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ANTI-INTEGRAL ELEMENTS AND COEFFICIENTS OF THEIR MINIMAL POLYNOMIALS

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Let R be a Noetherian domain and $R[X]$ a polynomial ring. Let α be a non-zero element of an algebraic field extension L of the quotient field K of R and let $\pi : R[X] \to R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$$

Then η_i $(1 \leq i \leq d)$ are uniquely determined by α. Let $I_{\eta_i} := R :_R \eta_i$ and $I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i}$, the latter of which is called a *generalized denominator ideal* of α. We say that α is an *anti-integral element* over R if $\operatorname{Ker}\pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. For $f(X) \in R[X]$, let $C(f(X))$ denote the ideal of R generated by the coefficients of $f(X)$. For an ideal J of $R[X]$, let $C(J)$ denote the ideal generated by the coefficients of the elements in J. If α is an anti-integral element, then $C(\operatorname{Ker}\pi) = C(I_{[\alpha]}\varphi_\alpha(X)R[X]) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Put $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Let $\tilde{J}_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1})$. If $J_{[\alpha]} \not\subseteq p$ for all $p \in \operatorname{Dp}_1(R) := \{p \in \operatorname{Spec}(R) \mid \operatorname{depth} R_p = 1\}$, then α is called a *super-primitive* element over R. It is known that a super-primitive element is an anti-integral element (cf.[7,(1.12)]). It is known that any algebraic element over a Krull domain R is anti-integral over R (cf.[7,(1.13)]). When α is a non-zero element in K, $\varphi_\alpha(X) = X - \alpha$. So we have $J_{[\alpha]} = I_{[\alpha]}(1, \alpha) = I_\alpha(1, \alpha) = I_\alpha + \alpha I_\alpha = I_\alpha + I_{\alpha^{-1}}$, where $I_\alpha := R :_R \alpha$, a *denominator ideal* of $\alpha \in K$.

In this paper, we use the following notation unless otherwise specified:

Let R be a Noetherian domain with quotient field K. Let L be an algebraic field extension of K and let α be a non-zero element in L which is of degree d over K. Let $\varphi_\alpha(X) := X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ denote the minimal polynomial of α over K (that is, $\eta_i \in K$). Put $A := R[\alpha]$ and $B := R[\eta_1, \ldots, \eta_d]$.

Our general reference for unexplained technical terms is [4].

§1. Ring-Extensions Generated by the Coefficients of a Polynomial.

The objective of this section is to investigate some relations between the ring-extensions A/R and B/R.

Lemma 1.1 (cf.[7,(3.4)], [1,Proposition 6]). *Assume that α is anti-integral over R. Then*
(1) *A is flat over R if and only if $J_{[\alpha]} = R$;*
(2) *A is faithfully flat over R if and only if $\tilde{J}_{[\alpha]} = R$.*

Lemma 1.2. *Assume that α is anti-integral over R. If $I_{[\alpha]}A = A$, then A is flat over R.*

Proof. Since $\tilde{J}_{[\alpha]} \supseteq I_{[\alpha]}$, the equality $I_{[\alpha]}A = A$ induces $\tilde{J}_{[\alpha]}A = A$. Hence A is flat over R by [1,Theorem 15].

Proposition 1.3. *Assume that α is anti-integral over R. If $I_{[\alpha]}A = A$, then $R \hookrightarrow B$ is an open immersion.*

Proof. Since $I_{[\alpha]}A = A$, A is flat over R by Lemma 1.2. So we have $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)R = R$ by Lemma 1.1. Take $p \in \mathrm{Spec}(R)$. If $I_{[\alpha]} \not\subseteq p$, then $\eta_i \in R_p$ for all i. Thus $B_p = R_p$. Assume that $I_{[\alpha]} \subseteq p$. Then $I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)R_p = R_p$ and hence $I_{[\alpha]}\eta_i R_p = R_p$ for some i. Thus we have $I_{[\alpha]}R_p = I_{\eta_i}R_p$. It follows that $I_{\eta_i}B_p = B_p$ and that $pB_p = B_p$. Hence B_p is flat over R_p. Since B and R are birational, $R \hookrightarrow B$ is an open immersion.

Theorem 1.4. *Assume that α is anti-integral over R. If A is flat over R, then B and $B[\alpha]$ are flat over R and $B[\alpha]$ is flat over B.*

Proof. Since A is flat over R, we have $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) = R$. Since $\eta_1, \ldots, \eta_d \in B$, we have $I_{[\alpha]}B = B$. Take $p \in \mathrm{Spec}(R)$. If $p \not\supseteq I_{[\alpha]}$, we have $B_p = R_p$ because $\eta_1, \ldots, \eta_d \in R_p$. If $p \supseteq I_{[\alpha]}$, then $pB = B$. Hence B is flat over R. Since $B[\alpha]$ is free B-module of rank d, $B \supseteq B[\alpha]$ is a flat extension. Therefore $R \subseteq B[\alpha]$ is also a flat extension.

Example 1.5. Let R be a polynomial ring $k[a, b]$ over a field k. An element α is a root of an irreducible polynomial $\varphi_\alpha(X) = X^2 + (1/a)X + 1/b$. Then $\varphi_\alpha(X)$ is the minimal polynomial of α and α is an anti-integral element over R because R is a Noetherian normal domain. We have $J_{[\alpha]} = (a, b)R \neq R$ and $B = R[1/a, 1/b]$. Since B is obtained by localizations, B is flat over R. But since $J_{[\alpha]} \neq R$, $A = R[\alpha]$ is not flat over R. Thus the converse statement of Theorem 1.4 is not always valid.

Theorem 1.6. *The following statements are equivalent :*
(1) *A is integral over R,*
(2) *B is integral over R.*

Proof. Let $\overline{R}$ denote the integral closure of R in K.
(2) $\Rightarrow$ (1) : Since B is integral over R, we have $B \subseteq \overline{R}$. Since α is integral over B, α is integeral over $\overline{R}$. So α is integral over R.
(1) $\Rightarrow$ (2) : Since R is Noetherian domain, $\overline{R}$ is a Krull domain. So $\overline{R} = \bigcap \overline{R}_P$ $P \in \mathrm{Ht}_1(\overline{R})$, where $\overline{R}_P$ is a DVR. Since α is anti-integral and integral over a DVR $\overline{R}_P$, we have $\varphi_\alpha(X) \in \overline{R}_P[X]$. Hence $\eta_i \in \overline{R}_P$ for all i. So $\eta_i \in \overline{R}$, which implies that B is integral over R.

Lemma 1.7 (cf.[1, The proof of Theorem 8]). *Assume that α is anti-integral over R. Then $\Omega_R(A) \cong A/I_{[\alpha]}\varphi'_\alpha(\alpha)A$, where $\varphi'_\alpha(X)$ denotes the derivative of $\varphi_\alpha(X)$ and $\Omega_R(A)$ denotes the module of differentials.*

Theorem 1.8. *Assume that α is anti-integral over R. If A is unramified over R, then $B[\alpha]$ is unramified over B and B is unramified over R.*

Proof. Note that $\Omega_R(A) \cong A/I_{[\alpha]}\varphi'_\alpha(\alpha)A$ by Lemma 1.7. Since A is unramified over R, we have $I_{[\alpha]}\varphi'_\alpha(\alpha)A = A$. Thus $I_{[\alpha]}\varphi'_\alpha(\alpha)A[\eta_1, \dots, \eta_d] = A[\eta_1, \dots, \eta_d] = B[\alpha]$. Since $\varphi'_\alpha(\alpha) \in A[\eta_1, \dots, \eta_d]$, $\varphi'_\alpha(\alpha)$ is an invertible element in $B[\alpha]$. Hence $B[\alpha]$ is unramified over B. Note here that $B[\alpha] = B[X]/\varphi_\alpha(X)B[X]$. So $B[\alpha]$ is flat over B. Moreover we know that $B[\alpha] = I_{[\alpha]}\varphi'_\alpha(\alpha)B[\alpha] \subseteq I_{[\alpha]}B[\alpha] \subseteq B[\alpha]$. Hence $I_{[\alpha]}B[\alpha] = B[\alpha]$. Since $B[\alpha]$ is flat over B and $B[\alpha]$ is integral over B, $B[\alpha]$ is faithfully flat over B. So we have $I_{[\alpha]}B = B$. Thus $R \hookrightarrow B$ is an open immersion by Proposition 1.3 and hence unramified.

§2. Constant Terms of Minimal Polynomials and Flat Elements.

In this section, we characterize the ring $A \cap K$ under the condition $I_{[\alpha]}A = A$.

We begin with recalling the following lemma which is easy to prove.

Lemma 2.1 (cf. [3,Lemma 3(2)]) *The equality $I_{[\alpha^{-1}]} = \eta_d I_{[\alpha]}$ holds.*

Lemma 2.2. *Let p be a prime ideal of R. If $pR[\alpha] = R[\alpha]$, then α^{-1} is integral over R_p.*

Proof. Since $pR[\alpha] = R[\alpha]$, we have $a_0 + a_1\alpha + \cdots + a_\ell\alpha^\ell = 1$ for some $a_i \in p$ $(0 \leq i \leq \ell)$. Thus α satisfies the equation : $a_\ell\alpha^\ell + \cdots + a_{\ell-1}\alpha + (a_0 - 1) = 0$. Hence α^{-1} satisfies the equation : $a_\ell + a_{\ell-1}\alpha^{-1} + \cdots + a_1\alpha^{\ell-1} + (a_0 - 1)(\alpha^{-1})^\ell = 0$. Since $a_0 - 1$ is a unit in R_p, we can conclude that α^{-1} is integral over R_p.

Proposition 2.3. *Assume that α is anti-integral over R. Consider the following statements :*
(1) $I_{[\alpha]}A = A$;
(2) $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$;
(3) $I_{[\alpha]} = I_{\eta_d}$ *and* $I_{\eta_d}(1, \eta_d)R = R$.
Then the following implications hold : (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) : Suppose that there exists $p \in \mathrm{Spec}(R)$ such that $I_{[\alpha]} + I_{[\alpha^{-1}]} \subseteq p$. Since $I_{[\alpha]}A = A$, α^{-1} is integral over R_p by Lemma 2.2. Since α is anti-integral over R, so is α^{-1} by [2,Theorem 6]. Hence α^{-1} is anti-integral and integral over R_p. Thus $\varphi_{\alpha^{-1}}(X) \in R_p[X]$ and hence $I_{[\alpha^{-1}]}R_p = R_p$, which contradicts the assumption $I_{[\alpha^{-1}]} \subseteq p$.
(2) $\Rightarrow$ (3) : Since $I_{[\alpha^{-1}]} = \eta_d I_{[\alpha]}$ by Lemma 2.1, we have $I_{[\alpha]} + I_{[\alpha^{-1}]} = I_{[\alpha]}(1, \eta_d)R = R$. So we have $I_{[\alpha]} = I_{\eta_d}$ and $J_{\eta_d} = I_{\eta_d}(1, \eta_d)R = R$.
The converse implication (3) $\Rightarrow$ (2) can be seen by tracing the above argument backward.

Example 2.4. The following example shows that the implication (2) $\Rightarrow$ (1) is not valid in general. Let R be a polynomial ring $k[a, b]$ over a field k. Let α is a solution of the equation : $\varphi_\alpha(X) := X^2 + (b/a^2)X + ((a-1)/a)^2 = 0$. Then α is anti-integral over R because R is a Noetherian normal domain. We have $I_{[\alpha]} = a^2R$, $\varphi_{\alpha^{-1}}(X) = X^2 + (b/(a-1)^2)X + (a/(a-1))^2$ and $I_{[\alpha^{-1}]} = (a-1)^2R$. Thus $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$. Moreover we have $J_{[\alpha]} = R$ and $\tilde{J}_{[\alpha]} = a^2(1, b/a^2)R = (a^2, b)R$. Since $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$, we have $\sqrt{\tilde{J}_{[\alpha]}} \neq \sqrt{I_{[\alpha]}}$. Hence $I_{[\alpha]}A \neq A$, which implies that the implication (2) $\Rightarrow$ (3) does not always hold.

An element $\alpha \in L$ is called *exclusive* over R if $R[\alpha] \cap K = R$ (cf. [6]).

Now we study the exclusiveness for a while. We start the following Lemma.

Lemma 2.5 ([6, Theorem 5]). *Assume that R contains an infinite field k and that α is super-primitive over R. Then the following statements*

are equivalent :
(1) α *is exclusive over* R ;
(2) $\bigcap_{i=1}^{d-1} I_{\eta_i} \subseteq I_{\eta_d}$;
(3) $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$ *or* $\tilde{J}_{[\alpha]} = R$.

Proposition 2.6. *Assume that α is super-primitive over R and that R contains an infinite field. If either* $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$ *or* $\tilde{J}_{[\alpha]} = R$, *then both α and α^{-1} are exclusive, i.e.,* $R[\alpha] \cap K = R[\alpha^{-1}] \cap K = R$.

Proof. By Lemma 2.5, we have the following equivalences :
(a) α is exclusive over $R \Leftrightarrow \mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$;
(b) α^{-1} is exclusive over $R \Leftrightarrow \mathrm{grade}(I_{[\alpha^{-1}]}(\eta_1/\eta_d, \dots, \eta_{d-1}/\eta_d, 1)) > 1$
$\Leftrightarrow \mathrm{grade}(I_{[\alpha]}(\eta_1, \dots, \eta_d)) > 1$,
where the last equivalence follows from Lemma 2.1. These equivalence induce our conclusion.

Proposition 2.7. *Assume that α is super-primitive over R. If A is faithfully flat over R, then α is exclusive.*

Proof. From Lemma 1.1, it follows the equivalence : $R[\alpha]$ is faithfully flat over $R \Leftrightarrow \tilde{J}_{[\alpha]} = R$. So we have our conclusion by Lemma 2.5.

Proposition 2.8. *Assume that α is super-primitive over R and that R contains an infinite field. If A_p is faithfully flat over R_p for each $p \in \mathrm{Dp}_1(R)$, then α is exclusive, i.e.,* $R[\alpha] \cap K = R$.

Proof. By Lemma 2.5, note that $R[\alpha]_p$ is faithfully flat over R_p for each $p \in \mathrm{Dp}_1(R) \Rightarrow \mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$ or $\tilde{J}_{[\alpha]} = R$, by Lemma 1.1. The latter condition give rise to the statement that α is exclusive over R by Lemma 2.5.

Lemma 2.9. *Assume that α is super-primitive over R. If $I_{[\alpha]}A = A$, then $B \subseteq A$.*

Proof. Since $I_{[\alpha]}A = A$, A is flat over R by Lemma 1.2. Take $P \in \mathrm{Dp}_1(A)$ and put $p := P \cap R$. Then $p \in \mathrm{Dp}_1(R)$. Since α is super-primitive over R, the ideal $I_{[\alpha]}R_p$ is a principal ideal. So there exists $a \in I_{[\alpha]}$ such that $I_{[\alpha]}R_p = aR_p$. Hence $aA_p = A_p$ by the assumption $I_{[\alpha]}A = A$. Since $I_{[\alpha]} \subseteq I_{\eta_i}$ by definition, putting $\eta_i = b_i/a$ with $b_i \in R$. Since a is an invertible element in A_p, we have $\eta_i \in A_p \subseteq A_P$. Thus $\eta_i \in \bigcap_{P \in \mathrm{Dp}_1(A)} A_P = A$. Therefore $B = R[\eta_1, \dots, \eta_d] \subseteq A$.

Theorem 2.10. *Assume that α is super-primitive over R. The following statements are equivalent :*
(1) $I_{[\alpha]}A = A$;
(2) $B \subseteq A$ *and* $I_{[\alpha]}B = B$.
If the condition (2) *holds, B is flat over R.*

Proof. (1) $\Rightarrow$ (2) : The first statement is shown in Lemma 2.9. The assumption $I_{[\alpha]}A = A$ implies that A is flat over R by Lemma 1.2 and that B is flat over R by Lemma 1.3. Hence $J_{[\alpha]} = R$. Since α is anti-integral over B and since α is integral over B, it follows that $I_{[\alpha]}^{(B)} = B$, where $I_{[\alpha]}^{(B)} = B[X] :_B \varphi_\alpha(X)$. Thus $I_{[\alpha]}B = I_{[\alpha]}^{(B)}$ because B is flat over R.
(2) $\Rightarrow$ (1) :Since $B \subseteq A$, $I_{[\alpha]}B = B$ induces $I_{[\alpha]}A = A$.

Proposition 2.11. *Assume that α is super-primitive over R and that R contains an infinite field. If $R[\eta_d]$ is flat over R, then $A \cap K \subseteq R[\eta_d]$.*

Proof. Since R and $R[\eta_d]$ have the same quotient field K, the element α is of degree d over both R and $R[\eta_d]$. Put $I_{[\alpha]}^{(R[\eta_d])} := \bigcap_{i=1}^{d} I_{\eta_i}^{(R[\eta_d])}$, where $I_{\eta_i}^{(R[\eta_d])} := R[\eta_d] :_{R[\eta_d]} \eta_i$. Then $I_{[\alpha]} \subseteq I_{[\alpha]}^{(R[\eta_d])}$, so that $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \dots, \eta_d) \subseteq J_{[\alpha]}^{(R[\eta_d])} = I_{[\alpha]}^{(R[\eta_d])}(1, \eta_1, \dots, \eta_d)$, where $J_{[\alpha]}^{(R[\eta_d])} := I_{[\alpha]}^{(R[\eta_d])}(1, \eta_1, \dots, \eta_d)$. Since α is super-primitive over R, we have $\mathrm{grade}(J_{[\alpha]}) > 1$. Since $R[\eta_d]$ is flat over R, we have $\mathrm{grade}(J_{[\alpha]}R[\eta_d]) > 1$ and hence $\mathrm{grade}(J_{[\alpha]}^{(R[\eta_d])}) > 1$. So α is super-primitive over $R[\eta_d]$. Since $\eta_d \in R[\eta_d]$, we have $\bigcap_{i=1}^{d-1} I_{\eta_i}^{(R[\eta_d])} \subseteq I_{\eta_d}^{(R[\eta_d])} = R[\eta_d]$. So applying Lemma 2.5 to the extension $A/R[\eta_d]$, we obtain $A \cap K \subseteq R[\eta_d][\alpha] \cap K = R[\eta_d]$.

Theorem 2.12. *Assume that α is super-primitive over both R and $R[\eta_d]$ and that R contains an infinite field. Consider the following statements :*
(1) $I_{[\alpha]}A = A$,
(2) $R[\eta_d] \subseteq A$, $I_{[\alpha]} = I_{\eta_d}$ *and* $R[\eta_d]$ *is flat over* R,
(3) $A \cap K = R[\eta_d] = B$.
Then the implications (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) *hold.*

Proof. (1) + (2) $\Rightarrow$ (3) : (1) implies that $B \subseteq A$ by Lemma 2.9. Since $R[\eta_d]$ is flat over R, $R[\eta_d] \supseteq A \cap K$ by Proposition 2.11. Hence we have $A \cap K = R[\eta_d] \supseteq B = R[\eta_1, \dots, \eta_d]$.
(1) $\Rightarrow$ (2) : We have $R[\eta_d] \subseteq B \subseteq R[\alpha]$ by Theorem 2.10, and $I_{[\alpha]} = I_{\eta_d}$ by

Proposition 2.3. Since $R \hookrightarrow R[\eta_d] \hookrightarrow B$ is an open immersion by Lemma 1.3, $R \hookrightarrow R[\eta_d]$ is flat. (2) $\Rightarrow$ (1) : Since $R[\eta_d]$ is flat over R, we have $I_{[\alpha]}^{(R[\eta_d]} = I_{\eta_d} R[\eta_d]$. Thus the fact $\eta_d \in R[\eta_d]$ implies that $I_{[\alpha]}^{(R[\eta_d]} = R[\eta_d]$. So it follows that $I_{\eta_d} A = A$ because $R[\eta_d] \subseteq A$. Since $I_{[\alpha]} = I_{\eta_d}$, we conclude $I_{[\alpha]} A = A$.

§3. Coefficients of Minimal Polynomials.

Remark 3.1. Assume that α is anti-integral over R and that $\eta_d \in R$. Then A is faithfully flat over R if and only if A is flat over R. Indeed, since $\eta_d \in R$, we have $I_{[\alpha]} = \bigcap_{i=1}^{d} I_{\eta_i} = \bigcap_{i=1}^{d-1} I_{\eta_i}$ and hence $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1}) = \tilde{J}_{[\alpha]}$. Hence $\tilde{J}_{[\alpha]} = R$. Thus our conclusion follows Lemma 1.1.

Proposition 3.2. *Assume that α is a super-primitive element of degree d over R. Assume that the polynomial $\varphi(X) := X^{d-1} + \eta_1 X^{d-2} + \cdots + \eta_{d-1}$ is irreducible in $K[X]$ and let β is a solution of $\varphi(X) = 0$. Assume more that $\eta_d \in R$. Then β is super-primitive over R, and $R[\alpha]$ is flat over R if and only if $R[\beta]$ is flat over R.*

Proof. Since $\eta_d \in R$, noting that $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$ by definition, we conclude that $I_{[\alpha]} = I_{[\beta]}$ and hence $J_{[\alpha]} = J_{[\beta]}$.

Theorem 3.3. *Assume that K contains a field of characteristic zero and that $\eta_d \in R$. Let β be a solution of $\varphi'_\alpha(X) = 0$. Then*
(1) *if α is super-primitive over R, then so is β,*
(2) *$R[\alpha]$ is flat over R if and only if $R[\beta]$ is flat over R.*

Proof. By the similar argument in the proof of Proposition 3.2, we have $J_{[\alpha]} = J_{[\beta]}$.

Example 3.4. Consider the case $d = 2$ in Theorem 3.3. Put $\varphi_\alpha(X) := X^2 + \eta X + a$ with $a \in R$. Let α is a solution of an equation $\varphi_\alpha(X) = 0$. Then α is flat element over R, that is, $R[\alpha]$ is flat over $R \Leftrightarrow \eta$ is a flat element over R. In this case, α is characterized by η.

Lemma 3.5. *If $I_{[\alpha]}$ is an invertible ideal of R, then α is a super-primitive element over R.*

Proof. For each $p \in \mathrm{Spec}(R)$, $(I_{[\alpha]})_p$ is a principal ideal of R_p. So the conclusion follows [7,(2.11)].

Proposition 3.6. *Assume that $I_{[\alpha]} = I_{\eta_i}$ and that η_i is a flat element over R for some i, then α is a flat element over R. Moreover if $i \neq d$, then A is faithfully flat over R.*

Proof. Let η_i is flat element over R. Then $J_{\eta_i} = I_{\eta_i}(1, \eta_i) = R$, so that η_i is super-primitive over R by Lemma 3.5. Since $I_{[\alpha]} = I_{\eta_i}$ and $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) \supseteq J_{\eta_i} = R$, we have $J_{[\alpha]} = R$. So α is a flat element over R. Assume that $i \neq d$. Then $\tilde{J}_{[\alpha]} \supseteq I_{\eta_i}(1, \eta_i) = R$ and hence $\tilde{J}_{[\alpha]} = R$.

Theorem 3.7. *Assume that R is a local ring with maximal ideal m. Then A is flat over R if and only if $I_{[\alpha]} = I_{\eta_i}$ and η_i is flat over R for some i.*

Proof. ($\Leftarrow$) is shown in Proposition 3.6.
($\Rightarrow$) We have only to show this in the case $I_{[\alpha]} \subseteq m$. Since $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) = R$ by the assumption, there exists i such that $\eta_i I_{[\alpha]} = R$. Thus $I_{[\alpha]} = I_{\eta_i}$. Since $I_{\eta_i}(1, \eta_i) = I_{[\alpha]}(1, \eta_i) = R$, I_{η_i} is an invertible ideal. So by Lemma 3.5, η_i is super-primitive over R. Thus we conclude that η_i is a flat element over R.

Remark 3.8. Let (R, m) be a local ring. If there exists a prime ideal p of R such that none of $\eta_1, \ldots, \eta_d$ is flat element over R_p, then $R[\alpha]$ is not flat over R. Such p is the one not containing $J_{[\alpha]}$.

Example 3.9. Let R be a local ring $k[a, b]_{(a,b)}$, where $k[a, b]$ is a polynomial ring over a field k.
(1) Let α is a solution of the equation: $\varphi_\alpha(X) := X^2 + (b/a)X + a/b = 0$. Then $\varphi_\alpha(X)$ is a minimal polynomial of α over K and α is anti-integral over R because R is a Noetherian normal domain. We have $I_{[\alpha]} = abR$ and $J_{[\alpha]} = I_{[\alpha]}(1, b/a, a/b)R = ab(1, b/a, a/b)R = (ab, b^2, a^2)R \neq R$. So $A := R[\alpha]$ is not flat over R. We see that $\eta_1 := b/a$ and $\eta_2 := a/b$ and that neither I_{η_1} nor I_{η_2} is equal to R. Note here that $I_{[\alpha]} \neq I_{\eta_1}$ and $I_{[\alpha]} \neq I_{\eta_2}$.
(2) Let α is a solution of the equation : $\varphi_\alpha(X) := X^3 + (b/a)X^2 + (a/b)X + 1/a = 0$. Then α is anti-integral over R as in (1). It follows that $1/a$ is a flat element. But $I_{[\alpha]} = abR$ is equal to non of $I_{b/a}$, $I_{a/b}$ and $I_{1/a}$. Since $J_{[\alpha]} \neq R$, $R[\alpha]$ is not flat over R.

Theorem 3.10. *Assume that $I_{[\alpha]}$ is an invertible ideal of R. If A is flat over R, then for each $p \in \operatorname{Spec}(R)$ there exists i such that η_i is a flat element over R_p and that $I_{[\alpha]}R_p = I_{\eta_i}R_p$.*

Proof. Since $I_{[\alpha]}$ is an invertible ideal, α is super-primitive over R by Lemma 2.16. So A is flat over R if and only if $J_{[\alpha]} = R$. Take $p \in \mathrm{Spec}(R)$. Localizing at p, we may assume that R is a local ring with maximal ideal m. Since $I_{[\alpha]}$ is invertible, we have $I_{[\alpha]} = aR$ and $\eta_i = b_i/a$ for some $a, b_i \in R$. Assume first that $a \notin m$. Then $\eta_i \in R$ and hence η_i is a flat element over R. Assume next that $a \in m$. Then $J_{[\alpha]} = R$ and hence there exists i such that $b_i \notin m$. So $\eta_i = b_i/a$ is a flat element and $I_{[\alpha]} = I_{\eta_i}$.

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