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ORTHOGONALITY IN THE CATEGORY OF COMPLEXES

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1. Definitions and Preliminaries.

In this section, $\mathcal{C}$ will be an abelian category. If $\mathcal{E}$ is a class of objects of $\mathcal{C}$, a morphism $\phi : E \to X$ of $\mathcal{C}$ is called an *$\mathcal{E}$-precover* of X if $E \in \mathcal{E}$ and if Hom $(F, E) \to$ Hom (F, X) is surjective for all $F \in \mathcal{E}$. If, moreover, any $f : E \to E$ such that $\phi = \phi \circ f$ is an automorphism of E then $\phi : E \to X$ is called an *$\mathcal{E}$-cover* of X. If an $\mathcal{E}$-cover of X exists, it is unique up to isomorphism. An $\mathcal{E}$-preenvelope and an $\mathcal{E}$-envelope $X \to E$ are defined dually.

Auslander and Smalø [3] and Auslander and Reiten [2] use the terminology left and right approximations and minimal left and right approximations for precovers, preenvelopes, covers and envelopes.

We sometimes name the $\mathcal{E}$-(pre)covers and $\mathcal{E}$-envelopes by the name of the class $\mathcal{E}$. For example, a flat cover in the category of left R-modules is an $\mathcal{E}$-cover where $\mathcal{E}$ is the class of flat left R-modules.

Using a simple version of the argument in ([8], Proposition 4.1) we see that if $E_i \to X_i$ are $\mathcal{E}$-covers for $i = 1, 2, \dots, n$ for a class of objects such that $\oplus_{i=1}^{n} E_i \in \mathcal{E}$ then $\oplus_{i=1}^{n} E_i \to \oplus_{i=1}^{n} X_i$ is an $\mathcal{E}$-cover. Similarly $\oplus_{i=1}^{n} X_i \to \oplus_{i=1}^{n} E_i$ is an $\mathcal{E}$-envelope if each of $X_i \to E_i$, $i = 1, 2, \dots, n$ is and if $\oplus_{i=1}^{n} E_i \in \mathcal{E}$.

An $\mathcal{E}$-precover $E \to X$ in a category $\mathcal{C}$ is not necessarily an epimorphism. But if $\mathcal{C}$ has enough projective objects and these are in $\mathcal{E}$ then such an $E \to X$ is an epimorphism, for then any epimorphism $P \to X$ with P projective can be factored through $E \to X$. Similarly, if $\mathcal{C}$ has enough injectives and if $\mathcal{E}$ contains the injective objects of $\mathcal{C}$ then an $\mathcal{E}$-preenvelope $X \to E$ is a monomorphism. If $\mathcal{E}$ is a class of objects of $\mathcal{C}$ such that every object X of $\mathcal{C}$ has an $\mathcal{E}$-preenvelope $X \to E$, then we can construct a complex

$$0 \to X \to E^0 \to E^1 \to E^2 \to \cdots$$

(not necessarily exact) such that all $E^i \in \mathcal{E}$ and such that Hom $(F, -)$ makes the complex exact when $F \in \mathcal{E}$. Such a complex will be called a right $\mathcal{E}$-resolution of X. Then we get the usual uniqueness up to homotopy. If every X also has an $\mathcal{E}$-envelope then there is a minimal such resolution $0 \to X \to E^0 \to E^1 \to \cdots$. This means that if

$$\begin{array}{ccccccccc} 0 \to & X \to & E^0 \to & E^1 \to & E^2 \to \cdots \\ & \| & \downarrow & \downarrow & \downarrow \\ 0 \to & X \to & E^0 \to & E^1 \to & E^2 \to \cdots \end{array}$$

is commutative, each of the maps $E^n \to E^n$ is an isomorphism.

Dually we have left $\mathcal{E}$-resolutions or minimal left $\mathcal{E}$-resolutions of the objects of $\mathcal{C}$ when every object has an $\mathcal{E}$-precover or every object has an $\mathcal{E}$-cover. If $\mathcal{F}$ and $\mathcal{E}$ are two classes of objects of $\mathcal{C}$ such that every object has an $\mathcal{F}$-precover and every object has an $\mathcal{E}$-preenvelope, we say that Hom $(-,-)$ is *right balanced* by $\mathcal{F} \times \mathcal{E}$ if for every left $\mathcal{F}$-resolution

$$\cdots \to F_1 \to F_0 \to X \to 0$$

of an object X of $\mathcal{C}$, Hom $(-,E)$ makes the sequence exact for all $E \in \mathcal{E}$ and if for every right $\mathcal{E}$-resolution $0 \to Y \to E^0 \to E^1 \to \cdots$ of an object Y, Hom $(F,-)$ makes the sequence exact for all $E \in \mathcal{E}$.

The usual argument then allows us to construct right derived functors of Hom $(-,-)$, say $(R^n\mathrm{Hom})(X,Y)$ using either left $\mathcal{F}$-resolutions of X or right $\mathcal{E}$-resolutions of Y.

If, on the other hand, Hom $(E,-)$ makes each such $\cdots \to F_1 \to F_0 \to X \to 0$ exact when $E \in \mathcal{E}$ and Hom $(-,F)$ makes each such $0 \to Y \to E^0 \to E^1 \to \cdots$ exact when $F \in \mathcal{F}$ then we say Hom $(-,-)$ is *left balanced* by $\mathcal{E} \times \mathcal{F}$ and define the left derived functors $(L_n\mathrm{Hom})(Y,X)$ (see [9] for examples and applications of these notions).

When all objects have $\mathcal{E}$-preenvelopes we define the right $\mathcal{E}$-dimension of an object to be the least n such that there is a right $\mathcal{E}$-resolution of the form $0 \to Y \to E^0 \to \cdots \to E^n \to 0$ if there is such an n and say that right $\mathcal{E}$-dimension is ∞ otherwise. In a dual manner we define the left $\mathcal{F}$-dimension of objects when every object has an $\mathcal{F}$-precover.

For a class of objects $\mathcal{E}$ of $\mathcal{C}$ we let $\mathcal{E}^\perp$ be the class of objects Y such that $\mathrm{Ext}^1(E,Y) = 0$ for all $E \in \mathcal{E}$, i.e. such that all short exact sequences $0 \to Y \to Z \to E \to 0$ with $E \in \mathcal{E}$ split. And we let ${}^\perp\mathcal{E}$ consist of objects X such that $\mathrm{Ext}^1(X,E) = 0$ for all $E \in \mathcal{E}$. Note that $\mathcal{E} \subset {}^\perp({}^\perp\mathcal{E})$ and $\mathcal{E} \subset (\mathcal{E}^\perp)^\perp$. The following proposition is called Wakamatsu's lemma by Auslander and Reiten [2].

Proposition 1.1. If the class $\mathcal{E}$ is closed under extensions and $\phi : E \to X$ is an $\mathcal{E}$-cover of an object X of $\mathcal{C}$ then Ker $(\phi) \in \mathcal{E}^{\perp}$ and if $\psi : X \to E$ is an $\mathcal{E}$-envelope then Coker $(\psi) \in {}^{\perp}\mathcal{E}$.

For a proof see [13].

Now let R be any ring and let $\mathcal{C}$ be the abelian category of complexes of left R-modules. An object

$$\cdots \to X^{-1} \overset{\partial^{-1}}{\to} X^0 \overset{\partial^0}{\to} X^1 \overset{\partial^1}{\to} X^2 \to \cdots$$

will be denoted X. We let $Z^n(X) = \text{Ker } (\partial^n)$ and $B^n(X) = \text{Im } (\partial^{n-1})$. We will use the notation suggested by Brown [5] and let $\mathcal{H}om\,(X,Y)$ denote the usual complex formed from the two complexes X and Y. Then $Z^0\mathcal{H}om\,(X,Y)$ will be the group Hom (X,Y) of cochain maps (or morphisms) from X to Y.

Since $\mathcal{C}$ has enough injectives and projectives, we can compute the right derived functors $\text{Ext}^i(X,Y)$ of Hom $(-,-)$. (This extension functor is not the same as that of Avramov and Foxby [4]).

We will use subscripts to distinguish complexes and use superscripts to denote homogeneous components, e.g. X_n^m denotes the degree m term of the complex X_n. So using the topologist's trick, X_{-n} can be thought of as indexed by a superscript of n.

Let $\mathcal{E}$ be the class of exact complexes of left R-modules. We will use Foxby and Avramov's terminology [4] and call a complex P *DG-projective* if each P^n is projective and if $\mathcal{H}om(P,E)$ is an exact complex for all $E \in \mathcal{E}$. A complex I is called *DG-injective* if each I^n is injective and if $\mathcal{H}om\,(E,I)$ is exact for all $E \in \mathcal{E}$. The classes of DG-projective and DG-injective complexes are closed under taking finite sums and taking summands.

For examples, if I is such that all I^n are injective and such that for some n_0, $I^n = 0$ for $n \leq n_0$ (i.e. I is bounded below), then I is DG-injective. If l.gl.dim $R < \infty$ then any complex I with all I^n injective is DG-injective (see Dold [7] and Avramov, Foxby [4]). A bounded above complex P with all P^n projective is DG-projective and if l.gl.dim $R < \infty$, any P with all P^n projective is DG-projective. For any n, $X[n]$ denotes the complex such that $X[n]^m = X^{n+m}$ and whose boundary operators are $(-1)^n\partial^{n+m}$. If X is DG-projective (injective), then so is $X[n]$ for any n.

Given a morphism $f : X \to Y$ of complexes, we let $M(f)$ denote the mapping cone and write the associated exact sequence as $0 \to Y \to M(f) \to X[1] \to 0$ (see Dold [7]). Here $M(f)^n = X^{n+1} \oplus Y^n$ and $\partial(x,y) =$

$(-\partial x, f(x) + \partial y)$ for $(x, y) \in X^{n+1} \oplus Y^n$. A morphism $f : X \to Y$ of complexes such that $H(f) : H(X) \to H(Y)$ is an isomorphism will be called a *quasi-isomorphism.*

If M is any injective left R-module, then the complex $\cdots \to 0 \to M \xrightarrow{\text{id}} M \to 0 \to \cdots$ is injective (with the first M in the n-th place). In fact, any injective complex is uniquely up to isomorphism the direct sum of such complexes (one such complex for each $n \in Z$).

2. The Main Theorem and Remarks.

Theorem. *If R is any ring and $\mathcal{C}$ is the abelian category of complexes of left R-modules and $\mathcal{E}$ is the class of exact complexes of left R-modules, then the following hold:*

a) $^{\perp}\mathcal{E}$ consists of the DG-projective complexes
b) $\mathcal{E}^{\perp}$ consists of the DG-injective complexes
c) $(^{\perp}\mathcal{E})^{\perp} = \mathcal{E}$
d) $^{\perp}(\mathcal{E}^{\perp}) = \mathcal{E}$
e) $\mathcal{E} \cap {}^{\perp}\mathcal{E}$ is the class projective complexes
f) $\mathcal{E}^{\perp} \cap \mathcal{E}$ is the class of injective complexes
g) every complex X has an $\mathcal{E}$-cover (so an exact cover)
h) every complex X has an $\mathcal{E}^{\perp}$-envelope (so a DG-injective envelope by b))
i) every object X of $\mathcal{C}$ has an $\mathcal{E}$-preenvelope
j) every object X of $\mathcal{C}$ has an $^{\perp}\mathcal{E}$-precover
k) Hom $(-,-)$ is right balanced by $^{\perp}\mathcal{E} \times \mathcal{E}^{\perp}$
l) Hom $(-,-)$ is left balanced by $\mathcal{E} \times \mathcal{E}$
m) the projective dimension of each $P \in {}^{\perp}\mathcal{E}$ is 0 or ∞
n) the injective dimension of each $I \in \mathcal{E}^{\perp}$ is 0 or ∞
o) if $0 \to Y \to I_0 \to I_{-1} \to I_{-2} \to \cdots$ is a minimal right $\mathcal{E}^{\perp}$-resolution of X then each of $I_{-1}, I_{-2}, \ldots$ are injective complexes and I_0 will be injective if and only if $Y \in \mathcal{E}$
p) if $\cdots \to E_2 \to E_1 \to E_0 \to X \to 0$ is a minimal left $\mathcal{E}$-resolution of X the terms $E_1, E_2, \ldots$ are injective complexes. E_0 will be injective if and only if $X \in \mathcal{E}^{\perp}$ (so by b), if and only if X is DG-injective).

The proofs of a) - p) will be given separately in the next section. We comment that part of the interest in the properties a) - p) arises from their occurence in other settings (with some modifications in their statements). For example, if R is Iwanaga Gorenstein (i.e. left and right noetherian and of finite self injective dimension on either side [16]) and if $\mathcal{L}$ is the class of left R-modules of finite projective dim and $\mathcal{C}$ the category of left R-modules, then with $\mathcal{L}$ replacing $\mathcal{E}$ in a) - p) we get the corresponding claims but where ${}^{\perp}\mathcal{L}$ and $\mathcal{L}^{\perp}$ are the classes of Gorenstein projective and injective modules (see [10], [11], [12], [13]). If R is artinian and Iwanaga Gorenstein we can take $\mathcal{C}$ to be the category of finitely generated left R-modules. Then with the corresponding $\mathcal{L}$ (which is denoted P^{∞} by Auslander and Reiten) we get a) - p) still hold. In Auslander and Reiten's terminology, g) says that P^{∞} is contravariantly finite in $\mathcal{C}$.

If $\mathcal{C}$ is the category of left R-modules for any ring R and $\mathcal{O}$ is the class of objects of $\mathcal{C}$, then with $\mathcal{O}$ replacing $\mathcal{E}$ in the theorem we get a series of familiar statements (some trivial). Here $\mathcal{O}^{\perp}$ is the class of injective modules and so h) says every module has an injective envelope. The statement g) becomes a triviality ($M \xrightarrow{\text{id}} M$ is an $\mathcal{O}$-cover of the module M) and i) is the usual balance giving two ways to compute the groups $\operatorname{Ext}^i_R(M, N)$ for modules M and N.

3. The Proofs and Other Results.

We will let $\mathcal{C}$ be the category of cochain complexes of left R-modules and let $\mathcal{E}$ be the class of exact such complexes throughout this section . So the term complex will mean a cochain complex of left R-modules.

Lemma 3.1. *If $I \in \mathcal{E}^{\perp}$ then each I^n is injective.*

Proof. Let $S \subset M$ be a submodule of a left R-module and let $f : S \to I^n$ be linear. Form the pushout

$$\begin{array}{ccc} S & \to & M \\ \downarrow & & \downarrow \\ I^n & \to & I^n \oplus_S M \end{array}$$

and then form the obvious complex

$$\bar{I}: \quad \cdots \to I^{n-1} \to I^n \oplus_S M \to I^{n+1} \oplus M/S \to I^{n+2} \to \cdots$$

We have an exact sequence of complexes

$$0 \to I \to \bar{I} \to E \to 0$$

where E is the exact complex

$$\cdots \to 0 \to M/S \stackrel{\text{id}}{\to} M/S \to 0 \to \cdots .$$

By hypothesis, this sequence splits (in the category of complexes). But then the existence of a retraction $I^n \oplus_S M \to I^n$ at the module level shows that $S \to I^n$ can be extended to $M \to I^n$. □

We will use the easy:

Lemma 3.2. *Let $f : X \to Y$ be a morphism of complexes. Then the exact sequence $0 \to Y \to M(f) \to X[1] \to 0$ associated with the mapping cone $M(f)$ splits in $\mathcal{C}$ if and only if f is homotopic to 0.*

Proof. If the sequence splits and if $x \mapsto (x, -s(x))$ is a section, the definition of the boundary operator in $M(f)$ gives that s is a homotopy between f and 0. Conversely, if s is such a homotopy, $x \mapsto (x, -s(x))$ is a section for $M(f) \to X[1]$. □

Corollary 3.3. *If X is a complex, then $\mathcal{H}om\,(X, I)$ is exact for a complex I such that I^m is injective for all $m \in \mathbb{Z}$ if and only if $Ext^1(X, I[n]) = 0$ for all $n \in \mathbb{Z}$.*

Proof. $\mathcal{H}om\,(X, I)$ being exact is equivalent to the claim that for each n each morphism $X \to I[n]$ is homotopic to 0.

For $n \in Z$, any exact sequence $0 \to I[n] \to Y \to X \to 0$ of complexes splits at the module level since each $I[n]^m$ is injective. So this sequence is isomorphic to a sequence $0 \to I[n] \to M(f) \to X \to 0$ where $f : X[-1] \to I[n]$ (or $f : X \to I[n+1]$) is a morphism of complexes. Since n is arbitrary, the claim then follows from the preceeding Lemma. □

Proposition 3.4. *$\mathcal{E}^{\perp}$ is the class of DG-injective complexes.*

Proof. If $I \in \mathcal{E}^{\perp}$, by Lemma 3.1 all I^n are injective. Then by the previous Corollary we get that $\mathcal{H}om\,(E, I)$ is exact for all $E \in \mathcal{E}$ and so I

is DG-injective. □

Dual arguments give

Proposition 3.5. *$^{\perp}\mathcal{E}$ is the class of DG-projective complexes.*

Proposition 3.6. $^{\perp}(\mathcal{E}^{\perp}) = \mathcal{E}$.

Proof. Clearly $\mathcal{E} \subset {}^{\perp}(\mathcal{E}^{\perp})$, so let $E \in {}^{\perp}(\mathcal{E}^{\perp})$. We show that E is exact. By the previous Corollary, $\mathcal{H}om\,(E, I)$ is exact when I (and so also any $I[n]$, $n \in \mathbb{Z}$) is DG-injective. Let I be a complex of injective modules concentrated at 0, i.e. such that $I^n = 0$ if $n \neq 0$. Then I is DG-injective and so $\mathcal{H}om\,(E, I)$ is exact. But $\mathcal{H}om\,(E, I)$ is the complex

$$\cdots \to \mathrm{Hom}\,(E^1, I^0) \to \mathrm{Hom}\,(E^0, I^0) \to \mathrm{Hom}\,(E^{-1}, I^0) \to \cdots$$

Since the injective module I^0 is arbitrary, we see that E is exact. □.

Remark. If $I \in \mathcal{E}^{\perp}$ then $\mathrm{Ext}^i(E, I) = 0$ for all $E \in \mathcal{E}$ and all $i \geq 1$. For if $0 \to S \to P \to E \to 0$ is an exact sequence of complexes with P projective (and so exact), then S is exact. So $0 = \mathrm{Ext}^1(S, I) = \mathrm{Ext}^2(E, I)$. Using this proceedure and induction we get $\mathrm{Ext}^i(E, I) = 0$ for all $i \geq 1$.

Similarly, if $P \in {}^{\perp}\mathcal{E}$, $\mathrm{Ext}^i(P, E) = 0$ for all $i \geq 1$. Then by Proposition 3.4 and these remarks we see that if $0 \to I' \to I \to I'' \to 0$ is an exact sequence of complexes and I' and I'' are DG-injective, then so is I. And if I' and I are DG-injective, then so is I''.

Similarly, if $0 \to P' \to P \to P'' \to 0$ is an exact sequence of complexes, and P' and P'' are DG-projective, so is P. And when P and P'' are DG-projective, so is P'.

Proposition 3.7. *$\mathcal{E} \cap {}^{\perp}\mathcal{E}$ is the class of projective complexes and $\mathcal{E}^{\perp} \cap \mathcal{E}$ is the class of injective complexes.*

Proof. Let $I \in \mathcal{E}^{\perp} \cap \mathcal{E}$. Let $I \subset J$ where J is an injective complex. Then if E is exact, $\mathrm{Ext}^1(E, J/I) \cong \mathrm{Ext}^2(E, J) = 0$. Hence by Proposition 3.4, J/I is DG-injective. But then since I is exact $\mathrm{Ext}^1(I, J/I) = 0$ and so $0 \to I \to J \to J/I \to 0$ splits. Hence I is a direct summand of J and so is injective. Similar arguments give that $\mathcal{E} \cap {}^{\perp}\mathcal{E}$ consists of all the projective complexes. □

Arguments dual to those given in the proof of Proposition 3.6 give

Proposition 3.8. $({}^{\perp}\mathcal{E})^{\perp} = \mathcal{E}$

Theorem 3.9 (Spaltenstein [18]). *For every complex X there are quasi-isomorphisms $P \to X$ and $X \to I$ where P is DG-projective and I is DG-injective.*

Corollary 3.10. *Every complex X has a DG-projective precover and a DG-injective preenvelope.*

Proof. If $P \to X$ is as in the theorem, we first note we can assume $P \to X$ is surjective. For $\mathcal{C}$ has enough projectives. If $Q \to X$ is surjective with Q projective, we can replace $P \to X$ with $P \oplus Q \to X$. Since $H(Q) = 0$ we still have a homology isomorphism. Hence we have an exact sequence $0 \to E \to P \to X \to 0$ of complexes with E exact and P DG-projective. If P' is also DG-projective, then $\mathrm{Ext}^1(P', E) = 0$ by Proposition 3.5 and so $\mathrm{Hom}(P', P) \to \mathrm{Hom}(P', X) \to 0$ is exact. Hence $P \to X$ is a DG-projective precover.

Similarly we get that every X has a DG-injective preenvelope $X \to I$ □

We see that such precovers $P \to X$ preenvelopes $X \to I$ are necessarily surjective and injective respectively since projective complexes are DG-projective and injective complexes are DG-injective.

Lemma 3.11. *If $E \in \mathcal{E}$ and $E \to I$ is an injective envelope, then $E \to I$ is a DG-injective envelope.*

Proof. Let $0 \to E \to I \to E' \to 0$ be exact. Since E and I are exact, so is E'. Hence if I' is DG-injective, $\mathrm{Ext}^1(E', I') = 0$ and so $\mathrm{Hom}\,(I, I') \to \mathrm{Hom}\,(E, I') \to 0$ is exact. So since I is DG-injective, $E \to I$ is a DG-injective preenvelope. But then since it is an injective envelope, we see that it is a DG-injective envelope. □.

Theorem 3.12. *Every complex X has a DG-injective envelope $X \to I$. Such an envelope is injective and is a quasi-isomorphism. A morphism $X \to I$ of complexes is a DG-injective envelope of X if and only if it is injective, if I is DG-injective, if $X \to I$ is a quasi-isomorphism, and if there are no exact subcomplexes $E \subset I$ such that $X \cap E = 0$ and $E \neq 0$.*

Proof. The proof that X has a DG-injective envelope is a straightforward modification of the proof of Theorem 6.1 in [13]. In that proof we replace the class of modules of finite projective dimension with the class of exact complexes and note that this class of complexes is closed under inductive limits.

Since every DG-injective preenvelope $X \to I$ is injective, so is such an envelope. So then if $X \to I$ is a DG-injective envelope, $I/X \in {}^{\perp}(\mathcal{E}^{\perp}) = \mathcal{E}$ by Proposition 3.6 and by Lemma 3.11. But I/X exact implies that $X \to I$ is a homology isomorphism.

For an alternate argument note that if $X \to I'$ is an injective homology isomorphism with I' DG-injective (which exists by Spaltenstein), then an envelope $X \to I$ is a retract of $X \to I'$. Hence I/X is a retract of the exact I'/X and so is exact.

If $X \to I$ is a DG-injective envelope (so $X \to I$ is an injection), let $E \subset I$ be an exact complex with $X \cap E = 0$. Since the injective envelope, say J, of E is the DG-injective envelope and since I is DG-injective, $E \to I$ can be extended to a map $f : J \to I$. f is injective since E is essential in J. But then $X \cap f(J) = 0$ since E is essential in $f(J)$. Since J is also exact, we see we can assume the original E is injective. Then I/E is also DG-injective. Since $X \to I$ is a homology isomorphism and since $H(E) = 0$, $X \to I/E$ is a homology isomorphism.

This gives an exact sequence $0 \to X \to I/E \to F \to 0$ with F exact. But then $X \to I/E$ is a DG-injective preenvelope. Hence there is a commutative diagram

$$\begin{array}{ccc} & & I \\ & \nearrow & \downarrow \\ X & \to & I/E \\ & \searrow & \downarrow \\ & & I \end{array}$$

with $I \to I/E$ the canonical surjection. Since $X \to I$ is an envelope, $I \to I/E \to I$ is an automorphism of I and so $E = 0$.

Conversely, if $X \to I'$ is an injective DG-injective preenvelope of X, which is a homology isomorphism, there is a commutative diagram

$$\begin{array}{ccc} & & I \\ & \nearrow & \downarrow \\ X & \rightarrow & I' \\ & \searrow & \downarrow \\ & & I \end{array}$$

Since $I \to I' \to I$ is an automorphism of I, $J = \text{Ker}\,(I' \to I)$ is a retract of I and so is DG-injective. Since $X \to I$ and $X \to I'$ are homology isomorphisms, J is exact, so $J \in \mathcal{E}^{\perp} \cap \mathcal{E}$. So by Proposition 3.7, J is injective. Since $X \cap J = 0$ we get that $J = 0$ and so $I' \to I$ is an isomorphism. Hence $X \to I'$ is an envelope. □

Corollary 3.13. *The DG-injective envelope $X \to I$ of a complex X is such that I is injective if and only if X is exact.*

Proof. By Lemma 3.11, the DG-injective envelope of an exact complex is injective.

Now let $X \to I$ be a DG-injective envelope and assume I is injective. If $0 \to X \to I \to E \to 0$ is exact then we know E is an exact complex by the preceeding theorem, then since I and E are exact, so is X. □

Corollary 3.14. *If $0 \to X \to I_0 \to I_{-1} \to I_{-2} \to \cdots$ is a minimal DG-injective resolution of X then each I_{-i} for $i \geq 1$ is injective. I_0 is injective if and only if X is exact.*

Proof. By Theorem 3.12, $X \to I_0$ is injective and a quasi-isomorphism, so I_0/X is exact. So by Lemma 3.11, I_{-1} is injective. Repeating this argument we get $I_{-2}, I_{-3}, \ldots$ injective.

If X is exact, then since I_0/X is exact, I_0 is exact. So $I_0 \in \mathcal{E}^{\perp} \cap \mathcal{E}$ and so by Proposition 3.7, I_0 is injective. Conversely, if I_0 is injective, then I_0 is exact and so X is exact. □

Proposition 3.15. *For a DG-injective complex I, the following are equivalent:*

a) if $E \subset I$ is an exact complex, then $E = 0$
b) if $J \subset I$ is an injective subcomplex, then $J = 0$

c) for each n, $Z^n(I)$ is essential in I^n.

Proof. a) $\to$ b) since injective complexes are exact.

To argue b) $\to$ c) assume $Z^n(I)$ is not essential in I^n. Then there is an injective submodule $M \subset I^n$ with $Z^n(I) \cap M = 0$ and $M \neq 0$. But then $\partial^n|M$ is an injection. But then $\cdots 0 \to M \to \partial^n(M) \to 0 \to \cdots$ is non-zero injective subcomplex of I.

c) $\Rightarrow$ b).

An injective complex $J \subset I$ is the direct sum of complexes of the form $\cdots 0 \to M \overset{\text{id}}{\to} M \to 0 \to \cdots$ where M is an injective module. But by c), every such M must be 0. Hence $J = 0$.

b) $\Rightarrow$ a).

If $E \subset I$ is exact, then as in the proof of Theorem 3.12 we see that we have $E \subset J \subset I$ with J an injective envelope of E. But then $J = 0$ implies $E = 0$. □

Definition. A DG-injective complex I is said to be minimal if I satisfies a), b), c) above.

Proposition 3.16. *A DG-injective complex is the direct sum of an injective complex and a minimal DG-injective complex. This direct sum decomposition is unique up to isomorphism.*

Proof. For each n, we find a maximal submodule $J^n \subset I^n$ such that $J^n \cap \operatorname{Ker}(I^n \to I^{n+1}) = 0$. Clearly J^n is injective. So I has the injective subcomplex

$$\cdots \to 0 \to J^n \to \partial^n(J^n) \to 0 \to \cdots$$

The sum of these complexes is direct and is isomorphic to their external direct product. Hence this sum is injective. Let I_1 be the sum. Then for some I_2, we have $I = I_1 + I_2$ (direct). By construction we see that I_2 is minimal. Now suppose we have another decomposition $I = I_1' + I_2'$ (direct) with I_1' injective and I_2' minimal. Then $I_1 \cap I_2' = 0$ for $S = I_1 \cap I_2'$ an injective envelope of S in I_1 projects injectively to a submodule of I_2' (using the projection $I \to I_2'$ provided by the decomposition). Since I_2' is minimal, we see that $S = 0$.

Hence I_1 projects injectively into I_1' say with image $J \subset I_1'$. Then J is a summand of I_1, so we have a decomposition $I = J + K + I_2'$ (direct). But since I_1 then projects isomorphically onto J we have $I = I_1 + K + I_2'$ (direct). But then $I_2 \cong I/I_1 \cong K + I_2'$. Since K is injective and I_2 is minimal, $K = 0$ and so I_1 projects isomorphically onto I_1'. Hence $I = I_1 + I_2'$ (direct) and so $I_2 \cong I/I_1 \cong I_2'$. □

Corollary 3.17. *Given any complex X, there exists a quasi-isomorphism $X \to I$ with I a minimal DG-injective complex.*

Proof. By Spaltenstein, there is a quasi-isomorphism $X \to I$ with I a DG-injective complex. If $I = I_1 + I_2$ (direct) with I_1 injective and I_2 minimal, then since $H(I_1) = 0$, $X \to I_2$ is the desired homology isomorphism. □

Theorem 3.18. *Every complex X has an exact cover $E \to X$. Every exact cover $E \to X$ is surjective and Ker $(E \to X)$ is DG-injective. A morphism $E \to X$ of complexes is an exact cover of X if and only if E is exact, $E \to X$ is surjective and Ker $(E \to X)$ is a minimal DG-injective complex. If $E \to X$ is an exact cover, E is injective if and only if X is DG-injective.*

Proof. Given X, by Spaltenstein's theorem there is a homology isomorphism $f : X \to I$ with I DG-injective. So we have an exact sequence $0 \to I \to M(f) \to X[1] \to 0$. Then $M(f)$ is exact. Let $E = M(f)[-1]$. Then we have a surjective morphism $E \to X$ with the kernel $I[-1]$ DG-injective.

We claim $E \to X$ is an exact precover. For if E' is an exact complex, $\mathrm{Ext}^1(E', I[-1]) = 0$ by Proposition 3.4, and so Hom $(E', E) \to$ Hom $(E', X) \to 0$ is exact. We then note that $\mathcal{E}$ is closed under inductive limits and modify the proof of Theorem 2.1 in [8] to argue that X has an exact cover.

If $\phi : E \to X$ is an exact cover it is clearly surjective. $I =$ Ker $(E \to X) \in \mathcal{E}^{\perp}$ by Proposition 1.1 and so is DG-injective. If I were not minimal, by Proposition 3.16 it would have an injective subcomplex $I_1 \neq 0$. Then if $E = I_1 + E_2$ (direct), the projection map $E \to E_2$ gives rise to a map $f : E \to E$ such that $\phi \circ f = \phi$ with f not an automorphism. Hence I must be minimal.

Conversely, suppose $\phi : E \to X$ is a surjective morphism with E exact and $I = \text{Ker}\,(\phi)$ a minimal DG-injective complex. Let $\bar{E} \to X$ be an exact cover. Then noting that $E \to X$ is an exact precover, we see there is a commutative diagram

$$\begin{array}{ccc} \bar{E} & & \\ \downarrow & \searrow & \\ E & \rightarrow & X \\ \downarrow & \nearrow & \\ \bar{E} & & \end{array}$$

But $\bar{E} \to E \to \bar{E}$ is an automorphism of E. So Ker $(E \to \bar{E})$ is a direct summand of E (so is exact) contained in $I = \text{Ker}\,(\phi)$. Since I is minimal, Ker $(E \to \bar{E}) = 0$ and so $E \to \bar{E}$ is an isomorphism. Hence $E \to X$ is a cover.

Now suppose X is DG-injective and $E \to X$ an exact cover with $I =$ Ker $(E \to X)$. Then since I is DG-injective and $0 \to I \to E \to X \to 0$ is exact, E is DG-injective. But since E is also exact, by Proposition 3.7, E is injective.

If on the other hand $E \to X$ is an exact cover and E is injective, then with I as above, the exactness of $0 \to I \to E \to X \to 0$ along with the fact that I and E are DG-injective give that X is DG-injective. □

Example. Let X be a complex concentrated at 0, so $X = \cdots 0 \to X^0 \to 0 \to \cdots$. If $0 \to X^0 \to X^1 \to X^2 \to X^3 \to \cdots$ is a minimal injective resolution of the module X^0, then the complex $E = \cdots 0 \to 0 \to X^0 \to X^1 \to X^2 \to X^3 \to \cdots$ with the map $E \to X$ which is the identity on X^0 is an exact cover of X since we see that the kernel $\cdots \to 0 \to 0 \to X^1 \to X^2 \to \cdots$ is a minimal DG-injective complex.

Corollary 3.19. *If $\cdots \to E_2 \to E_1 \to E_0 \to X \to 0$ is a minimal exact left resolution of a complex X, then each E_i, $i \geq 1$ is injective and E_0 is injective if and only if X is DG-injective.*

Proof. Immediate from the above theorem. □

Theorem 3.20. *In $\mathcal{C}$, Hom $(-,-)$ is right balanced by ${}^{\perp}\mathcal{E} \times \mathcal{E}^{\perp}$.*

Proof. If $0 \to Y \to I_0 \to I_{-1} \to I_{-2} \to \cdots$ is any right $\mathcal{E}^{\perp}$-resolution of

Y and $P \in {}^{\perp}\mathcal{E}$, we must show that Hom $(P, -)$ leaves the sequence exact. From the construction of the sequence $0 \to Y \to I_0 \to I_{-1} \to I_{-2} \to \cdots$ we see that we only need show that if $Y \to I$ is a DG-injective preenvelope and if $C = \text{Coker}\,(Y \to I)$, then Hom $(P, I) \to$ Hom $(P, C) \to 0$ is exact. Given another such DG-injective preenvelope $Y \to \bar{I}$ with cokernel $\bar{C}$, by the definition of preenvelopes we see that if the diagram

$$\begin{array}{ccccccc} 0 \to & Y \to & \bar{I} \to & \bar{C} \to 0 \\ & \| & \downarrow & \downarrow \\ 0 \to & Y \to & I \to & C \to 0 \end{array}$$

is commutative, then the vertical maps give a homotopy equivalence between the two complexes (of complexes). Hence if Hom $(P, -)$ leaves either exact, it leaves the other exact. So we only need show Hom $(P, -)$ leaves an exact sequence $0 \to Y \to I \to E \to 0$ exact where E is DG-injective and E is exact (and such a sequence exists by Theorem 3.12). Now let $Q \to E$ be a surjection with Q a projective complex. Then $K = \text{Ker}\,(Q \to E)$ is exact so $\text{Ext}^1(P, E) = 0$. Hence any map $P \to E$ can be lifted to a map $P \to Q$. Since any map $Q \to E$ has a lifting $Q \to I$ we see that any map $P \to E$ can be lifted to a map $P \to I$. Thus Hom $(P, I) \to$ Hom $(P, E) \to 0$ is exact.

A dual argument shows that if $0 \to D \to P \to X \to 0$ is exact with $P \to X$ a DG-projective precover then whenever I is DG-injective, Hom $(P, I) \to$ Hom $(D, I) \to 0$ is exact. □

Lemma 3.21. *Let I be a DG-injective complex and let $id : I \to I$ give the exact sequence $0 \to I \to M(\text{id}) \to I[1] \to 0$. Then $M(\text{id})$ is injective and $M(\text{id}) \to I[1]$ is an exact precover. If I is minimal then $I \to M(\text{id})$ is an injective envelope and $M(\text{id}) \to I[1]$ is an exact cover.*

Proof. Since id is a homology isomorphism, $M(\text{id})$ is exact. Since I and $I[1]$ are DG-injective, so is $M(\text{id})$. Then by Proposition 3.7 $M(\text{id})$ is injective. Since $\text{Ext}^1(E, I) = 0$ for any exact E, Hom $(E, M(\text{id})) \to$ Hom $(E, I[1]) \to 0$ is exact. So $M(\text{id}) \to I[1]$ is an exact precover.

Now suppose I is minimal. If $M(\text{id})$ is not an injective envelope of I, then there is an injective subcomplex $J \subset M(\text{id})$ with $I \cap J = 0$ and $J \neq 0$. But J maps isomorphically to a subcomplex of $I[1]$. Since J is exact and $I[1]$ is minimal, we get a contradiction.

When I is minimal we get that $M(\text{id}) \to I[1]$ is an exact cover by

Theorem 3.18. □

Corollary 3.22. *A DG-injective complex I has injective dimension 0 or ∞.*

Proof. Immediate. □

Proposition 3.23. *If P is a DG-projective complex and if $id : P \to P$ has the associated exact sequence $0 \to P \to M(\mathrm{id}) \to P[1] \to 0$, then $M(\mathrm{id})$ is projective and $P \to M(\mathrm{id})$ is an exact preenvelope of P.*

Proof. A dual argument. □

Corollary 3.24. *A DG-projective complex has projective dimension 0 or ∞.*

Proof. Immediate. □

Lemma 3.25. *Let X, Y be complexes, $f : X \to Y$ a morphism and let the exact sequence $0 \to Y[-1] \to M(\mathrm{id})[-1] \to Y \to 0$ be gotten from $id : Y \to Y$. Then $f : X \to Y$ has a lifting $Y \to M(\mathrm{id})[-1]$ if and only if f is homotopic to 0.*

Proof. Given such a lifting, let s be the composition of $Y \to M(\mathrm{id})[-1]$ and the projection map $M(\mathrm{id})[-1] \to Y[-1]$. From the definition of the boundary operator in $M(\mathrm{id})[-1]$ it follows that s is a homotopy between f and 0.

Conversely such an s gives a lifting $Y \to M(\mathrm{id})[-1]$ in the obvious manner. □

Proposition 3.26. *Every complex X has an $\mathcal{E}$-preenvelope.*

Proof. Let $\mathcal{N}$ be an infinite cardinal number with $\mathrm{Card}\, X \leq \mathcal{N}$ and $\mathrm{Card}\,(R) \leq \mathcal{N}$.

Now let E be an exact complex and S a subcomplex of E with $\mathrm{Card}(S) \leq \mathcal{N}$. We claim there is an exact subcomplex $E' \subset E$ with $S \subset E'$ and $\mathrm{Card}\,(E') \leq \mathcal{N}$. For let $S_0 = S$. Each element of $Z(S_0)$ is in $B(E)$. If we choose $y_x \in E$ with $\partial y_x = x$ for each $x \in Z(E)$ and let S_1 be generated by all y_x and by S_0, we get Card $(S_1) \leq \mathcal{N}$, $S_0 \subset S_1$ and $Z(S_0) \subset B(S_1)$. If we repeat this proceedure with S_1 replacing S_0, we get a corresponding S_2 with $S_1 \subset S_2$. So then we get a sequence $S = S_0 \subset S_1 \subset S_2 \subset \cdots$ of subcomplexes with $Z(S_n) \subset B(S_{n+1})$ and Card $(S_n) \leq \mathcal{N}$ for all n. If $E' = \bigcup_{n=0}^{\infty} S_n$ then E' satisfies our requirements.

Now let U be some set with Card $U \geq \mathcal{N}$. We consider the set of all pairs (F, f) where F is an exact complex, $F \subset U$ (as a set) and $f : X \to F$ is a morphism. If $E = \Pi F$ (the product over the set of (F, f)) and if $\phi : X \to E$ is the morphism such that $\phi : X \to E$ composed with the projection map onto the (F, f) component is f we see from the preceeding that $\phi : X \to E$ is an exact preenvelope. □

Proposition 3.27. *In $\mathcal{C}$, Hom $(-,-)$ is left balanced by $\mathcal{E} \times \mathcal{E}$.*

Proof. By the preceeding Proposition and by Theorem 3.18 we get the required resolutions. The other requirement of balance is immediate. □

4. Minimal Injective Resolutions (see [4] or [15] where the term DG-injective resolution is used).

It is known that if X is bounded below, then X has a so-called "minimal injective resolution". This means that there is a quasi-isomorphism $X \to I$ where I is a minimal DG-injective complex. Such an I will also be bounded below.

If I_1, I_2 are bounded below minimal DG-injective complexes, any quasi-isomorphism $I_1 \to I_2$ is an isomorphism. This fact can be used to prove that a "minimal injective resolution" $X \to I$ as above is unique up to isomorphism.

In this section we show that as a result of Spaltenstein's work we no longer need to restrict complexes to those which are bounded below to get these results.

We use quotation marks with the term "minimal injective resolution" since the definition above is not in agreement with our usage in this paper. In the rest of the section we will refer to these resolutions by giving their properties.

Lemma 4.1. *If I is a minimal DG-injective complex and $E \to I$ is an* *exact cover with kernel K, then K is a minimal DG-injective complex and $K \to E$ is an injective envelope of K. And conversely if K is a minimal DG-injective complex and $K \to E$ is an injective envelope with cokernel I, then I is a minimal DG-injective complex and $E \to I$ is an exact cover.*

Proof. Given a minimal DG-injective complex I, let $E = M(\mathrm{id})[-1]$ for $\mathrm{id} : I \to I$. Then we have the exact sequence

$$0 \to I[-1] \to M(\mathrm{id})[-1] \to I \to 0$$

Clearly $I[-1]$ is a minimal DG-injective complex. Also $E = M(\mathrm{id})[-1]$ is exact (and injective since E is also DG-injective). Then by Theorem 3.18 $E \to I$ is an exact cover. So letting $K = I[-1]$ we get the first part of the claim.

If $K \to E$ were not an injective envelope, there would be an injective subcomplex $J \subset E$ with $K \cap J = 0$, $J \neq 0$. But then $E \to I$ would map J isomorphically onto an injective (so exact) subcomplex of I. Since I is minimal, we get a contradiction. So $K \to E$ is an injective envelope.

For the second part of the Proposition, let $I = K[1]$ and construct

$$0 \to I[-1] \to M(\mathrm{id})[-1] \to I \to 0$$

as above. Then $I[-1] = K$ and by the preceeding $K = I[-1] \to M(\mathrm{id})[-1] = E$ is an injective envelope and we also have $I = K[1]$ (i.e. the cokernel of $K \to E$) is a minimal DG-injective complex. □

Lemma 4.2. *Let I be a DG-injective complex and let $f : X \to I$ be a morphism of complexes. Then the following are equivalent:*

a) f is homotopic to 0.
b) f can be factored through an exact cover $E \to I$.
c) f can be factored through some exact complex F.
d) f can be factored through some injective complex J.
e) $H(f) = 0$.

Proof. For $\mathrm{id} : I \to I$ we have the exact sequence $0 \to I[-1] \to M(\mathrm{id})[-1] \to I \to 0$. By Lemma 3.21, $M(\mathrm{id})[-1] \to I$ is an exact cover of I. By Lemma 3.25 f has a lifting to $M(\mathrm{id})[-1]$ (i.e. f can be factored through $M(\mathrm{id})[-1] \to I$) if and only if homotopic to 0. Hence a) and b) are equivalent.

Since by Lemma 3.21, the exact cover $E = M(\text{id})[-1] \to I$ of I is injective (and of course exact), b) implies c) and d).

But each of c) and d) implies b) since $E \to I$ is a cover (and since injective complexes I are exact).

Each of a), b), c) and d) implies e). And e) implies d) by Lemmas 3.21 and 3.25. □

Corollary 4.3. *If I is a minimal DG-injective complex and $S =$ End(I) (the endomorphism ring of I) then any $g \in S$ which is homotopic to 0 is in the Jacobson radical of S.*

Proof. By the preceeding Lemma, any such g can be factored through $E \to I$ where $E \to I$ is an exact cover of I. Hence, using the notation of Lemma 4.1 we get a commutative diagram

$$\begin{array}{ccccccc} 0 \to & K \to & E \to & I \to 0 \\ & \| & \downarrow & \downarrow \text{id} + g \\ 0 \to & K \to & E \to & I \to 0 \end{array}$$

Since $K \to E$ is an injective envelope, $E \to E$ is an isomorphism. Hence $\text{id} + g$ is an isomorphism. Since the set of $g \in S$ homotopic to 0 is a two-sided ideal of S, the claim is established. □

Theorem 4.4. *If I_1 and I_2 are minimal DG-injective complexes, any quasi-isomorphism $f : I_1 \to I_2$ is an isomorphism.*

Proof. Let $I_1 \to J$ be an injective where J is an injective complex. Then $I_1 \to I_2 \oplus J$ is an injective homology isomorphism, so we have an exact sequence

$$0 \to I_1 \to I_2 \oplus J \to E \to 0$$

with E exact. This splits by Proposition 3.4. Hence we get a homology isomorphism $I_2 \to I_2 \oplus J \to I_1$ where $I_2 \oplus J \to I_1$ is a retraction guaranteed by the splitting. Call this morphism g. Then with $\text{id} : I_1 \to I_1$ we have $g \circ f - \text{id}$ can be factored through the exact complex J. Hence $g \circ f - \text{id}$ is homotopic to 0. Hence by Corollary 4.3 $\text{id} + (g \circ f - \text{id}) = g \circ f$ is an automorphism of I_1.

Similarly, there is a morphism $h : I_1 \to I_2$ so that $h \circ g$ is an automorphism of I_2.

Hence f is an isomorphism of complexes. □

Theorem 4.5. *Given any complex X, there is a homology isomorphism $X \to I$ where I is a minimal DG-injective complex. If $\phi_1 : X \to I_1$ and $\phi_2 : X \to I_2$ are two such morphisms, then there is a morphism $f : I_1 \to I_2$ such that $f \circ \phi_1$ is homotopic to ϕ_2. Any such f is an isomorphism. Furthermore f is unique up to homotopy.*

Proof. In Corollary 3.17 it was established that given X there is a homology isomorphism $X \to I$ with I a minimal DG-injective complex.

Now let $\phi_1 : X \to I_1$ and $\phi_2 : X \to I_2$ be as in the theorem. Let $X \to J$ be an injective envelope of X. Then by familiar arguments we see that $X \to I_1 \oplus J$ and $X \to I_2 \oplus J$ are DG-injective preenvelopes, so there is a commutative diagram

$$\begin{array}{ccc} & & I_1 \oplus J \\ & \nearrow & \\ X & & \downarrow \\ & \searrow & \\ & & I_2 \oplus J \end{array}$$

Let f be the composition $I_1 \to I_1 \oplus J \to I_2 \oplus J \to I_2$. Then f is a quasi-isomorphism and $f \circ \phi_1 - \phi_2$ can be factored through J. Hence by Lemma 4.2, $f \circ \phi_1$ is homotopic to ϕ_2 and by Theorem 4.4 f is an isomorphism. The uniqueness of f follows from Lemma 4.2. □

Remarks. Given a complex X and a DG-injective envelope $X \to I$, a decomposition $I = I_1 + I_2$ (direct) with I_1 injective and I_2 minimal gives a "minimal injective resolution" $X \to I_2$. Then I_1 plays the role of Auslander's δ-invariant in his theory of maximal Cohen-Macaulay approximations (see [6] and [17]). In order that $I_1 = 0$ it is necessary that X have no exact subcomplexes $E \neq 0$. It is not known if this condition guarantees that $I_1 = 0$ but we conjecture that this is so.

If $Y \to I$ is a "minimal injective resolution" of Y the hypercohomology groups $\mathrm{Ext}^n(X, Y)$ are the homology modules of $\mathcal{H}om\,(X, I)$ where X is any complex (these are not the groups $\mathrm{Ext}^n(X, Y)$ used in this paper).

The results in this paper can be used to show that these hypercohomology groups can be computed using a complete resolution (resembling Tate's complete resolutions (see [5])) associated with Y. Let $0 \to I \to E \to Y \to 0$

be an exact cover. Let

$$0 \to I \to E_{-1} \to E_{-2} \to \cdots$$

be an injective resolution of I and let $\cdots \to E_1 \to E_0 \to I \to 0$ be a minimal left $\mathcal{E}$-resolution of I. Pasting along I, we have the complex

$$\cdots \to E_2 \to E_1 \to E_0 \to E_{-1} \to E_{-2} \to \cdots$$

Then the homology groups of

$$\cdots \to \operatorname{Hom}(X, E_1) \to \operatorname{Hom}(X, E_0) \to \operatorname{Hom}(X, E_{-1}) \to \cdots$$

are isomorphic to the hypercohomology groups $\operatorname{Ext}^i(X, Y)$ as in [14].

If $R \neq 0$ then the category of complexes of left R-modules has infinite global dimension even if R has finite left global dimension (i.e. the category of left R-modules has finite global dimension). However if l.gl.dim $R = n < \infty$, then the right DG-injective dimension of every complex is at most n.

Similarly, if R is Iwanaga Gorenstein and the injective dimension of R as an R-module is n on the left and right, then every module has right Gorenstein injective dimension at most n even if l.gl.dim $R = \infty$. (see [10] for the definition of Gorenstein injective modules). This is another of the many similarities which hold between the categories of complexes over various rings and the categories of modules over Iwanaga Gorenstein rings.

In both situations there is a natural way to go from an infinite dimensional homological setting to a finite dimensional relative homological setting.

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