\theta_4}^8) \tag{6.11}$$

$$F(\tau) = \frac{1}{2}({\theta_3}^4 + {\theta_4}^4)({\theta_3}^4 + {\theta_2}^4)({\theta_4}^4 - {\theta_2}^4) \tag{6.12}$$

The distriminant of the equation (6.10) is equal to

$$D(\tau) = \frac{\pi^{12}}{12^3}(E(\tau)^3 - F(\tau)^2) = \pi^{12}\eta(\tau)^{24} = \pi^{12}\Delta(\tau) \tag{6.13}$$

Proof. It is easy to see that $e_2 + e_3 + e_4 = 0$ from (6.4), (6.7) and (6.8), and that

$$g_2 = -4(e_3e_4 + e_4e_2 + e_2e_3) = \frac{\pi^4}{24}({\theta_2}^8 + {\theta_3}^8 + {\theta_4}^8)$$

$$g_3 = 4e_2e_3e_4 = \frac{\pi^6}{432}({\theta_3}^4 + {\theta_4}^4)({\theta_3}^4 + {\theta_2}^4)({\theta_4}^4 - {\theta_2}^4)$$

by mnemonics mentioned above. If $X + Y + Z = 0$, then putting

$$s_2 = XY + YZ + ZX \quad \text{and} \quad s_3 = XYZ$$

we have

$$\{(X-Y)(Y-Z)(Z-X)\}^2 = -(4{s_2}^3 + 27{s_3}^2)$$

In fact the polynomial

$$\Phi(t) = (t-X)(t-Y)(t-Z) = t^3 + s_2t - s_3$$

in the variable t has the discriminant $d = -(4{s_2}^3 + 27{s_3}^2)$. If we put $X = {\theta_3}^4$, $Y = -{\theta_4}^4$ and $Z = -{\theta_2}^4$, then $s_2 = -\frac{1}{2}(X^2 + Y^2 + Z^2) = -E(\tau)$

and $s_3=(\theta_2\theta_3\theta_4)^4=16\eta(\tau)^{12}$ by (2.3). So

$$4F(\tau)^2 = 4(E(\tau)^3 - 1728\eta(\tau)^{24})$$

This shows that $E^3 - F^2 = 12^3\eta(\tau)^{24} = 12^3\Delta(\tau)$. The discriminant of (6.10) being $D(u) = g_2{}^3 - 27g_3{}^2 = (\pi^{12}/12^3)(E^3 - F^2)$, this shows the theorem.

Theorem 6.2. *The functions $E(\tau)$ and $F(\tau)$ are the modular form for the full modular group $\Gamma(1)=SL_2(Z)$ of degree 4 and 6, respectively. (In fact, these are equal to $E_2(\tau)$ and $E_3(\tau)$, respectively in the notation of Serre* [3].) *And it holds that*

$$E(\tau)/\eta(\tau)^8 = \gamma_2(\tau) \tag{6.14}$$

$$F(\tau)/\eta(\tau)^{12} = \gamma_3(\tau) \tag{6.15}$$

where $\gamma_2(\tau) = \sqrt[3]{j(\tau)}$ and $\gamma_3(\tau) = \sqrt{j(\tau) - 12^3}$ are the Weber's modular functions.

Proof. The first part can be read from the formulae (1.11) and (1.13) of [4]. The formula (6.14) is the same as (2.21) of [4]. Also the formula (6.15) is derived;

$$\begin{aligned} F(\tau)/\eta(\tau)^{12} &= \frac{1}{2}(1+\beta^4)(1+\gamma^4)(\beta^4-\gamma^4)\alpha^{12} \\ &= (\alpha^{12} + \frac{1}{2}(\alpha^3\beta\gamma)^4)(\beta^4-\gamma^4) = (\alpha^{12}+8)(\beta^4-\gamma^4) \end{aligned}$$

because $\beta^4 + \gamma^4 = 1$ and $\alpha^3\beta\gamma = 2$ from (2.2) and (2.3) of [4], where $\alpha = \theta_3/\eta$, $\beta = \theta_4/\theta_3$ and $\gamma = \theta_2/\theta_3$. Thus (6.15) follows from (2.22) of [4].

The absolute invariant $J(\tau)$ of (6.10) is given by

$$J(\tau) = \frac{g_2{}^3}{g_2{}^3 - 27g_3{}^2} = \frac{\pi^{12}}{12^3}\frac{E(\tau)^3}{\pi^{12}\Delta(\tau)} = \frac{1}{12^3}\frac{E(\tau)^3}{\Delta(\tau)} \tag{6.16}$$

7 In the theorem 4.1, differentiating the formula (4.17), we have

$$P'(u) = 2f_p'(u)f_p(u)$$

On the other hand, by (4.16), $P'(u) = -2Q(u) = -2f_2(u)f_3(u)f_4(u)$. Thus we have

$$f_p'(u) = -f_q(u)f_r(u) \tag{7.1}$$

	0	1	$\tau+1$	τ
Θ_1'	θ_1'	0	$-i\varepsilon'\theta_3$	$\varepsilon'\theta_4$
Θ_2'	0	$-\theta_1'$	$-\varepsilon'\theta_4$	$-i\varepsilon'\theta_3$
Θ_3'	0	0	$i\varepsilon\theta_1'$	$-i\varepsilon'\theta_2$
Θ_4'	0	0	$-i\varepsilon'\theta_2$	$i\varepsilon\theta_1'$

Table 2: (with $\varepsilon = q^{-\frac{1}{4}}$ and $\varepsilon' = \frac{\pi}{2}q^{-\frac{1}{4}}$)

where $\{p,q,r\} = \{2,3,4\}$. For two functions $f(u)$ and $g(u)$, we put

(7.2) $$[f,g] = f'(u)g(u) - f(u)g'(u)$$

Clearly $[f,g] = -[g,f]$. And we have the following theorem.

Theorem 7.1. *For* $\{p,q,r\}=\{2,3,4\}$

(7.3) $$[\Theta_1,\Theta_p] = \frac{\pi}{2}\theta_p{}^2\Theta_q(u)\Theta_r(u)$$

Proof. An easy calculation shows this theorem. That is,

$$f_p' = \frac{\theta_1'}{\theta_p}\frac{[\Theta_p,\Theta_1]}{\Theta_1{}^2} \quad \text{and} \quad f_qf_r = \frac{\theta_1'^2}{\theta_q\theta_r}\frac{\Theta_q\Theta_r}{\Theta_1{}^2}$$

As $\theta_1' = \frac{\pi}{2}\theta_2\theta_3\theta_4$, so one obtains the desired formula.

Remark. It would be an amusing puzzle to obtain the values $\Theta_p'(\tau_q)$ for $p,q\in\{2,3,4\}$, where $\tau_1 = 0$ and other ones are given in §6. Here we give its table(Table 2).

8 We define the following three elliptic function for the parallelotopes related to $[2,2\tau]$, which are called Jacobi's elliptic functions.

(8.1) $$S(u) = \frac{\theta_3}{\theta_2}\frac{\Theta_1}{\Theta_4}, \quad C(u) = \frac{\theta_4}{\theta_2}\frac{\Theta_2}{\Theta_4}, \quad \text{and} \quad D(u) = \frac{\theta_4}{\theta_3}\frac{\Theta_3}{\Theta_4}$$

From the quadratic relations (6.1), it follows that they satisfy

(8.2) $$S(u)^2 + C(u)^2 = 1$$

(8.3) $$k^2S(u)^2 + D(u)^2 = 1$$

where $k = \theta_2{}^2/\theta_3{}^2$. From (7.3), it follows that

$$S'(u) = \frac{\pi}{2}\theta_3{}^2 C(u)D(u) = KC(u)D(u) \tag{8.4}$$

where we abbreviate $K = \frac{\pi}{2}\theta_3{}^2$ for a certain reason. Differentiations of (8.2) and (8.3) give

$$\begin{aligned} C'(u) &= -S'S/C = -KS(u)D(u) \\ D'(u) &= -k^2S'S/D = -k^2KS(u)C(u) \end{aligned}$$

As $C'(u) = \theta_4\theta_2{}^{-1}\Theta_2{}^{-2}[\Theta_2, \Theta_4]$ and $D'(u) = \theta_4\theta_3{}^{-1}\Theta_4{}^{-2}[\Theta_3, \Theta_4]$, we have

$$\begin{cases} [\Theta_4, \Theta_2] = \frac{\pi}{2}\theta_3{}^2\Theta_1(u)\Theta_3(u) \\ [\Theta_4, \Theta_3] = \frac{\pi}{2}\theta_2{}^2\Theta_1(u)\Theta_2(u) \end{cases} \tag{8.5}$$

The remaining $[\Theta_3, \Theta_2]$ is easily obtained from $[\Theta_1, \Theta_4]$ of (7.3) replacing u by $u + 1$. That is

$$[\Theta_3, \Theta_2] = \frac{\pi}{2}\theta_4{}^2\Theta_1(u)\Theta_4(u) \tag{8.6}$$

For the function $S(u)$, we have $S(0) = 0$ and $S(1) = 1$. And $C(0) = D(0) = 1$. In fact, if the parameter τ is purely imaginary with posotive imaginary part, the function $S(u)$ is monotone increasing from 0 to 1 in the interval $[0, 1]$. In the complete elliptic integral of the first kind

$$K = \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} \tag{8.7}$$

we substitute $y = S(u)$. Then

$$dy = \frac{\pi}{2}\theta_3{}^2\sqrt{(1-y^2)(1-k^2y^2)}\, du$$

Thus we have

$$K = \int_0^1 \frac{\pi}{2}\theta_3{}^2 du = \frac{\pi}{2}\theta_3{}^2$$

That is

$$K = K(k) = \frac{\pi}{2}\theta_3(\tau)^2 \tag{8.8}$$

9 For a non-zero complex number ω and a complex variable v, we put

(9.1) $$\wp(v) = \omega^{-2}P(\frac{v}{\omega}) = v^{-2} + c_2\omega^{-4}v^2 + c_4\omega^{-6}v^4 + \cdots$$

This is the $\wp$-function of Weierstrass for the parallelotope $\omega[2, 2\tau]=[2\omega_2, 2\omega_4]$, where $\omega_2 = \omega$ and $\omega_4 = \tau\omega$, and we put $\omega_3=\omega_2 + \omega_4$. As $\wp'(v) = \omega^{-3}P'(\frac{v}{\omega})$, we have

(9.2) $$\wp'(v)^2 = \omega^{-6}P'(\frac{v}{\omega})^2 = 4\wp(v)^3 - g_2\omega^{-4}\wp(v) - g_3\omega^{-6}$$

If we write this differential equation as

(9.3) $$\wp'(v)^2 = 4\wp(v)^4 - g_2^*\wp(v) - g_3^*$$

then, putting

(9.4) $$\rho = \frac{\pi\eta(\tau)^2}{6\ \omega}$$

we have

(9.5) $$\begin{cases} g_2^* = 108\rho^4\gamma_2(\tau) = 2^2 3^3\rho^4\gamma_2(\tau) \\ g_3^* = 216\rho^6\gamma_3(\tau) = 2^3 3^3\rho^6\gamma_3(\tau) \end{cases}$$

and also

(9.6) $$\begin{aligned} D^* &= g_2^{*3} - 27g_3^{*2} = 2^6 3^9(\gamma_2^3 - \gamma_3^2)\rho^{12} \\ &= (6\rho)^{12} = (\frac{\pi\eta(\tau)^2}{\omega})^{12} = (\frac{\pi}{\omega})^{12}\Delta(\tau) \end{aligned}$$

As for the half-period values $e_p^* = \wp(\omega_p)$, we have, putting $\rho_1 = \pi/6\omega$,

(9.7) $$e_2^* = \omega^{-2}P(\tau_2) = 3\rho_1^2(\theta_3^4 + \theta_4^4) = 3\rho^2\alpha(\tau)^4(1 + \beta(\tau)^4)$$

(9.8) $$e_3^* = 3\rho_1^2(-\theta_4^4 + \theta_2^4) = 3\rho^2\alpha(\tau)^4(-\beta(\tau)^4 + \gamma(\tau)^4)$$

(9.9) $$e_4^* = 3\rho_1^2(-\theta_2^4 - \theta_3^4) = 3\rho^2\alpha(\tau)^4(-\gamma(\tau)^4 - 1)$$

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