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INFINITE GALOIS THEORY OF SEMISIMPLE RINGS

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Introduction. Throughout, a simple ring will mean a (right) Artinian two-sided simple ring with identity, and a semisimple ring will mean a direct sum of finite number of simple rings. In [4], N. Jacobson presented an infinite outer Galois theory for division rings. In [9] and [13], the theory was extended to simple rings by T. Nagahara and H. Tominaga. On the other hand in [16], O. E. Villamayor and D. Zelinsky presented a finite Galois theory for separable G-extensions of commutative semi-connected rings. In [5], K. Kishimoto and T. Nagahara presented a finite Galois theory for G-extensions of semi-connected rings which is a generalization of [16] to non-commutative rings. In [11], T. Nagahara and the present author presented an infinite outer Galois theory for semisimple rings which is a generalization of [8. Corolary 1.4] applying the same methods as in the [5;16]. This paper is the continuation of the preceding paper [11]. In this paper, we shall present some Galois theory for semisimple rings which are not outer. Moreover, some results in [8;9] will be generalized to semisimple rings.

In what follows, we shall summarize the notations and definitions which will be used very often in the subsequent study. Throughout the present paper, B will mean a semisimple ring with $P = \{e_1, \dots, e_n\}$ the set of all central primitive idempotents of B, and A a semisimple subring of B containing the identity element of B. By G, we denote the group of all A-ring automorphisms of B. For a subset K of G and subsets T and S of B, we shall use the following conventions:

$$T(K) = \{b \in T \mid \sigma(b) = b \text{ for all } \sigma \in K\}$$
$$K(T) = \{\sigma \in K \mid \sigma(t) = t \text{ for all } t \in T\}$$
$$V_T(S) = \{t \in T \mid ts = st \text{ for all } s \in S\}$$
$$Z(T) = V_T(T)$$
$$\langle T \rangle = \text{the set of all inner automorphisms induced by the regular elements of T}$$
$$K|_T = \text{the restriction of } K \text{ to } T$$
$$KT = \{\sigma(a) \mid \sigma \in K, a \in T\}$$

Moreover, if T is a subring containing S, then we shall use the following conventions:

$$\mathrm{Aut}(T/S) = \text{the group of all } S\text{-ring automorphisms of } T$$
$$id_T = \text{identity map on } T$$

Next, let $T' \supseteq T$ be intermediate semisimple rings of B/A. By $[T']$, we denote the cardinal number of the set of central primitive idempotents in T'. The subring T is called *regular in* T' if T, $V_{T'}(T)$ are semisimple rings and $[T'] = [V_{T'}(T)]$. As is easily seen, if T' is a simple ring and T is regular in T' then T is also a simple ring. In general, if T is regular in T' then $[T] \leq [T']$. The extension T'/T is said to be *Galois* (or *infinite Galois*) if T is regular in T' and $T'(\mathrm{Aut}(T'/T)) = T$. If T'/T is Galois and $V_{T'}(T) = Z(T')$, then we call that T'/T is *outer Galois*. T'/T is said to be *(left) locally finite* if $T[F]$ is finitely generated as a left T-module (abr. left finite over T) for each finite subset F of T'. If for each finite subset F of T' there exists an intermediate semisimple ring N of $T'/T[F]$ such that N/T is Galois and left finite, then T'/T is said to be *locally Galois*. If for each finite subset F of T' there exists an intermediate semisimple ring N' of $T'/T[F]$ such that N'/T is Galois and left finite, and $\mathrm{Aut}(T'/T)|_{N'} \supseteq \mathrm{Aut}(N'/T)$ then we shall say that T'/T is $\mathrm{Aut}(T'/T)$*-locally Galois*.

Now, let K be a subgroup of G. By K^*, we denote the set of all elements σ in G such that for each $e \in P$, $\sigma|_{Be} = \tau|_{Be}$ for some $\tau \in K$. If $K = K^*$, then K will be called to be *fat*. The definition of *"fat"* have been appeared in [16] firstly. Moreover, if $B(K)$ is regular in B and $K \supseteq \langle V_B(B(K))\rangle$ then K is called *regular*. At last, we shall introduce a finite topology on G in which the collection of sets of form $\{\tau \in G \mid \tau(x_i) = \sigma(x_i)\}$ is a basis of the open sets, where $\{x_i\}$ is a finite subset of B and σ is a fixed element of G. And for a subset K of G, by ClK, we denote the closure of K in G.

In §1 of this paper, we will consider some Galois extensions of semisimple rings with Galois groups which are locally compact. The principal theorem of this section is the next: Let B/A be Galois and locally finite, and let $V_B(A)/Z(B)$ be left finite. Then G is locally compact and there exists a 1-1 dual correspondence in the usual sense of Galois theory between fat and closed regular subgroups of G and regular intermediate rings of B/A. This theorem is a generalization of [9. Corollary 4.2] to semisimple rings and a generalization of [11. Main Theorem] to non outer case. In §2–§4 we shall deal with locally Galois extension. The Galois groups which will be considered in these sections are both locally compact and not locally compact. In §2, we shall see that if $V_B(A)/Z(B)$ is left finite and B/A is locally finite then the conditions Galois, locally Galois, and G-locally Galois are all equivarent. In §3, we shall deal with intermediate rings which are locally Galois over A. In §4, we shall deal with subrings whose extensions B are locally Galois. In the last section §5, we shall present some Galois theory of semisimple rings with Galois groups which are not locally compact. These are some generalizations of §1.

Now, if $V_B(A)$ is semisimple and $m = [V_B(A)]$ then the set of idempotents in $V_B(A)$ is of the cardinal number 2^m. When this is the case, noting $Z(B) \subseteq Z(V_B(A))$ we see that the idempotents of $Z(V_B(A))$ are contained in $Z(B)$ if and only if $m = n(= [B])$.

In this note, we assume that $V_B(A)$ is semisimple and $[V_B(A)] = n(= [B])$. (In another words, A is regular in B.) Moreover, we write $V_B(A) = V_1 \oplus \cdots \oplus V_n$, where the V_i are simple rings. Then P is the set of all central primitive idempotents of $V_B(A)$. Thus, we may assume that for each i, $V_i = V_{Be_i}(Ae_i)$ and e_i is a unit of V_i. (In this case, the Ae_i are simple rings since each Ae_i is semisimple and $Z(Ae_i) \subseteq Z(V_i) =$ field.) Moreover, we shall use the following conventions: $B_i = Be_i$, $A_i = Ae_i$, and $G_i = \mathrm{Aut}(B_i/A_i)$. $C = Z(B)$, $V = V_B(A)$, and $H = V_B(V)$. And we set the following. $B_i = \sum_{s,t} D_i e_{st}^{(i)}$ where the $\{e_{st}^{(i)} \mid s,t\}$ are matrix units of B_i and the $D_i = V_{B_i}(\{e_{st}^{(i)} \mid s,t\})$ division rings. $V_i = \sum_{p,q} D'_i g_{pq}^{(i)}$ where the $\{g_{pq}^{(i)} \mid p,q\}$ are matrix units of V_i and the $D'_i = V_{V_i}(\{g_{pq}^{(i)} \mid p,q\})$ division rings.

1 Galois theory of semisimple rings with Galois Groups which are Locally Compact.

In [9], T. Nagahara and H. Tominaga have obtained a Galois theory for simple rings as following:

Let B/A be Galois as simple rings and locally finite, and V/C left finite. Then G is locally compact and there exists a 1-1 dual correspondence in the usual sense of Galois theory between closed regular subgroups of G and regular intermediate rings of B/A. (Cf. [9. Lemma 1.7, Theorem 4.2 and Corollary 4.2])

In this section, we extend this theory to semisimple rings. Firstly by making use of the same methods as in the proof of [5. Lemma 9], we obtain the next proposition.

Proposition 1.1. *If K is a subgroup of G, then $B_i(K(\{e_i\})|_{B_i}) = B(K)e_i$ for all i. In particular, if B/A is Galois then, for each i, B_i/A_i is a Galois extension of simple rings.*

Remark 1.2. *Let T be an intermediate ring of B/A. Then the following conditions are equivalent.*

(1) *T is regular in B.*

(2) *T is semisimple and the $V_{B_i}(Te_i)$ are simple rings.*

(3) *The Te_i are regular in B_i.*

The next lemma is used very often in the subsequent study.

Lemma 1.3. (1) *Let $A = Af_1 \oplus \cdots \oplus Af_m$ where the Af_j are simple rings and the f_j are identity elements of Af_j. Then $f_j \in C$.*

(2) *If B/A is locally finite then all the B_i/A_i are locally finite.*

(3) *If B/A is Galois and A is a simple ring. Then for any permutation σ on the set $\{1, \ldots, n\}$ there exists $\delta_\sigma \in G$ such that $\delta_\sigma(e_i) = e_{\sigma(i)}$ for all i.*

(4) *Let $A = Af_1 \oplus \cdots \oplus Af_m$ where the Af_j are simple rings and the f_j are identity elements of Af_j. If all the Bf_j/Af_j are locally Galois, then B/A is locally Galois.*

(5) *Let $A = Af_1 \oplus \cdots \oplus Af_m$ where the Af_j are simple rings and the f_j are identity elements of Af_j. If all the Bf_j/Af_j are $\mathrm{Aut}(Bf_j/Af_j)$-locally Galois, then B/A is G-locally Galois.*

(6) *Let T, T', T^*, be subrings of B. Let $T = \sum_{i,j} De_{ij}$ where $\{e_{ij}|i,j\}$ is a matrix units and $D = V_T(\{e_{ij}|i,j\})$ a division ring. If T' is an Artinian ring and $T \supseteq V_{T'}(T^*) \supseteq \{e_{ij}|i,j\}$, then $V_{T'}(T^*)$ is a simple ring.*

(7) *Let T be a simple subring of B and $T = \sum_{i,j} De_{ij}$ where $\{e_{ij}|i,j\}$ is a matrix units and $D = V_T(\{e_{ij}|i,j\})$ a division ring. If $T \supseteq T' \supseteq \{e_{ij}|i,j\}$ and T' is Artinian, then T' is a simple ring.*

Proof. (1) Clearly, $f_j \in Z(A) \subseteq Z(V) = Z(V_1) \oplus \cdots \oplus Z(V_n)$. Otherwise each idempotent of $Z(V)$ is an idempotent of C since the cardinal number of the set of idempotents in $Z(V) = Z(V_1) \oplus \cdots \oplus Z(V_n)$ and the cardinal number of the set of idempotents in $C = Z(B_1) \oplus \cdots \oplus Z(B_n)$ are both 2^n. Thus $f_j \in C$.

(2) Let F be a finite subset of B_1. Then $A[F]/A$ is left finite. Thus $A[F]e_1/A_1$ is left finite and $A[F]e_1 = A_1[F]$. Thus B_1/A_1 is locally finite. Similarly all the B_i/A_i are locally finite.

(3) Since G is transitive on the set P, there exists a subset $\{\delta_1, \dots, \delta_n\}$ of G such that $\delta_i(e_i) = e_{\sigma(i)}$. Now we define a map $\delta_\sigma : B \to B$ by

$$\delta_\sigma(b) = \delta_1(be_1) + \cdots + \delta_n(be_n)$$

for all $b \in B$. Then δ_σ is an A-automorphism of B and $\delta_\sigma(e_i) = e_{\sigma(i)}$ for all i.

(4) Let F be a finite subset of B. Then, for each j, there exists an intermediate semisimple ring N_j of $Bf_j/Af_j[Ff_j]$ such that N_j/Af_j is Galois and left finite. Thus, $N_1 \oplus \cdots \oplus N_m/A$ is Galois and left finite, and $N_1 \oplus \cdots \oplus N_m \supseteq F$.

(5) Let F be a finite subset of B. Then, for each j, there exists an intermediate semisimple ring N_j of $Bf_j/Af_j[Ff_j]$ such that N_j/Af_j is Galois and left finite, and $\mathrm{Aut}(Bf_j/Af_j)|_{N_j} \supseteq \mathrm{Aut}(N_j/Af_j)$. Thus, $N_1 \oplus \cdots \oplus N_m/A$ is Galois and left finite, and $N_1 \oplus \cdots \oplus N_m \supseteq F$. Moreover,

$$\begin{aligned} G|_{N_1\oplus\cdots\oplus N_m} &= \mathrm{Aut}(Bf_1/Af_1)|_{N_1} \times \cdots \times \mathrm{Aut}(Bf_m/Af_m)|_{N_m} \\ &\supseteq \mathrm{Aut}(N_1/Af_1) \times \cdots \times \mathrm{Aut}(N_m/Af_m) \\ &= \mathrm{Aut}(N_1 \oplus \cdots \oplus N_m/A). \end{aligned}$$

(6) As is easily seen, $V_{T'}(T^*) = \sum_{i,j} D'e_{ij}$ where $D' = V_{V_{T'}(T^*)}(\{e_{ij}|i,j\}) \subseteq D$. Let d be a nonzero element of D'. Then, $d^{-1} \in T'$ since $d \in T'$ and T' is an Artinian ring. Thus $d^{-1} \in V_{T'}(T^*)$. Thus $d^{-1} \in D'$. Thus D' is a division ring.

(7) If $T^* = C$ then $V_{T'}(T^*) = T'$. Thus by (6), T' is a simple ring.

By Lemma 1.3 (3), and by making use of the same methods as in the proof of [5. §3. I. (iii)], we obtain the following proposition.

Proposition 1.4. *Let B/A be a Galois extension and A a simple ring. Then, for any permutation σ on the set $\{1, \dots, n\}$ there exists $\delta_\sigma \in G$ such that $\delta_\sigma(e_i) = e_{\sigma(i)}$ for all i. Moreover, let δ_σ be an element of G as above, $K = \{\delta_\sigma \mid \sigma$ runs over all the permutations on the set $\{1, \dots, n\}\}$, and $K' = G_1 \times \cdots \times G_n$. Then $G = K'K = KK'$.*

Lemma 1.5. *Let B/A be Galois and T a regular intermediate ring of B/A. We assume that for each i, $B_i(G_i(Te_i)) = Te_i$ and $G_i|_{Te_i} \supseteq$*

$G'(Te_i/A_i)$, where $G'(Te_i/A_i)$ signifies the set of all A_i-ring isomorphisms of Te_i into B_i. Then $B(G(T)) = T$.

Proof. Let $f_1, \dots, f_r$ be central primitive idempotents of T. Then

$$B(G(T)) = Bf_1(\mathrm{Aut}(Bf_1/Tf_1)) \oplus \cdots \oplus Bf_r(\mathrm{Aut}(Bf_r/Tf_r)).$$

Thus we may assume that T and A are simple rings. We set $T^* = Te_1 \oplus \cdots \oplus Te_n$. Since $B_i(G_i(Te_i)) = Te_i$, $B(G(T)) \subseteq T^*$. On the other hand $T \simeq Te_i$ $(t \leftrightarrow te_i)$ since T is a simple ring, for every i. Obviously, there exists a T-automorphism τ of T^* such that $\tau(e_i) = e_{i+1}$ $(1 \leq i \leq n-1)$ and $\tau(e_n) = e_1$. Then we have $T^*(\tau) = T$. By Proposition 1.4, there is an A-automorphism μ of B such that $\mu|_{\{e_1,\dots,e_n\}} = \tau|_{\{e_1,\dots,e_n\}}$. Hence $\mu^{-1}\tau|_{\{e_1,\dots,e_n\}} = id_{\{e_1,\dots,e_n\}}$. Since $G_i|_{Te_i} \supseteq G'(Te_i/A_i)$, there is an ν in G such that $\nu|_{T^*} = \mu^{-1}\tau|_{T^*}$. Then $\mu\nu \in G$ and $\mu\nu|_{T^*} = \tau$. Thus $\mu\nu \in G(T)$ and $T^*(\mu\nu) = T$. Hence $B(G(T)) = T$.

Proposition 1.6. *Let B/A be Galois and locally finite, and let V/C be left finite. Let T be a regular intermediate ring of B/A. Then B/T is Galois.*

Proof. By Proposition 1.1 and Lemma 1.3 (2), for each i, B_i/A_i is Galois and locally finite, and $V_i/Z(B_i)$ left finite. Thus, by [9. Lemma 2.2, Theorem 2.3], each B_i/A_i is locally Galois and $V_i/Z(B_i)$ left finite. Thus, by [8. Theorem 4.2 (ii)], the B_i/Te_i are Galois. Moreover, by [8. Theorem 4.2 (i)], $G_i|_{Te_i} \supseteq G'(Te_i/A_i)$ for all i, where the $G'(Te_i/A_i)$ signifie the sets of all A_i-ring isomorphisms of Te_i into B_i. Thus, by Lemma 1.5, B/T is Galois.

Lemma 1.7. *Let K be a regular subgroup of G. Then each $K(\{e_i\})|_{B_i}$ is a regular subgroup of G_i for all i.*

Proof. By the definition of regular, each $B_i(K(\{e_i\})|_{B_i})$ and $V_{B_i}(B_i(K(\{e_i\})|_{B_i}))$ are simple rings for all i. Moreover, by the hypothesis, $K \supseteq \langle V_B(B(K))\rangle$. Since each $e_i \in Z(B)$, $K(\{e_i\}) \supseteq \langle V_B(B(K))\rangle$. Otherwise $V_B(B(K)) = V_{B_i}(B(K)e_i) \oplus V_{B(1-e_i)}(B(K)(1-e_i))$. Thus

$$\begin{aligned} K(\{e_i\})|_{B_i} &\supseteq \langle V_B(B(K))\rangle|_{B_i} \\ &= \langle V_{B_i}(B(K)e_i)\rangle \\ &= \langle V_{B_i}(B_i(K(\{e_i\})|_{B_i}))\rangle \end{aligned}$$

Lemma 1.8. *Let K be a fat and closed subgroup of G. Then each*

$K(\{e_i\})|_{B_i}$ is a closed subgroup of G_i for all i.

Proof. Let σ be an element of $G_i \setminus K(\{e_i\})|_{B_i}$. Now, we define a map σ_1 of B to B by

$$\sigma_1(b) = \sigma(be_i) + b(1 - e_i)$$

for all $b \in B$. Then $\sigma_1 \in G$ and $\sigma_1|_{B_i} = \sigma$. Thus $\sigma_1 \notin K(\{e_i\})$. Now, $\sigma_1 \notin K$ since $\sigma_1(e_i) = \sigma(e_i) = e_i$. Because K is closed, there is a finite subset X of B such that

$$\{\tau \in G | \tau(x) = \sigma_1(x) \text{ for all } x \in X\} \cap K = \phi.$$

Now, we shall show that

$$\{\tau \in G_i | \tau(xe_i) = \sigma(xe_i) \text{ for all } x \in X\} \cap (K(\{e_i\})|_{B_i}) = \phi.$$

The proof is by contradiction. Assume the assertion were false. Then we would have that there is some element

$$\tau_0 \in \{\tau \in G_i | \tau(xe_i) = \sigma(xe_i) \text{ for all } x \in X\} \cap (K(\{e_i\})|_{B_i}).$$

We define a map τ_1 of B to B by

$$\tau_1(b) = \tau_0(be_i) + b(1 - e_i).$$

Then

$$\tau_1|_{B_i} = \tau_0 \in K(\{e_i\})|_{B_i} \subseteq K|_{B_i}$$

and

$$\tau_1|_{B(1-e_i)} = id|_{B(1-e_i)} \in K|_{B(1-e_i)}.$$

Thus, $\tau_1 \in K$ since K is fat. Moreover $\tau_1(x) = \sigma_1(x)$ for all $x \in X$. This is a contradiction. Thus each $K(\{e_i\})|_{B_i}$ is closed in G_i.

Lemma 1.9. *Let K be a fat subgroup of G. If $G_i(B_i(K(\{e_i\})|_{B_i})) = K(\{e_i\})|_{B_i}$ for all i, then $G(B(K)) = K$.*

Proof. We may assume that $K\{e_1\} = \{e_1, \ldots, e_m\}$ $(m \leq n)$. Then $\sum_{i=1}^{m} e_i = e \in B(K)$. Let $\sigma \in G(B(K))$. Since $\sigma(e) = e$, there is some i such that $\sigma(e_1) = e_i$ where $1 \leq i \leq m$. Now, there is an element τ of K such that $\tau(e_i) = e_1$. Since $\tau\sigma \in G(B(K))$ and $\tau\sigma(e_1) = e_1$, $\tau\sigma|_{B_i} \in G_1(B(K)e_1) = G_1(Be_1(K(\{e_1\})|_{Be_1})) = K(\{e_1\})|_{Be_1}$. Thus, $\tau\sigma|_{Be_1} \in K|_{Be_1}$ and $\sigma|_{Be_1} \in K|_{Be_1}$. Similarly, $\sigma|_{B_i} \in K|_{B_i}$ for all i. Thus, $\sigma \in K$

since K is fat.

Proposition 1.10. *Let B/A be Galois and locally finite, and V/C left finite. Let K be a fat and closed regular subgroup of G. Then $G(B(K)) = K$.*

Proof. By Lemma 1.7 and Lemma 1.8, the $K(\{e_i\})|_{B_i}$ are closed regular subgroup of G_i. Otherwise, by Proposition 1.1 and Lemma 1.3 (2), the B_i/A_i are Galois and locally finite. Moreover, the $V_i/Z(B_i)$ are left fintie. Thus by [9. Lemma 1.7 and Corollary 4.2], $G_i(B_i(K(\{e_i\})|_{B_i})) = K(\{e_i\})|_{B_i}$. Thus, by Lemma 1.9, $G(B(K)) = K$ since K is fat.

Proposition 1.11. *If all the G_i are locally compact then G is locally compact.*

Proof. Since G is a topology group, it suffices to prove for special one element only. We put $id_B \in G$. Since the G_i are locally compact, there exist some finite subsets $F_i \subseteq B_i$ such that $Cl\{\tau_i \in G_i|\tau_i(b_i) = id_{B_i}(b_i)$ for all $b_i \in F_i\}$ are compact for all i. Here, we may assume that for each i, $F_i \ni e_i$. Let $\{\tau_i \in G_i|\tau_i(b_i) = id_{B_i}(b_i)$ for all $b_i \in F_i\} = U_i$ for all i. Now

$$\begin{aligned} & Cl\{\tau \in G \mid \tau(b) = id_B(b) \text{ for all } b \in \cup_i F_i\} \\ &= \{\tau \in G \mid \tau(b) = id_B(b) \text{ for all } b \in \cup_i F_i\} \\ &= U_1 \times \cdots \times U_n \\ &= ClU_1 \times \cdots \times ClU_n \end{aligned}$$

Thus $Cl\{\tau \in G \mid \tau(b) = id_B(b)$ for all $b \in \cup_i F_i\}$ is compact. Hence G is locally compact.

Theorem 1.12. *Let B/A be Galois and locally finite, and V/C left finite. Then G is locally compact and the following conditions are satisfied;*
(1) *Let T be a regular intermediate ring of B/A. Then $B(G(T)) = T$, and $G(T)$ is a fat and closed regular subgroup.*
(2) *Let K be a subgroup which is fat and closed regular. Then $G(B(K)) = K$. Moreover, $B(K)$ is regular in B.*

Proof. At first, for any intermediate ring T, $G(T)$ is fat closed and $G(T) \supseteq \langle V_B(G(T))\rangle$. Let T be a regular intermediate ring of B/A. Then, by Proposition 1.6, $B(G(T)) = T$. Hence, for each i, $B_i((G(T)(\{e_i\}))|_{B_i}) = Te_i$ and $V_{B_i}(B_i((G(T)(\{e_i\}))|_{B_i})) = V_{B_i}(Te_i)$ by Proposition 1.1. Thus for each i, $B_i((G(T)(\{e_i\}))|_{B_i})$ and $V_{B_i}(B_i((G(T)(\{e_i\}))|_{B_i}))$ are simple rings. Thus $G(T)$ is a regular subgroup. Next, let K be a regular subgroup of G and $B(K) = T$. Then T is regular in B. Let K be a fat and closed regular subgroup of G. Then $G(B(K)) = K$ by Proposition 1.10. At last, we shall prove that G is locally compact. By Lemma 1.3 (2), the B_i/A_i are locally finite, and $V_i/Z(B_i)$ are left finite for all i. Thus, by [9. Lemma 1.7], the G_i are locally compact. Thus, by Proposition 1.11, G is locally compact.

Theorem 1.13. *Let B/A be Galois and locally finite and V/C left finite. Let $V = D'_1 \oplus \cdots \oplus D'_n$, where each D'_i is a division ring. Then G is locally compact and the following conditions are satisfied;*

(1) *For each intermediate ring T of B/A, T is semisimple and $B(G(T)) = T$. Moreover $G(T)$ is a fat and closed regular subgroup.*
(2) *Let K be a subgroup which is fat and closed regular, then $G(B(K)) = K$.*

Proof. By Proposition 1.1 and [9. Lemma 2.2 and Theorem 2.3], we know that all the B_i/A_i are locally Galois. Let T be an intermediate ring of B/A. By [8. Theorem 1.1], each Te_i is a simple ring. Thus, T is a semisimple ring, since T is a subdirect sum of $Te_1, \ldots, Te_n$. Moreover each $V_{B_i}(Te_i)$ is a division ring since $D'_i \supseteq V_{B_i}(Te_i)$. Thus, by Theorem 1.12, this theorem is proved.

Remark 1.14. Let B/A be locally finite. As is shown in the proof of Theorem 1.12, if V/C is left finite, then G is locally compact. Conversely, if G is locally compact, then each G_i is locally compact. Whence, by [9. Lemma 1.7], each $V_i/Z(B_i)$ is left finite. Thus V/C is left finite. Therefore, it follows that G is locally compact if and only if V/C is left finite. This is a generalization of [9. Lemma 1.7] to semisimple rings.

In the last of this section, we consider the extensions of automorphisms. The following Proposition 1.15 and Corollary 1.16 are some generalizations of [8. Theorem 4.2 (i)] to semisimple rings.

Proposition 1.15. *Let B/A be Galois and locally finite, and V/C left finite. For each regular intermediate rings T_1, T_2 of $B/A_1 \oplus \cdots \oplus A_n$ every A-ring isomorphism of T_1 onto T_2 can be extended to an automorphism of B.*

Proof. We may assume that A is a simple ring. Let σ be an A-ring isomorphism of T_1 onto T_2. Then $\sigma(P) = P$ since P the set of all central primitive idempotents of T_1 and T_2. By Lemma 1.3 (3), there exists an element $\tau \in G$ such that $\tau(e_i) = \sigma(e_i)$ for all i. Then each $\tau^{-1}\sigma|_{T_1e_i}$ is an A_i-ring isomorphism of T_1e_i onto T_2e_i. Now, by Proposition 1.1 and [9. Lemma 2.2], the B_i/A_i are locally Galois. Thus, by [8. Theorem 4.2], there exist the automorphisms δ_i of B_i which are extensions of $\tau^{-1}\sigma|_{T_1e_i}$. Now we define a map $\delta : B \to B$ by

$$\delta(b) = \delta_1(be_1) + \cdots + \delta_n(be_n)$$

for all $b \in B$. Then $\delta|_{T_1} = \tau^{-1}\sigma$. Thus $\tau\delta$ is an automorphism of B and $\tau\delta|_{T_1} = \sigma$.

Corollary 1.16. *Let B/A be Galois and locally finite, and V/C left finite. Moreover, let A be a simple ring. Then for each intermediate simple rings T_1, T_2 of B/A which are regular in B, every A-ring isomorphism of T_1 onto T_2 can be extended to an automorphism of B.*

Proof. Let σ be an A-ring isomorphism of T_1 onto T_2. Now $T_1 \simeq T_1e_i$ $(b_1 \leftrightarrow b_1e_i)$ and $T_2 \simeq T_2e_i$ $(b_2 \leftrightarrow b_2e_i)$ for all i where $b_1 \in T_1$ and $b_2 \in T_2$ since T_1 and T_2 are simple rings. Thus the isomorphisms $\sigma_i : T_1e_i \to T_2e_i$, $(\sigma_i(b_1e_i) = \sigma(b_1)e_i)$ are well-defined. Now, we define a map

$$\sigma' : T_1e_1 + \cdots + T_1e_n \to T_2e_1 + \cdots + T_2e_n$$

by

$$\sigma'(t_1e_1 + \cdots + t_ne_n) = \sigma_1(t_1e_1) + \cdots + \sigma_n(t_ne_n)$$

for all $t_i \in T_1$. Then σ' is an extension of σ. Now, by Proposition 1.15, there exists an element $\tau \in G$ which is an extension of σ' and thus $\tau|_{T_1} = \sigma$.

2 The some relations of Galois, locally Galois, and G-locally Galois.

In [9], T. Nagahara and H. Tominaga have obtained the following theorems.

(α) *B/A is G-locally Galois as simple rings if and only if B/A is Galois and locally Galois as simple rings.*

(β) *If B and A are simple rings, B/A locally finite, and V/C left finite then, the conditions Galois, locally Galois, and G-locally Galois are all equivarent.*

In this section we will generalize these theorems to semisimple rings (Theorem 2.3 and Theorem 2.17) and we shall present some conditions which induce the condition G-locally Galois (Proposition 2.5 and Proposition 2.6).

Lemma 2.1. *Let B/A be Galois and let each B_i/A_i be G_i-locally Galois. Then B/A is G-locally Galois.*

Proof. By Lemma 1.3 (5), we may assume that A is a simple ring. Then, by Lemma 1.3 (3), there exist the automorphisms δ_i of B_1 onto B_i such that $\delta_i(ae_1) = ae_i$ $(a \in A)$, for all i. Here we may assume that $\delta_1 = id_{B_1}$. Let F be a finite subset of B. Let $F' = \cup_i \delta_i^{-1}(Fe_i)$. Then there exists an intermediate simple ring N of $B_1/A_1[F']$ such that N/A_1 is Galois and left finite, and $G_1|_N \supseteq \mathrm{Aut}(N/A_1)$. Let $N_i = \delta_i(N)$. Then the N_i/A_i are Galois and left finite, and $G_i|_{N_i} \supseteq \mathrm{Aut}(N_i/A_i)$. Thus $N_1 \oplus \cdots \oplus N_n/A_1 \oplus \cdots \oplus A_n$ is Galois and left finite, and $N_1 \oplus \cdots \oplus N_n \supseteq F$. Now we define a map $\delta_{1,2} : N_1 \oplus \cdots \oplus N_n \to N_1 \oplus \cdots \oplus N_n$ by

$$\delta_{1,2}(b_1 + \cdots + b_n) = \delta_2(b_1) + \delta_2^{-1}(b_2) + b_3 + \cdots + b_n$$

for all $b_i \in N_i$. Then $\delta_{1,2} \in \mathrm{Aut}(N_1 \oplus \cdots \oplus N_n/A)$. Let b be an element of $(N_1 \oplus \cdots \oplus N_n)(\mathrm{Aut}(N_1 \oplus \cdots \oplus N_n/A))$. Since $N_1 \oplus \cdots \oplus N_n/A_1 \oplus \cdots \oplus A_n$ is Galois, we get the expression $b = a_1e_1 + \cdots + a_ne_n$ $(a_i \in A)$. Then

$$\begin{aligned} a_1e_1 + \cdots + a_ne_n &= \delta_{1,2}(a_1e_1 + \cdots + a_ne_n) \\ &= a_1e_2 + a_2e_1 + \cdots + a_ne_n. \end{aligned}$$

Thus $a_1e_2 = a_2e_2$. Similarly $a_1e_i = a_ie_i$ for all i. Thus $b = a_1 \in A$. Thus $N_1 \oplus \cdots \oplus N_n/A$ is Galois and left finite. Now let τ be an element of $\mathrm{Aut}(N_1 \oplus \cdots \oplus N_n/A)$. Then there exists some permutation σ on the set $\{1, \ldots, n\}$ such that $\tau^{-1}(e_i) = e_{\sigma(i)}$. Let $\delta = \sum_i \delta_{\sigma(i)}\delta_i^{-1}|_{B_i}$. Then $\delta \in G$, $\delta(N_1 \oplus \cdots \oplus N_n) = N_1 \oplus \cdots \oplus N_n$, and $\delta\tau(N_i) = N_i$ for all i. Since $G_i|_{N_i} \supseteq \mathrm{Aut}(N_i/A_i)$, there exists $\tau_i \in G_i$ such that $\tau_i|_{N_i} = \delta\tau|_{N_i}$ for each i. Now we define a map $\tau' : B \to B$ by

$$\tau' = \tau_1(be_1) + \cdots + \tau_n(be_n)$$

for all $b \in B$. Then $\tau' \in G$ and $\tau'|_{N_1 \oplus \cdots \oplus N_n} = \delta\tau$. Thus $\delta^{-1}\tau' \in G$ and $\delta^{-1}\tau'|_{N_1 \oplus \cdots \oplus N_n} = \tau$. Thus B/A is G-locally Galois.

Lemma 2.2. *Let B/A be locally Galois. Then the B_i/A_i are locally Galois as simple rings.*

Proof. It suffices to prove that B_1/A_1 is locally Galois as simple rings. Let F be a finite subset of B_1. Then there exists a semisimple subring N of B such that N/A is Galois and left finite, and $N \supseteq A[F, \{e_{st}^{(1)} \mid s, t\}, \{g_{pq}^{(1)} \mid p, q\}]$. Since $N \ni e_1$, Ne_1/A_1 is Galois and $B_1 \supseteq Ne_1 \supseteq A_1$. Obviously $Ne_1 \supseteq F$ and Ne_1/A_1 is left finite. By Lemma 1.3 (7), Ne_1 is a simple ring since $B_1 \supseteq Ne_1 \supseteq A_1[\{e_{st}^{(1)} \mid s, t\}]$. Moreover, by Lemma 1.3 (6), $V_{Ne_1}(A_1)$ is a simple ring since $V_{B_1}(A_1) \supseteq V_{Ne_1}(A_1) \supseteq \{g_{pq}^{(1)} \mid p, q\}$. Thus B_1/A_1 is locally Galois as simple rings.

Theorem 2.3. *B/A is G-locally Galois if and only if B/A is Galois and locally Galois.*

Proof. The only if part is clear. Let B/A be Galois and locally Galois. Then, by Proposition 1.1 and Lemma 2.2, the B_i/A_i are Galois and locally Galois. Thus, by [9. Theorem 2.3], the B_i/A_i are G_i-locally Galois. Then, by Lemma 2.1, B/A is G-locally Galois.

Corollary 2.4. *Let B/A be G-locally Galois. Then the B_i/A_i are G_i-locally Galois as simple rings.*

Proof. By Theorem 2.3, B/A is Galois and locally Galois. Thus, by Proposition 1.1 and Lemma 2.2, all the B_i/A_i are Galois and locally Galois. Thus, by [9. Theorem 2.3], the B_i/A_i are G_i-locally Galois.

The next proposition is a generalization of [9. Theorem 2.4] to semisimple rings.

Proposition 2.5. *Let B/A be Galois and locally finite, and $V/Z(V)$ left finite. Then B/A is G-locally Galois.*

Proof. By Proposition 1.1 and Lemma 1.3 (2), the B_i/A_i are Galois and locally finite. Moreover the $V_i/Z(V_i)$ are left finite. Thus, by [9. Theorem 2.4], the B_i/A_i are G_i-locally Galois. Thus, by Lemma 2.1, B/A is G-locally Galois.

Proposition 2.6. *Let B/A be Galois and each B_i/A_i locally Galois. Then B/A is G-locally Galois.*

Proof. By Proposition 1.1, the B_i/A_i are Galois. Thus, by [9. Theorem 2.3], the B_i/A_i are G_i-locally Galois. Thus, by Lemma 2.1, B/A is G-locally Galois.

Next, we give some lemmas and propositions to obtain some generalization of (β) to semisimple rings. At first, we give the next proposition. This proposition is a generalization of [8. Theorem 1.1] to semisimple rings.

Proposition 2.7. *Let B/A be locally Galois. Then the following conditions are equivalent.*
(i) *Every intermediate ring of B/A is a semisimple ring.*
(ii) $V = D'_1 \oplus \cdots \oplus D'_n$ *where the D'_i are division rings.*

Proof. (ii)$\Rightarrow$(i). Let T be an intermediate ring of B/A. By Lemma 2.2, the B_i/A_i are locally Galois as simple rings. By [8. Theorem 1.1], the Te_i are simple rings since the $V_i = D'_i$ are division rings. Thus T is a semisimple ring since T is a subdirect sum of $Te_1, \ldots, Te_n$.

(i)$\Rightarrow$(ii). We fix for some i. As is easily seen, every intermediate ring of B_i/A_i is a semisimple ring. Let T_i be an intermediate ring of B_i/A_i. By the assumption, T_i is a semisimple ring. We now assume that $T_i = S_1 \oplus \cdots \oplus S_m$ where the S_j are simple rings and $m > 1$. Let the f_j are identity elements of S_j. Now let $T' = \sum_{u \leq v} f_u B_i f_v$ and $I = \sum_{u<v} f_u B_i f_v$. Then T' is an intermediate ring of B_i/A_i, I a nonzero ideal of T', and $I^m = 0$. Thus T' is not a semisimple ring. This is a contradiction. Thus every intermediate ring of B_i/A_i is a simple ring for each i. Thus, by [8. Theorem 1.1], the V_i are division rings. Thus $V = D'_1 \oplus \cdots \oplus D'_n$ where the D'_i are division rings.

Corollary 2.8. *Let B be a simple ring. If every intermediate ring of B/A is a semisimple ring, then every intermediate ring of B/A is a simple ring.*

Proof. We may prove this corollary by the same methods as in the proof of (i)$\Rightarrow$(ii) of Proposition 2.7.

Lemma 2.9. *Let B/A be locally Galois. Let N be a regular intermediate ring of B/A which is left finite over A then B/N is locally Galois.*

Proof. Let $\{f^{(i)}_{uv} \mid u,v\}$ be the matrix units of $V_{B_i}(Ne_i)$. Let F be a finite subset of B. Then there exists an intermediate semisimple ring N' of $B/N[F, \{f^{(i)}_{uv}, g^{(i)}_{pq} \mid u,v,p,q,i\}]$ such that N'/A is a finite Galois. Now, by Lemma 1.3 (6), the $V_{N'e_i}(Ne_i)$ are simple rings since $V_{B_i}(Ne_i) \supseteq V_{N'e_i}(Ne_i) \supseteq \{f^{(i)}_{uv} \mid u,v\}$. Thus $N'e_i$ are also simple rings. Moreover, by Lemma 1.3 (6), the $V_{N'e_i}(A_i)$ are simple rings since $V_i \supseteq V_{N'e_i}(A_i) \supseteq$

$\{g_{pq}^{(i)} \mid p, q\}$. Now, by [9. Lemma 1.5], the $V_{N'e_i}(A_i)/Z(N'e_i)$ are left finite since the $N'e_i/A_i$ are left finite. Thus $V_{N'}(A)/Z(N')$ is left finite. Thus, by Proposition 1.6, N'/N is finite Galois. Thus B/N is locally Galois.

Corollary 2.10. *Let B/A be locally Galois. Let A^* be an intermediate semisimple ring of B/A such that A^* is left finite over A and the $V_{B_i}(A^*e_i)$ are division rings. Then each intermediate ring of B/A^* is a semisimple ring.*

Proof. By Lemma 2.9, B/A^* is locally Galois. Thus, by Proposition 2.7, each intermediate ring of B/A^* is a semisimple ring.

The next lemma is a generalization of [8. Corollary 1.1] to semisimple rings.

Lemma 2.11. *Let B be outer Galois and finite over A. If N is an intermediate ring of $B/A_1 \oplus \cdots \oplus A_n$ and N/A is Galois, then $G|_N \subseteq \mathrm{Aut}(N/A)$.*

Proof. At first, by Proposition 2.7 and Corollary 2.8, the Ne_i are simple rings. Let $\{f_1, \dots, f_r\}$ be the set of all central primitive idempotents of A. Then $G = \mathrm{Aut}(Bf_1/Af_1) \times \cdots \times \mathrm{Aut}(Bf_r/Af_r)$ and $N = Nf_1 \oplus \cdots \oplus Nf_r$. Thus $GN = \mathrm{Aut}(Bf_1/Af_1)Nf_1 \oplus \cdots \oplus \mathrm{Aut}(Bf_r/Af_r)Nf_r$. Since each Bf_i/Af_i satisfies the assumptions of this lemma, we may assume that A is a simple ring. Now, let σ be an element of G. By Lemma 1.3 (3), there exists $\tau \in \mathrm{Aut}(N/A)$ such that $\tau(e_i) = \sigma(e_i)$, since N is a Galois extension of a simple ring A. Now, by [8. Theorem 4.2 (i)], there exist automorphisms

$$\delta_i : B_i \to B_i$$

which are extensions of $\sigma^{-1}\tau|_{Ne_i} : Ne_i \to \sigma^{-1}(N)e_i$, since each B_i/A_i is outer Galois and finite. Then $\delta_i \in G_i$. By [8. Corollary 1.1], $\delta_i(Ne_i) = Ne_i$ since each Ne_i/A_i is Galois. Thus $\sigma(Ne_i) = \tau(Ne_i) = N\tau(e_i)$. Thus $\sigma(Ne_i) \subseteq N$. Thus, $\sigma(N) = \sigma(Ne_1 \oplus \cdots \oplus Ne_n) \subseteq N$.

Lemma 2.12. *Let B/A be locally finite outer Galois. Let N be an intermediate ring of $B/A_1 \oplus \cdots \oplus A_n$. If N/A is Galois and left finite then $G|_N = \mathrm{Aut}(N/A)$.*

Proof. By [11. Lemma 5.2], there exists an intermediate ring N' of B/N such that N'/A is (outer)Galois and left finite, and $\mathrm{Aut}(N'/A) \supseteq G|_{N'}$. By Lemma 2.11, $\mathrm{Aut}(N/A) \supseteq \mathrm{Aut}(N'/A)|_N$. Thus $\mathrm{Aut}(N/A) \supseteq$

$G|_N$. Otherwise, by Proposition 1.15, $G|_N \supseteq \mathrm{Aut}(N/A)$. Thus $G|_N = \mathrm{Aut}(N/A)$.

The next proposition is a generalization of Lemma 2.11 to infinite outer case.

Proposition 2.13. *Let B/A be locally finite outer Galois. Let N be an intermediate ring of $B/A_1 \oplus \cdots \oplus A_n$. If N/A is Galois, then $G|_N = \mathrm{Aut}(N/A)$.*

Proof. Now, N/A is locally finite outer Galois. Let F be a finite subset of N. Then, by [11. Lemma 5.2], there exists an intermediate ring T of N/A such that $T \supseteq A[F, e_1, \ldots, e_n]$, T/A is left finite, and $\mathrm{Aut}(T/A) \supseteq \mathrm{Aut}(N/A)|_T$. Whence T/A is Galois and left finite. Thus, by Lemma 2.12, $G|_T = \mathrm{Aut}(T/A)$. Thus for any $\sigma \in G$, $\sigma(F) \subseteq T \subseteq N$. Thus $G|_N \subseteq \mathrm{Aut}(N/A)$. On the other hand, by Proposition 1.15, $G|_N \supseteq \mathrm{Aut}(N/A)$. Thus $G|_N = \mathrm{Aut}(N/A)$.

Lemma 2.14. *Let $B \supseteq T^* \supseteq T' \supseteq T \supseteq A$ and $T = Te_1 \oplus \cdots \oplus Te_n$ where each Te_i is a simple ring. Let T^*, T' be semisimple rings and all the $V_{T^*e_i}(A_i), V_{T'e_i}(A_i)$ simple rings. Moreover we assume That $V_B(T) = C$.*
(i) *Let*

$$G_{T'} = \{\sigma \in \mathrm{Aut}(T'/A) \mid \sigma|_T \in \mathrm{Aut}(T/A)\}$$

$$G_{T^*} = \{\tau \in \mathrm{Aut}(T^*/A) \mid \tau|_T \in \mathrm{Aut}(T/A)\}.$$

If T^*/A, T'/A are Galois and left finite then $G_{T'} = G_{T^*}|_{T'}$.
(ii) *If T^*/T' is left finite then $| \{\tau \in \mathrm{Aut}(T^*/A) \mid \tau|_{T'} = \sigma\} | < \infty$ for all $\sigma \in \mathrm{Aut}(T'/A)$.*

Proof. We may assume that A is a simple rings.

(i) By Proposition 1.15, $G_{T^*}|_{T'} \supseteq G_{T'}$. Let $\sigma \in G_{T^*}$. Then, by Proposition 1.15, there exists an element $\tau \in \mathrm{Aut}(T'/A)$ such that $\tau|_T = \sigma|_T$ since $\sigma|_T \in \mathrm{Aut}(T/A)$. Thus, by Proposition 1.15, there exists an element $\tau' \in \mathrm{Aut}(T^*/A)$ such that $\tau'|_{T'} = \tau$. Now, $(\tau')^{-1}\sigma \in \mathrm{Aut}(T^*/A)$ and $(\tau')^{-1}\sigma|_T = id_T$. Thus ${\tau'}^{-1}\sigma \in \mathrm{Aut}(T^*/T)$. Now, by [8. Theorem 4.2 (ii)], The T^*e_i/Te_i and $T'e_i/Te_i$ are outer Galois and finite. Thus T^*/T and T'/T are outer Galois and left finite since $T = Te_1 \oplus \cdots \oplus Te_n$. By Lemma 2.11, T' is $\mathrm{Aut}(T^*/T)$-invariant. Thus $(\tau')^{-1}\sigma(T') = T'$ and $\sigma(T') = T'$. Thus $\sigma|_{T'} \in G_{T'}$. Thus $G_{T^*}|_{T'} = G_{T'}$.

(ii) Let $\tau_1, \tau_2 \in \mathrm{Aut}(T^*/A)$, and $\tau_1|_{T'} = \tau_2|_{T'} = \sigma$. Since $\tau_1^{-1}\tau_2|_{T'} = id_{T'}$, $\tau_1^{-1}\tau_2 \in \mathrm{Aut}(T^*/T')$. Now, by [9. Lemma 1.6], $\mathrm{Aut}(T^*/T')$ is a finite set since T^*/T' is left finite and $V_{T^*}(T') = Z(T^*)$. Thus $\{\tau \in \mathrm{Aut}(T^*/A) \mid \tau|_{T'} = \sigma\}$ is a finite set.

The following Lemma 2.15 and Proposition 2.16 are some generalizations of [8. Lemma 4.2 and Theorem 4.1] to semisimple rings.

Lemma 2.15. *Let B/A be locally Galois and $V = C$. Then B/A is Galois.*

Proof. Let $T' \supseteq T$ be the semisimple intermediate rings of $B/A_1 \oplus \cdots \oplus A_n$ such that T'/A and T/A are Galois and left finite. Then, by Lemma 2.12, $\mathrm{Aut}(T'/A)|_T = \mathrm{Aut}(T/A)$. Moreover, by Lemma 2.14.(ii), $\mathrm{Aut}(T/A)$ is a finite group since T/A is outer Galois and left finite. Now let G' be the inverse limit of the system

$$\{\mathrm{Aut}(T/A) \mid B \supseteq T \supseteq A_1 \oplus \cdots \oplus A_n \text{ such that } T/A \text{ is Galois and left finite}\}.$$

Then, G' may be regarded as an automorphism group of B. Since each $\mathrm{Aut}(T/A)$ is finite and there hold $\mathrm{Aut}(T'/A)|_T = \mathrm{Aut}(T/A)$ for $T' \supseteq T$, [2. Chapter 8. Corollary 3.9], yields at once $G'|_T = \mathrm{Aut}(T/A)$. Thus $B(G') = A$ and B/A is Galois.

Proposition 2.16. *Let B/A be locally Galois and V/C left finite. Then B/A is Galois.*

Proof. At first, there exists an intermediate semisimple ring T of B/A such that $T = Te_1 \oplus \cdots \oplus Te_n$, $V_B(T) = C$, and T/A is Galois and left finite, since B/A is locally Galois and V/C is left finite. Then, by Lemma 2.9, B/T is locally Galois. Thus, by Lemma 2.15, B/T is Galois. Now, let σ be an element of $\mathrm{Aut}(T/A)$ and T' an intermediate ring of B/T such that T'/A is Galois and left finite. Moreover, let $G_{T',\sigma} = \{\tau \in \mathrm{Aut}(T'/A) \mid \tau|_T = \sigma\}$. Then, by Proposition 1.15 and Lemma 2.14.(ii), $0 < |G_{T',\sigma}| < \infty$. Moreover, by Lemma 2.14 (i), $G_{T^*,\sigma}|_{T'} = G_{T',\sigma}$ for any subring $T^* \supseteq T' \supseteq T$ such that T^*/A and T'/A are both Galois and left finite. Let G_σ be the inverse limit of the system

$$\{G_{T',\sigma} \mid B \supseteq T' \supseteq T \text{ such that } T'/A \text{ is Galois and left finite}\}.$$

Then, G_σ may be regarded as a subset of automorphisms of B. By [2. Chapter 8. Theorem 3.6], $G_\sigma \neq \phi$ and $G_\sigma|T = \sigma$. Thus for any elements of

$\mathrm{Aut}(T/A)$ can be extended to an automorphism of B. Thus B/A is Galois since B/T and T/A are Galois.

Theorem 2.17. *Let V/C be left finite. Then the following conditions are equivalent.*
(i) *B/A is Galois and locally finite.*
(ii) *B/A is locally Galois.*
(iii) *B/A is G-locally Galois.*

Proof. By Proposition 2.16, (ii) induces (i). By the definition of G-locally Galois, (iii) induces (ii). Now we assume the condition (i). Then, by Proposition 2.5, B/A is G-locally Galois.

Remark 2.18. Let B/A be Galois and left finite. Then, by Proposition 1.1 and [15 Theorem 1], B is separable over $A' = A_1 \oplus \cdots \oplus A_n$, in the sense of [3. §2]. Moreover, as is easily seen, A' is separable over A. Thus, by [3. Proposition 2.5], B is separable over A. Hence, if B/A is locally Galois then, B is the direct limit of the system of the subrings which are separable over A.

3 The subrings which are locally Galois over A.

If B/A is locally Galois as simple rings, then H/A is Galois([9. Lemma 2.4]). In this section we shall present some generalizations of this lemma to semisimple rings.

Lemma 3.1. *Let B/A be locally Galois, and let N be a regular intermediate ring of $B/A_1 \oplus \cdots \oplus A_n$ which is left finite over A. Then for each finite subset $F \subseteq B$, there exists a semisimple subring $N' \supseteq N[F]$ such that N'/A is Galois and left finite, and* $\mathrm{Aut}(N'/A)|_N \supseteq \mathrm{Aut}(N/A)$.

Proof. Let $\{f_{uv}^{(i)} \mid u, v\}$ be the matrix units of $V_{B_i}(Ne_i)$. Then there exists a semisimple subring $N' \supseteq N[F, \{e_{st}^{(i)}, g_{pq}^{(i)}, f_{uv}^{(i)} \mid p, q, u, v, s, t, i\}]$ such that N'/A is Galois and left finite. Then for each i, by lemma 1.3.(6) and (7), $N'e_i$ and $V_{N'e_i}(A_i)$ are simple rings, since $N'e_i \supseteq \{e_{st}^{(i)}, g_{pq}^{(i)} \mid s, t, p, q, \}$. Thus the $V_{N'e_i}(A_i)/Z(N'e_i)$ are left finite by [9. Lemma 1.5]. Moreover, by Lemma 1.3.(6), the $V_{N'e_i}(Ne_i)$ are simple rings since $V_{N'e_i}(Ne_i) \supseteq \{f_{uv}^{(i)} \mid u, v\}$. Thus N is regular in N'. Hence by Proposition 1.15, $\mathrm{Aut}(N'/A)|_N \supseteq \mathrm{Aut}(N/A)$.

Next, we consider the following lemma. This lemma is a generalization of [8.Lemma 4.1].

Lemma 3.2. *Let B/A be locally Galois, and let N be a regular intermediate ring of $B/A_1 \oplus \cdots \oplus A_n$ which is left finite over A, and A^* a semisimple subring of N. Let $H^* = V_B(V_B(A^*))$. If $\tau \in \mathrm{Aut}(N/A)$ and $\tau(A^*) = A^*$, then $\tau(N \cap H^*) \subseteq H^*$.*

Proof. The proof is by contradiction. Assume the assertion were false. Then we would have that there is a some element $a \in N \cap H^*$ such that $\tau(a) \notin H^*$. Then there exists $v \in V_B(A^*)$ such that $v\tau(a) \neq \tau(a)v$. By Lemma 3.1, there exists a semisimple subring $N' \supseteq N[v]$ such that N'/A is Galois and left finite, and $\mathrm{Aut}(N'/A)|_N \supseteq \mathrm{Aut}(N/A)$. Thus there exists $\delta \in \mathrm{Aut}(N'/A)$ such that $\delta|_N = \tau$. Since $\delta^{-1}(A^*) = A^*$, $\delta^{-1}(v) \in V_B(A^*)$. Then $\delta^{-1}(v)a = a\delta^{-1}(v)$. Thus $v\tau(a) = \delta(\delta^{-1}(v)a) = \delta(a\delta^{-1}(v)) = \tau(a)v$. This is a contradiction.

Lemma 3.3. *Let B/A be locally Galois. Let A^* be a regular intermediate ring of $B/A_1 \oplus \cdots \oplus A_n$ which is Galois and left finite over A. Let $H^* = V_B(V_B(A^*))$. If A is regular in H^* then H^*/A is Galois, locally Galois, and $\mathrm{Aut}(H^*/A)$-locally Galois.*

Proof. At first, we shall show that H^*/A is locally Galois. By Lemma 2.9, B/A^* is locally Galois. Thus, by Lemma 2.2, the B_i/A^*e_i are locally Galois as simple rings. Thus, by [9. Lemma 2.4], the H^*e_i are simple rings. Let $U^{(i)}$, $V^{(i)}$, and $W^{(i)}$ be the matrix units of the H^*e_i, $V_{H^*e_i}(A_i)$, and $V_{B_i}(A^*e_i)$, respectively. Let F be a finite subset of H^*. Then, by Lemma 3.1, there exists a semisimple subring $N \supseteq A^*[F, \{g_{pq}^{(j)} \mid p,q,j\}, U^{(i)}, V^{(i)}, W^{(i)} (1 \leq i \leq n)]$ such that N/A is Galois and left finite, and $\mathrm{Aut}(N/A)|_{A^*} \supseteq \mathrm{Aut}(A^*/A)$. Here, $N = Ne_1 \oplus \cdots \oplus Ne_n$, and $N \cap H^* = (N \cap H^*)e_1 \oplus \cdots \oplus (N \cap H^*)e_n$. Now, we shall prove that all the Ne_i, $(N \cap H^*)e_i$, $V_{B_i}(Ne_i)$, $V_{Ne_i}(A^*e_i)$, $V_{(N\cap H^*)e_i}(A_i)$, and $V_{(N\cap H^*)e_i}(A^*e_i)$ are simple rings. Since $Ne_i \supseteq \{g_{pq}^{(i)} \mid p,q\}$, the $V_{B_i}(Ne_i)$ are division rings. Thus the $Z(Ne_i)$ are fields and the Ne_i are simple rings. By Lemma 1.3 (6), the $V_{Ne_i}(A^*e_i)$ are simple rings since $V_{B_i}(A^*e_i) \supseteq V_{Ne_i}(A^*e_i) \supseteq W^{(i)}$. By Lemma 1.3 (6), the $V_{(N\cap H^*)e_i}(A_i)$ are simple rings since $V_{H^*e_i}(A_i) \supseteq V_{(N\cap H^*)e_i}(A_i) \supseteq V^{(i)}$. By Lemma 1.3 (7), the $(N \cap H^*)e_i$ are simple rings since $H^*e_i \supseteq (N \cap H^*)e_i \supseteq U^{(i)}$ and the $(N \cap H^*)e_i/A_i$ are left finite. Since the $V_{H^*e_i}(A^*e_i)$ are fields, the $V_{(N\cap H^*)e_i}(A^*e_i)$ are fields. Hence A,

A^*, and N are regular in $N \cap H^*$, N, and B, respectively. Moreover, A^* is regular in $N \cap H^*$. Now, by Proposition 1.6, N/A^* is Galois. Thus by Lemma 3.2, $\mathrm{Aut}(N/A^*)|_{(H^* \cap N)} \subseteq \mathrm{Aut}(H^* \cap N/A^*)$. Thus $N \cap H^*/A^*$ is Galois. Next, let $\sigma \in \mathrm{Aut}(A^*/A)$. Then there exists $\tau \in \mathrm{Aut}(N/A)$ such that $\tau|_{A^*} = \sigma$. Then, by Lemma 3.2, $\tau(H^* \cap N) \subseteq H^* \cap N$. Thus any element of $\mathrm{Aut}(A^*/A)$ can be extended to an automorphism of $H^* \cap N$. Thus we may induce that $H^* \cap N/A$ is Galois and left finite, since $H^* \cap N/A^*$ and A^*/A are Galois and left finite. Thus H^*/A is locally Galois. Moreover, by [9. Lemma 1.5], the $V_{H^*e_i}(A_i)/V_{H^*e_i}(A^*e_i)$ are left finite since A^*/A are left finite. Thus $V_{H^*}(A)/V_{H^*}(A^*)$ is left finite. Thus $V_{H^*}(A)/Z(H^*)$ is left finite. Thus, by Theorem 2.17, H^*/A is Galois and $\mathrm{Aut}(H^*/A)$-locally Galois.

Next theorem is a generalization of [9. Lemma 2.4] to semisimple rings.

Theorem 3.4. *Let B/A be locally Galois. Then H/A is Galois, locally Galois, and* $\mathrm{Aut}(H/A)$*-locally Galois.*

Proof. Let $A^* = A_1 \oplus \cdots \oplus A_n$. Then $H = V_B(V_B(A^*))$, A^* is regular in B, A^*/A is Galois and left finite, and A is regular in H. Thus, by Lemma 3.3, H/A is Galois, locally Galois, and $\mathrm{Aut}(H/A)$-locally Galois.

Theorem 3.5. *Let B/A be locally Galois. Let A^* be a semisimple intermediate ring of B/A which contains $\{g_{pq}^{(i)} | p, q, i\}$. Let $H^* = V_B(V_B(A^*))$. If A^*/A is Galois and left finite, then H^*/A is Galois, locally Galois, and* $\mathrm{Aut}(H^*/A)$*-locally Galois.*

Proof. Since $A^* \supseteq \{g_{pq}^{(i)} | p, q, i\}$, the $V_{B_i}(A^*e_i)$ are division rings. Thus A^* is regular in B. Hence, by the same argument as in the proof of Lemma 3.3, the H^*e_i are simple rings. Moreover, by Lemma 1.3 (6), the $V_{H^*e_i}(A_i)$ are simple rings since $H^* \supseteq \{g_{pq}^{(i)} | p, q, i\}$. Thus A is regular in H^*. Hence, by Lemma 3.3, H^*/A is Galois, locally Galois, and $\mathrm{Aut}(H^*/A)$-locally Galois.

The following theorem is a generalization of [9. Lemma 4.2] to semisimple rings.

Theorem 3.6. *Let B/A be locally Galois. Then for each finite subset F of B, there exists an intermediate semisimple ring T of $B/H[F]$ which satisfing the following conditions:*

(i) *$V_T(A)/Z(T)$ and T/H are left finite.*

(ii) *T/A is Galois, locally Galois, and* $\mathrm{Aut}(T/A)$*-locally Galois.*

Proof. Without loss of generality, we may assume that A is a simple ring. Since B/A is locally Galois, there exists an intermediate semisimple ring A^* of $B/A[F, \{g_{pq}^{(i)} \mid p, q, i\}]$ such that A^*/A is Galois and left finite. Let $V_B(V_B(A^*)) = H^*$. Then, $H^* \supseteq F$. Moreover, by Theorem 3.5 and of this proof, we know that H^*/A is $\mathrm{Aut}(H^*/A)$-locally Galois, $V_{H^*}(A)/Z(H^*)$ is left finite, and the $V_{H^*e_i}(He_i) = V_{H^*e_i}(A_i)$ are simple rings. Now, by Theorem 3.4, H/A is Galois. Thus by Lemma 1.3,(3), there exist the automorphisms σ_i of $\mathrm{Aut}(H/A)$ such that $\sigma_i(e_1) = e_i$. Thus by Proposition 1.15, there exist the automorphisms δ_i of $\mathrm{Aut}(H^*/A)$ such that $\delta_i|_H = \sigma_i$. Let $F' = \cup_i \delta_i^{-1}(Fe_i)$. Now, by Lemma 2.2, B_1/A_1 is locally Galois. Thus by [9. Lemma 4.2], there exists an intermediate simple ring T_1 of $B_1/He_1[F']$ such that T_1/He_1 is left finite and T_1/A_1 is Galois extension. Now, there exists a finite subset E of B such that $T_1 = He_1[E]$. Since B/A is locally Galois, there exists an intermediate semisimple ring A^{**} of $B/A^*[E]$ such that A^{**}/A is Galois and left finite. Let $H^{**} = V_B(V_B(A^{**}))$. Then, $H^{**} \supseteq T_1$. Moreover, by the same argument as in the above, we obtain that H^{**}/A is $\mathrm{Aut}(H^{**}/A)$-locally Galois, $V_{H^{**}}(A)/Z(H^{**})$ is left finite, and the $V_{H^{**}e_i}(He_i) = V_{H^{**}e_i}(A_i)$ are simple rings. Moreover, the $V_{H^{**}e_i}(H^*e_i)$ are division rings since $H^* \supseteq \{g_{pq}^{(i)} \mid p, q, i\}$. Thus by Proposition 1.15, there exist the automorphisms δ_i' of $\mathrm{Aut}(H^{**}/A)$ such that $\delta_i'|_{H^*} = \delta_i$. Here, we may assume that $\delta_1' = id_{H^{**}}$. Let $T_i = \delta_i'(T_1)$. Then, for each i, T_i/A_i is Galois extension and T_i/He_i is left finite since $\delta_i'(He_1) = He_i$ and $\delta_i'(A_1) = A_i$. Let $T = T_1 \oplus \cdots \oplus T_n$. Then $T \supseteq H[F]$, T/H is left finite, and $T/A_1 \oplus \cdots \oplus A_n$ is Galois extension. Now we define a map $\delta_{1,2} : T \to T$ by

$$\delta_{1,2}(t_1 + \cdots + t_n) = \delta_2'(t_1) + \delta_2'^{-1}(t_2) + t_3 + \cdots + t_n$$

for all $t_i \in T_i$. Then $\delta_{1,2} \in \mathrm{Aut}(T/A)$. Let b be an element of $T(\mathrm{Aut}(T/A))$. Since $T/A_1 \oplus \cdots \oplus A_n$ is Galois, we get the expression $b = a_1e_1 + \cdots + a_ne_n$ ($a_i \in A$). Then

$$\begin{aligned} a_1e_1 + \cdots + a_ne_n &= \delta_{1,2}(a_1e_1 + \cdots + a_ne_n) \\ &= a_1e_2 + a_2e_1 + \cdots + a_ne_n. \end{aligned}$$

Thus $a_1e_2 = a_2e_2$. Similarly $a_1e_i = a_ie_i$ for all i. Thus $b = a_1 \in A$. Thus T/A is Galois. Moreover, since T/H is left finite, $V_T(H)$ is finitely generated over $Z(T)$. Thus $V_T(A)/Z(T)$ is left finite since $V_T(H) = V_T(A)$. Thus by Theorem 2.17, T/A is Galois, locally Galois, and G-locally Galois.

4 The subrings whose extensions B are locally Galois.

In section 3, we considered the intermediate rings which are locally Galois over A. Next, in this section we shall deal with intermediate rings whose extensions B are locally Galois.

Lemma 4.1. *Let B/A be locally Galois and V/C left finite. If T is a regular intermediate ring of B/A, then B/T is a Galois extension.*

Proof. By Proposition 2.16, B/A is Galois. Thus, by Proposition 1.6, B/T is Galois.

Lemma 4.2. *Let B/A be locally Galois and V/C left finite. If T is a regular intermediate ring of $B/A_1 \oplus \cdots \oplus A_n$, then B/T is locally Galois.*

Proof. By Lemma 2.2, the B_i/A_i are locally Galois. Thus, by [8. Theorem 4.2 (ii)], the B_i/Te_i are locally Galois. Thus, by Lemma 1.3 (4), B/T is locally Galois.

The next theorem is a generalization of [8. Theorem 4.2 (ii)] to semisimple rings.

Theorem 4.3. *Let B/A be locally Galois and V/C left finite. If T is a regular intermediate ring of B/A, then B/T is Galois, locally Galois, and $G(T)$-locally Galois.*

Proof. By Lemma 4.1, B/T is Galois. By Lemma 4.2, $B/Te_1 \oplus \cdots \oplus Te_n$ is locally Galois. Thus $B/Te_1 \oplus \cdots \oplus Te_n$ is locally finite. Thus B/T is locally finite. Thus, by Theorem 2.17, B/T is locally Galois and $G(T)$-locally Galois.

Corollary 4.4. *Let B/A be locally Galois and V/C left finite. Let A^* be an intermediate semisimple ring of B/A such that all the $V_{B_i}(A^*e_i)$ are division rings. Then each intermediate ring of B/A^* is a semisimple ring.*

Proof. By Theorem 4.3, B/A^* is locally Galois. Thus, by Proposition 2.7, each intermediate ring of B/A^* is a semisimple ring.

Combining Theorem 3.5 and Theorem 4.3, we get the next theorem. This theorem is a generalization of Lemma 2.9.

Theorem 4.5. *Let B/A be locally Galois. Let A^* be an intermediate ring of B/A which is left finite over A. Let $H^* = V_B(V_B(A^*))$. If A' is an intermediate ring of H^*/A which is regular in B, then B/A' is locally*

Galois.

Proof. For each i, let $\{f_{uv}^{(i)} \mid u, v\}$ be a matrix units of $V_{B_i}(A'e_i)$. Let F be any finite subset of B. Then there exists an intermediate semisimple ring A^{**} of $B/A^*[F, \{f_{uv}^{(i)}, g_{pq}^{(i)} \mid u, v, p, q, i\}]$ such that A^{**}/A is Galois and left finite. Let $H^{**} = V_B(V_B(A^{**}))$. Then, by Theorem 3.5, H^{**}/A is locally Galois. And by the some argument as in the proof of Lemma 3.3, $V_{H^{**}}(A)/Z(H^{**})$ are left finite, and the $H^{**}e_i$ and $V_{H^{**}e_i}(A_i)$ are simple rings. Moreover, by Lemma 1.3 (6), the $V_{H^{**}e_i}(A'e_i)$ are simple rings since $V_{B_i}(A'e_i) \supseteq V_{H^{**}e_i}(A'e_i) \supseteq \{f_{uv}^{(i)} \mid u, v\}$. Thus, by Theorem 4.3, H^{**}/A' is locally Galois and $H^{**} \supseteq F$. Thus B/A' is locally Galois.

The next theorem is a generalization of [9. Theorem 4.3 (i)] to semisimple rings.

Theorem 4.6. *If B/A is locally Galois then B/H is $G(H)$-locally Galois.*

Proof. By Theorem 4.5, B/H is locally Galois. Moreover B/H is Galois since H is a centralizer of V in B. Thus, by Theorem 2.3, B/H is $G(H)$-locally Galois.

Proposition 4.7. *Let B/A be G-locally Galois. Let N be a regular intermediate ring of B/A which is left finite over A, then B/N is $G(N)$-locally Galois.*

Proof. Let $\{f_{uv}^{(i)} \mid u, v\}$ be the matrix units of $V_{B_i}(Ne_i)$. Let F be a finite subset of B. Then there exists an intermediate semisimple ring N' of $B/N[F, \{f_{uv}^{(i)}, g_{pq}^{(i)} \mid u, v, p, q, i\}]$ such that N'/A is a finite Galois and $G|_{N'} \supseteq \mathrm{Aut}(N'/A)$. Then, by the same argument as in the proof of Lemma 2.9, N'/N is Galois and left finite. Moreover $G|_{N'} \supseteq \mathrm{Aut}(N'/A) \supseteq \mathrm{Aut}(N'/N)$. Thus $G(N)|_{N'} \supseteq \mathrm{Aut}(N'/N)$. Thus B/N is $G(N)$-locally Galois.

5 Galois theory of semisimple rings with Galois groups which are not locally compact.

In §1, we considered about Galois extensions whose Galois groups are locally compact. In this section we shall consider some Galois extensions whose Galois groups are not locally compact. The theorems in this section are generalizations of [9. §3 and §4] to semisimple rings. At first, we give the some definitions which are used in this section. Let K be a subgroup of G and T be an intermediate ring of B/A. If each $V_{B_i}(B(K)e_i)$ is simple and $K \supseteq \langle V_B(B(K))\rangle$ then K is called a *$(*)$-regular subgroup.* If K is $(*)$-regular and V is finitely generated as a right $V_B(B(K))$-module, then K is called *$(*_f)$-regular.* If K is regular and $(*_f)$-regular then K is called *f-regular.* T will be called to be *f-regular* if T is regular and V is finitely generated as right $V_B(T)$-module(abr. right finite). Moreover, B/A is said to be *hereditarily Galois*(abr. h-Galois) if the following two conditions are fulfilled:

(i) For each regular intermediate ring A' of B/A which is left finite over A, B/A' is Galois and the $V_{B_i}(V_{B_i}(A'e_i))$ are simple.

(ii) If B' is a regular intermediate ring of B/H which is left finite over H, then for each i, the cardinal number of linearly independent He_i-left basis of $B'e_i$ is equal to the cardinal number of linearly independent $V_{B_i}(B'e_i)$-right basis of V_i and $V_B(V_B(B')) = B'$.

Remark 5.1. Let B/A be locally finite. Let T be a regular intermediate ring of B/A. Then T is an f-regular intermediate ring of B/A if and only if there exists an intermediate ring A' of B/A such that A'/A is left finite and $V_B(V_B(A')) \supseteq T$.

Combining Theorem 4.5 and the preceding Remark 5.1, we get the next conclusion.

Conclusion 5.2. *Let B/A be locally Galois. If A' is an f-regular intermediate ring of B/A, then B/A' is locally Galois.*

Now, we shall present the next proposition which are analogous to that of Lemma 1.7 and Lemma 1.8.

Proposition 5.3. (i) *If K is $(*)$-regular in G then each $K(\{e_i\})|_{B_i}$ is $(*)$-regular in G_i.*

(ii) *If K is $(*_f)$-regular in G then each $K(\{e_i\})|_{B_i}$ is $(*_f)$-regular in G_i.*

(iii) *If K is f-regular in G then each $K(\{e_i\})|_{B_i}$ is f-regular in G_i.*

Proof. (*i*) By the same methods as in the proof of Lemma 1.7, we obtain the (*i*).

(*ii*) Since $V_i = Ve_i$ and $V_{B_i}(K(\{e_i\})|_{B_i}) = V_B(B(K))e_i$, V_i is finitely generated as right $V_{B_i}(K(\{e_i\})|_{B_i})$-module. Thus, by (*i*), each $K(\{e_i\})|_{B_i}$ is $(*_f)$-regular in G_i.

(*iii*) By Lemma 1.7, each $K(\{e_i\})|_{B_i}$ is regular in G_i. By (*ii*), each $K(\{e_i\})|_{B_i}$ is $(*_f)$-regular in G_i. Thus, each $K(\{e_i\})|_{B_i}$ is f-regular in G_i.

The next proposition is a generalization of [9. Theorem 2.8] to semisimple rings.

Proposition 5.4. *If B/A is G-locally Galois, then any $(*_f)$-regular subgroup of G is f-regular.*

Proof. Let K be a $(*_f)$-regular subgroup of G. Then, by Corollary 2.4 and Proposition 5.3 (*ii*), each B_i/A_i is G_i-locally Galois and each $K(\{e_i\})|_{B_i}$ is $(*_f)$-regular in G_i. Thus, by [9. Theorem 2.8], each $K(\{e_i\})|_{B_i}$ is f-regular in G_i. Thus the $B_i(K(\{e_i\})|_{B_i})$ and the $V_{B_i}(B_i(K(\{e_i\})|_{B_i}))$ are simple rings. Thus $B(K)$ and $V_B(B(K))$ are semisimple rings. Thus K is regular and $(*_f)$-regular. Thus K is f-regular.

The next proposition is a generalization of [9. Corollary 2.7] to semisimple rings.

Proposition 5.5. *If B/A is Galois and locally finite, and V/C is left finite. Then any $(*)$-regular subgroup of G is regular.*

Proof. Let K be a $(*)$-regular subgroup of G. Then, by Proposition 5.3 (*iii*), each $K(\{e_i\})|_{B_i}$ is $(*)$-regular in G_i. Thus, by [9. Corollary 2.7], each $K(\{e_i\})|_{B_i}$ is regular in G_i. Thus, the $B_i(K(\{e_i\})|_{B_i})$ and the $V_{B_i}(B_i(K(\{e_i\})|_{B_i}))$ are simple rings. Thus $B(K)$ and $V_B(B(K))$ are semisimple rings. Thus K is regular.

As is easily seen, $V_B(V_B(T)) = V_{B_1}(V_{B_1}(Te_1)) \oplus \cdots \oplus V_{B_n}(V_{B_n}(Te_n))$ and $V_B(T) = V_{B_1}(Te_1) \oplus \cdots \oplus V_{B_n}(Te_n)$ for any intermediate ring T of B/A. Thus, by birtue of Proposition 1.1, we may induce next lemma. It is an analogous to that of Proposition 1.1, Lemma 2.2, and Corollary 2.4.

Lemma 5.6. *If B/A is h-Galois then the B_i/A_i are h-Galois.*

The next proposition is an analogous to that of Lemma 2.9 and Proposition 4.7.

Proposition 5.7. *If B/A is locally finite and h-Galois then so is B/A' for each regular intermediate ring A' of B/A which is left finite over A.*

Proof. Let A^* be a regular intermediate ring of B/A' which is left finite over A'. Then A^* is a regular intermediate ring of B/A which is left finite over A since A'/A is left finite. Thus B/A' satisfies the condition (i) of h-Galois. Next, by Lemma 1.3 (2) and Lemma 5.6, the B_i/A_i are locally finite and h-Galois. Thus, by [9. Corollary 3.1], the $B_i/A'e_i$ are locally finite and h-Galois. Thus B/A' satisfies the condition (ii) of h-Galois.

Lemma 5.8. *Let B/A be locally finite and h-Galois. If T is an f-regular intermediate ring of B/A then B/T is locally finite.*

Proof. By Lemma 1.3 (2) and Lemma 5.6, the B_i/A_i are locally finite h-Galois. Thus, by [9. Corollary 3.3 (i)], the B_i/Te_i are locally finite. Thus $B/Te_1 \oplus \cdots \oplus Te_n$ is locally finite. Since $Te_1 \oplus \cdots \oplus Te_n/T$ is left finite, B/T is locally finite.

Lemma 5.9. *Let B/A be locally finite and h-Galois. If K is fat closed f-regular then $G(B(K)) = K$.*

Proof. By Lemma 1.8 and Proposition 5.3 (i), each $K(\{e_i\})|_{B_i}$ is closed f-regular in G_i. Thus, by [9. Corollary 3.3], $G_i(B_i(K(\{e_i\})|_{B_i})) = K(\{e_i\})|_{B_i}$. Thus, by Lemma 1.9, $G(B(K)) = K$.

The following theorem is a generalization of [9. Theorem 3.4] to semisimple rings and some generalization of Theorem 1.12.

Theorem 5.10. *Let B/A be locally finite and h-Galois, and let B be countably generated as H-left module. Then there exists a 1-1 dual correspondence between fat and closed f-regular subgroups of G and f-regular intermediate rings of B/A, in the usual sense of Galois theory.*

Proof. By Lemma 1.3 (2) and Lemma 5.6, the B_i/A_i are locally finite h-Galois. Moreover each B_i has a countable linearly independent He_i-left basis. Let T be an f-regular intermediate rings of B/A. Then, by [9. Theorem 3.4], $B_i(G_i(Te_i)) = Te_i$ since each Te_i is an f-regular intermediate ring of B_i/A_i. Thus, by Lemma 1.5 and [9.Theorem 3.5], $B(G(T)) = T$. Let K be a fat and closed f-regular subgroups of G. Then, by Lemma 5.9, $G(B(K)) = K$. Moreover, by the definitions of f-regular subgroup and f-regular intermediate ring, $G(T)$ is an f-regular subgroup and $B(K)$ is an f-regular intermediate ring. Moreover, as is easily seen that $G(T)$ is fat and closed.

The following theorem is a generalization of [9.Theorem 3.3] to semisimple rings and a partial generalization of Proposition 5.7.

Theorem 5.11. *Let B/A be locally finite and h-Galois, and let B be countably generated as H-left module. If T is an f-regular intermediate ring of B/A then B/T is locally finite and h-Galois.*

Proof. At first, by Lemma 5.8, B/T is locally finite. Let T' be a regular intermediate ring of B/T which is left finite over T. Then $V_B(T)/V_B(T')$ is right finite since T'/T is left finite and B/T is locally finite. Since T is f-regular, $V/V_B(T)$ is right finite. Thus $V/V_B(T')$ is right finite. Thus T' is an f-regular intermediate ring of B/A. Thus, by Theorem 5.10, B/T' is Galois. Otherwise, by Lemma 1.3 (2) and Lemma 5.6, the B_i/A_i are locally finite h-Galois. Moreover each B_i has a countable linearly independent He_i-left basis. Thus, by [9. Theorem 3.3], the B/Te_i are h-Galois. Thus, the $V_{B_i}(V_{B_i}(T'e_i))$ are simple rings and the conditions (ii) of the definition of h-Galois are satisfied.

In the next, we consider the extensions of automorphisms. The following Propositions and Corollaries for extensions of automorphisms are some generalizations of Proposition 1.15 and Corollary 1.16.

Lemma 5.12. *Let B/A be Galois and A', A^* be regular intermediate rings of $B/A_1 \oplus \cdots \oplus A_n$. We assume that for each i, and each A_i-ring isomorphism f_i of $A'e_i$ onto A^*e_i there exists an automorphism g_i of B_i which is extension of f_i. Then for each A-ring isomorphism f of A' onto A^* is extended to an automorphism of B.*

Proof. We may assume that A is a simple ring. At first, $f(P) = P$ since P is the set of all central primitive idempotents of A' and A^*. Since B/A is Galois, there exists an element $\tau \in G$ such that $\tau(e_i) = f(e_i)$ for all i. Then each $\tau^{-1}f|_{A'e_i}$ is an A_i-ring isomorphism of $A'e_i$ onto A^*e_i. Thus there exists an automorphism δ_i of B_i which is an extension of $\tau^{-1}f|_{A'e_i}$ for each i. Now we define a map $\delta : B \to B$ by

$$\delta(b) = \delta_1(be_1) + \cdots + \delta_n(be_n)$$

for all $b \in B$. Then $\delta|_{A'} = \tau^{-1}f$. Thus $\tau\delta$ is an automorphism of B and $\tau\delta|_{A'} = f$.

The following Proposition 5.13 and Corollary 5.14 are generalizations of [9. Corollary 3.7] to semisimple rings.

Proposition 5.13. *Let B/A be locally finite and h-Galois, and let A' and A^* be regular intermediate rings of $B/A_1 \oplus \cdots \oplus A_n$ which are left finite over A. If σ is an A-ring isomorphism of A' onto A^* then σ is contained in $G|_{A'}$.*

Proof. By Lemma 1.3 (2) and Lemma 5.6, the B_i/A_i are locally finite h-Galois. Hence, by [9. Corollary 3.7], for each i, and for each A_i-ring isomorphism f_i of $A'e_i$ onto A^*e_i is extended to an automorphism of B_i. Thus, by Lemma 5.12, this proposition is proved.

Corollary 5.14. *Let B/A be locally finite and h-Galois, and let A' and A^* be regular intermediate simple rings of B/A which are left finite over A. If σ is an A-ring isomorphism of A' onto A^* then σ is contained in $G|_{A'}$.*

Proof. For each i, $A' \simeq A'e_i$ $(b_1 \leftrightarrow b_1e_i)$ and $A^* \simeq A^*e_i$ $(b_2 \leftrightarrow b_2e_i)$ where $b_1 \in A'$ and $b_2 \in A^*$, since A' and A^* are simple rings. Thus the isomorphisms $\sigma_i : A'e_i \to A^*e_i$, $(\sigma_i(b_1e_i) = \sigma(b_1)e_i)$ are well-defined. Now, we define a map

$$\sigma' : A'e_1 + \cdots + A'e_n \to A^*e_1 + \cdots + A^*e_n$$

by

$$\sigma'(t_1e_1 + \cdots + t_ne_n) = \sigma_1(t_1e_1) + \cdots + \sigma_n(t_ne_n)$$

for all $t_i \in A'$. Then σ' is an extension of σ. Now, by Proposition 5.13, there exists an element $\tau \in G$ which is an extension of σ' and thus $\tau|_{A'} = \sigma$.

The following Proposition 5.15 and Corollary 5.16 are generalizations of [9. Theorem 3.5] to semisimple rings.

Proposition 5.15. *Let B/A be locally finite and h-Galois, and let B be countably generated as H-left module. Let A' and A^* be f-regular intermediate rings of $B/A_1 \oplus \cdots \oplus A_n$. If σ is an A-ring isomorphism of A' onto A^* then σ is contained in $G|_{A'}$.*

Proof. By Lemma 1.3 (2) and Lemma 5.6, the B_i/A_i are locally finite h-Galois. Moreover each B_i has a countable linearly independent He_i-left basis. Thus, by [9. Theorem 3.5] and Lemma 5.12, this theorem is proved.

Corollary 5.16. *Let B/A be locally finite and h-Galois, and let B be countably generated as H-left module. Let A' and A^* be f-regular intermediate simple rings of B/A. If σ is an A-ring isomorphism of A' onto A^* then σ is contained in $G|_{A'}$.*

Proof. By the same argument as in the proof of Corollary 5.14, we may prove this corollary.

In the last of this paper, we shall deal with G-locally Galois extensions. This extension is one example of h-Galois.

Lemma 5.17. *If B/A is G-locally Galois, then B/A is h-Galois.*

Proof. Let A' be an intermediate regular ring of B/A which is left finite over A. Then B/A' is Galois by Proposition 4.7. By [9. Corollary 2.2, Corollary 2.3, Corollary 2.5] and Corollary 2.4, another conditions of the definition of h-Galois are proofed.

The next proposition is a generalizations of [9. Theorem 4.2] to semisimple rings, and one example of Theorem 5.10.

Proposition 5.18. *Let B/A be G-locally Galois and B be countably generated as H-left module. Then the following conditions are satisfied;*

(1) *Let T be an intermediate f-regular ring of B/A. Then $B(G(T)) = T$, and $G(T)$ is a fat, closed, and f-regular subgroup.*
(2) *Let K be a subgroup which is fat, closed, and $(*_f)$-regular, then $G(B(K)) = K$. Moreover $B(K)$ is an f-regular intermediate ring of B/A.*

Proof. By Lemma 5.17, B/A is h-Galois. Thus, by birtue of Proposition 5.4 and Theorem 5.10, this proposition is proved.

The next corollary is a generalization of [9. Corollary 4.1] to semisimple rings. And this corollary is an analogous of Theorem 5.11.

Corollary 5.19. *Let B/A be G-locally Galois and B be countably generated as H-left module. Then B/T is $G(T)$-locally Galois for each intermediate f-regular ring of B/A.*

Proof. By Proposition 5.18, B/T is Galois. By Conclusion 5.2, B/T is locally Galois. Thus, by Theorem 2.3, B/T is $G(T)$-locally Galois.

The next theorem is a generalization of [9. Theorem 4.4] to semisimple rings and some generalization of Proposition 2.16.

Theorem 5.20. *Let B/A be locally Galois and B be countably generated as H-left module. Then B/A is G-locally Galois.*

Proof. Let B be generated by the countable set $\{x_1, x_2, \dots\}$ as H-left module. Then we can construct inductively an ascending chain of semisimple rings

$$A^{(1)} \subseteq A^{(2)} \subseteq A^{(3)} \subseteq \cdots$$

such that $A^{(k)} \supseteq \{g_{pq}^{(i)} \mid p, q, i\}$, $A^{(k)} \supseteq \{x_1, \dots x_k\}$, and $A^{(k)}/A$ is Galois and left finite. Let $H^{(k)} = V_B(V_B(A^{(k)}))$. Then, by Theorem 3.5 and of this proof, the $H^{(k)}/A$ are $\mathrm{Aut}(H^{(k)}/A)$-locally Galois and the $V_{H^{(k)}}(A)/Z(H^{(k)})$ are left finite. Thus, by Proposition 1.15, $\mathrm{Aut}(H^{(k+1)}/A)|_{H^{(k)}} \supseteq \mathrm{Aut}(H^{(k)}/A)$ and $\cup_k H^{(k)} = B$. Thus for any k, every element of $\mathrm{Aut}(H^{(k)}/A)$ can be extended to an automorphism of B. Thus B/A is Galois. Thus, by Theorem 2.3, B/A is G-locally Galois.

Combining Proposition 2.5 and Theorem 5.20, we get the next corollary. This corollary is a generalization of Theorem 2.17

Corollary 5.21. *If $V/Z(V)$ is left finite, B/A is locally finite, and B is countably generated as H-left module then, the conditions Galois, locally Galois, and G-locally Galois are all equivalent.*

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