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THE PROPERTY SPECIAL (DF) FOR UNIT-REGULAR RINGS

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In this paper, we shall consider about the property special (DF) for unit-regular rings. A unit-regular ring R is said to have the property special (DF) provided that nP is directly finite for every directly finite projective R-module P and each positive integer n. It is unknown that the property special (DF) holds or not for unit-regular rings.
In §1, we shall give the sufficient condition about the property special (DF) for unit-regular rings (Theorem 1.5), and using this resulut, we show that all factor rings of the following unit-regular rings have the property special (DF):

(1) regular rings whose primitive factor rings are artinian

(2) unit-regular rings satisfying general comparability

(3) $\aleph_0$-continuous regular rings

(4) simple directly finite regular rings satisfying weak comparability (Corollary 1.8)

(5) unit-regular rings satisfying s-comparability (Corollary 1.11).

In §2, we shall give the characterization of the property special (DF) for unit-regular rings (Theorem 2.2).

Throughout this paper, R is a ring with identity and R-modules are unitary right R-modules.

§1. Preliminaries and the property special (DF).

Notation. For two R-modules P and Q, we use $P \lesssim Q$ (resp. $P \lesssim_{\oplus} Q$) to mean that P is isomorphic to a submodule of Q (resp. a direct summand of Q). For a submodule P of an R-module Q, $P \leq_{\oplus} Q$ means that P is a direct summand of Q. For a cardinal number k and an R-module P, kP denotes a direct sum of k-copies of P.

Definition. An R-module P is *directly finite* provided that P is not isomorphic to a proper direct summand of itself. If P is not directly finite, then P is said to be *directly infinite.* A ring R is said to be *unit-regular* ring if, for each $x \in R$, there exists a unit element (i.e. invertible element)

u of R such that $xux = x$.

All basic results concerning regular rings can be found in a K.R. Goodearl's book [3].

Lemma 1.1 ([1, Lemma 3.3]). *Let R be a regular ring, and let A and B be finitely generated projective R-modules with $A \lesssim nB$ for some positive integer n. Then there exists a decomposition $A = A_1 \oplus A_2 \oplus \ldots \oplus A_n$ such that $B \gtrsim A_1 \gtrsim A_2 \gtrsim \ldots \gtrsim A_n$.*

Lemma 1.2 ([5,Lemma 1]). *Let R be a unit-regular ring, and P be a projective R-module with a cyclic decomposition $P = \oplus_{i \in I} P_i$. Then the following conditions* (a) $\sim$ (c) *are equivalent:*

(a) *P is directly infinite.*

(b) *There exists a nonzero principal right ideal X of R such that $X \lesssim \oplus_{I \setminus \{i_1, \ldots, i_n\}} P_i$ for all finite subsets $\{i_1, \ldots, i_n\}$ of I.*

(c) *There exists a nonzero principal right ideal X of R such that $\aleph_0 X \lesssim_{\oplus} P$.*

Definition. A regular ring R is said to have the property *special* (DF) provided that nP is directly finite for every directly finite projective R-module P and each positive integer n.

Note that a regular ring R has the property special (DF) if and only if $2P$ is directly finite for every directly finite projective R-module P.

Lemma 1.3 ([3,Proposition 2.18]). *Let I be a two-sided ideal in a regular ring R, and let $f_1, f_2, \ldots$ be a finite or countably infinite sequence of orthogonal idempotents in R/I. Then there exist orthogonal idempotents $e_1, e_2, \ldots \in R$ such that $\bar{e}_n = f_n$ for all n. Moreover, if $f_1 + \ldots + f_k = 1$ for some positive integer k, then the e_n can be chosen so that $e_1 + \ldots + e_k = 1$.*

Lemma 1.4 ([3,Proposition 2.19]). *Let I be a two-sided ideal in a regular ring R, and let $A_1, \ldots, A_n$ be finitely generated projective R-modules such that the modules A_i/A_iI are pairwise isomorphic. Then there exist decompositions $A_i = B_i \oplus C_i$ for each i such that the modules B_i are pairwise isomorphic and each $C_i = C_iI$.*

Theorem 1.5. *Let R be a unit-regular ring. Assume that there exists a positive integer n such that for all $x, y \in R$*

$$(2n)(xR) \lesssim 2(yR) \text{ implies } xR \lesssim yR$$

Then R/I has the property special (DF) *for all two-sided ideals I of R.*

Proof. We can take a positive integer m such that $n \leq 2^{m-1}$.
(Claim I). Assume that there exists a nonzero principal right ideal X of R such that

$$X \lesssim 2Q_1, \qquad (1)$$
$$X \lesssim 2Q_2, \qquad (2)$$
$$\dots$$

and

$$X \lesssim 2Q_{m+1} \qquad (m+1)$$

,where the Q_i is a finitey generated projective R-module. Then we claim that $X \lesssim Q_1 \oplus Q_2 \oplus \dots \oplus Q_{m+1}$. From Lemma 1.1 and (1), there exists a decomposition $X = X_1 \oplus X_1^*$ with $Q_1 \gtrsim X_1 \gtrsim X_1^*$, and so $2X_1^* \lesssim X$. By (2), we see that $X_1^* \lesssim X \lesssim 2Q_2$. Using Lemma 1.1, there exists a decomposition $X_1^* = X_2 \oplus X_2^*$ with $Q_2 \gtrsim X_2 \gtrsim X_2^*$, and so $4X_2^* \lesssim 2X_1^* \lesssim X$. Continuing this procedure to (m), there exists a decomposition $X = X_1 \oplus X_2 \oplus \dots \oplus X_m \oplus X_m^*$ such that $2^m X_m^* \lesssim X$ and $X_1 \oplus X_2 \oplus \dots \oplus X_m \lesssim Q_1 \oplus Q_2 \oplus \dots \oplus Q_m$. We have that $X \lesssim 2Q_{m+1}$ by $(m+1)$, and so there exists a decomposition $X = X_{m+1} \oplus X_{m+1}^*$ with $Q_{m+1} \gtrsim X_{m+1} \gtrsim X_{m+1}^*$. Hence $(2n)X_m^* \leq 2^m X_m^* \lesssim X \lesssim 2X_{m+1}$. By the assumption, we have that $X_m^* \lesssim X_{m+1} \lesssim Q_{m+1}$ and $X = X_1 \oplus \dots \oplus X_m \oplus X_m^* \lesssim Q_1 \oplus \dots \oplus Q_m \oplus Q_{m+1}$ as desired.
(Claim II). Set $\bar{R} = R/I$, and so $\bar{R}$ is a unit-regular ring. We claim that $\bar{R}$ has the property special (DF). Let $\bar{P}$ be a projective $\bar{R}$-module with a cyclic decomposition $\bar{P} = \oplus_{i \in I} \bar{P}_i$, and assume that $2\bar{P}$ is directly infinite. From Lemma 1.2, there exists a nonzero principal right ideal $\bar{X}$ of $\bar{R}$ such that

$$\bar{X} \lesssim 2\bar{Q}_1,$$
$$\bar{X} \lesssim 2\bar{Q}_2,$$
$$\dots$$

$$\bar{X} \lesssim 2\bar{Q}_{m+1} \quad \text{and}$$
$$\bar{Q}_1 \oplus \bar{Q}_2 \oplus \ldots \oplus \bar{Q}_{m+1} \leq P,$$

where the $\bar{Q}_i$ is a finite direct sum of $\bar{P}_i'$s. Now we put $\bar{Q}_i = \bar{P}_1^i \oplus \ldots \oplus \bar{P}_k^i$ for $i = 1, \ldots, m+1$. Then $\bar{X} \lesssim 2(\bar{P}_1^i \oplus \ldots \oplus \bar{P}_k^i)$, and so using [3, Theorem 2.8], $\bar{X} \cong (\bar{P}_{11}^i \oplus \bar{P}_{12}^i) \oplus \ldots \oplus (\bar{P}_{k1}^i \oplus \bar{P}_{k2}^i)$ for some submodules $\bar{P}_{j1}^i$ and $\bar{P}_{j2}^i$ of $\bar{P}_j^i$ for $j = 1, \ldots, k$. Putting $\bar{C}_j^i = \bar{P}_{j1}^i \cap \bar{P}_{j2}^i$, we have decompositions $\bar{P}_{j1}^i = \bar{Y}_{j1}^i \oplus \bar{C}_j^i$ and $\bar{P}_{j2}^i = \bar{Y}_{j2}^i \oplus \bar{C}_j^i$ such that $\bar{P}_{j1}^i + \bar{P}_{j2}^i = \bar{Y}_{j1}^i \oplus \bar{Y}_{j2}^i \oplus \bar{C}_j^i \leq \bar{P}_j^i$. Hence there exists a decomposition $\bar{X} = \oplus_{j=1}^k (\bar{X}_{j1}^i \oplus \bar{X}_{j2}^i \oplus \bar{D}_{j1}^i \oplus \bar{D}_{j2}^i)$ such that

$$\bar{X}_{j1}^i \cong \bar{Y}_{j1}^i,$$
$$\bar{X}_{j2}^i \cong \bar{Y}_{j2}^i \quad \text{and}$$
$$\bar{D}_{j1}^i \cong \bar{D}_{j2}^i \cong \bar{C}_j^i.$$

Putting $\bar{X}^j = \bar{X}_{j1}^i \oplus \bar{X}_{j2}^i \oplus \bar{D}_{j1}^i \oplus \bar{D}_{j2}^i$, we have a decomposition $X^j = X_{j1}^i \oplus X_{j2}^i \oplus D_{j1}^i \oplus D_{j2}^i$ $(\leq R)$ from Lemma 1.3. Note that $\bar{D}_{j1}^i \cong \bar{D}_{j2}^i$. Using Lemma 1.4 and [3,Proposition 2.17], we have decompositions $D_{j1}^i = E_{j1}^i \oplus F_{j1}^i$ and $D_{j2}^i = E_{j2}^i \oplus F_{j2}^i$ such that

$$E_{j1}^i \cong E_{j2}^i,$$
$$F_{j1}^i = F_{j1}^i I \quad \text{and} \quad F_{j2}^i = F_{j2}^i I.$$

Hence there exists a principal right ideal X_i of R such that $X_i = \oplus_{j=1}^k X_j \lesssim \oplus_{j=1}^k 2(X_{j1}^i \oplus X_{j2}^i \oplus F_{j1}^i \oplus F_{j2}^i \oplus E_{j1}^i)$ and $\bar{X}_i = \bar{X}$ for $i = 1, \ldots, m+1$. Noting that $\bar{X} = \bar{X}_1 = \ldots = \bar{X}_{m+1} (\neq 0)$, there exist decompositions $X_i = B_i \oplus C_i$ such that the B_i's are pairwise isomorphic and each $C_i = C_i I$ by Lemma 1.4. Then we see that $B_i \not\lesssim I$. Putting $B_1 = eR$ for some idempotent e in R, we have that $\bar{e}\bar{R} = \bar{X}_i = \bar{X}$, and we have decompositions

$$eR \lesssim \oplus_{j=1}^k 2(X_{j1}^1 \oplus X_{j2}^1 \oplus F_{j1}^1 \oplus F_{j2}^1 \oplus E_{j1}^1),$$
$$eR \lesssim \oplus_{j=1}^k 2(X_{j1}^2 \oplus X_{j2}^2 \oplus F_{j1}^2 \oplus F_{j2}^2 \oplus E_{j1}^2), \ldots$$

and

$$eR \lesssim \oplus_{j=1}^k 2(X_{j1}^{m+1} \oplus X_{j2}^{m+1} \oplus F_{j1}^{m+1} \oplus F_{j2}^{m+1} \oplus E_{j1}^{m+1}).$$

Using (Claim I), we have that

$$eR \lesssim \oplus_{i=1}^{m+1} \oplus_{j=1}^k (X_{j1}^i \oplus X_{j2}^i \oplus F_{j1}^i \oplus F_{j2}^i \oplus E_{j1}^i),$$

and so

$$\begin{aligned}(0 \neq)\bar{X} = \bar{e}\bar{R} &\lesssim (\oplus_{i=1}^{m+1} \oplus_{j=1}^{k} (\bar{X}_{j1}^{i} \oplus \bar{X}_{j2}^{i} \oplus \bar{E}_{j1}^{i})) \\ &\lesssim (\oplus_{i=1}^{m+1} \oplus_{j=1}^{k} (\bar{Y}_{j1}^{i} \oplus \bar{Y}_{j2}^{i} \oplus \bar{C}_{j}^{i})) \\ &\lesssim (\oplus_{i=1}^{m+1} \oplus_{j=1}^{k} \bar{P}_{j}^{i}) \\ &\lesssim \bar{Q}_1 \oplus \ldots \oplus \bar{Q}_{m+1}.\end{aligned}$$

There exists a decomposition $\bar{P} = \bar{Q}_1 \oplus \ldots \oplus \bar{Q}_{m+1} \oplus \bar{P}'$ for some submodule $\bar{P}'$ of $\bar{P}$, and we can apply above discussion to $\bar{P}'$ by Lemma 1.2. Continuing this procedure, we see that $\aleph_0 \bar{X} \lesssim_{\oplus} \bar{P}$ and so $\bar{P}$ is directly infinite as desired. This theorem is complete.

Corollary 1.6. *Let R be a unit-regular ring with the property that for all finitely generated projective R-modules A and B*

$$nA \lesssim nB \text{ for some positive integer } n \text{ implies } A \lesssim B.$$

Then R/I has the property special (DF) *for all two-sided ideals I of R.*

Note. It is OPEN PROBREM 27[3] that the assumption of Corollary 1.6 holds or not for unit-regular rings R. We see that unit-regular rings R have the property special (DF) if this open problem holds.

From Corollary 1.6 and [3, OPEN PROBLEM 27(p.347)], we see that all factor rings of the following rings (1) $\sim$ (3) have the property special (DF):

(1) regular rings whose primitive factor rings are artinian.
(2) unit-regular rings satisfying general comparability.
(3) $\aleph_0$-continuous regular rings.

Now we shall show that simple directly finite regular rings satisfying weak comparability have the property special (DF).

The following definition was given by K.C. O'Meara[6].

Definition. A regular ring R satisfies *weak comparability* if each nonzero $y \in R$, there exists a positive integer n such that for all $x \in R$

$$n(xR) \lesssim R \text{ mplies } xR \lesssim yR.$$

Notation. Given finitely generated projective R-modules P and Q we write $P \prec Q$ to mean that P is isomorphic to a proper submodule of Q.

Lemma 1.7 ([2, Corollary 4.4]). *Let R be a stably finite simple regular ring. Then the following conditions are equivalent:*

(a) *R satisfies weak N^*-comparability.*

(b) *R satisfies the doubling condition.*

(c) *R satisfies weak comparability.*

(d) *The class of finitely generated projective R-modules is strictly unperforated.*

(e) *For any $x, y \in R$ and a positive integer n, if $n(xR) \prec n(yR)$, then $xR \prec yR$.*

Corollary 1.8. *Let R be a simple directly finite regular ring satisfying weak comparability. Then R has the property special* (DF).

Proof. It is well-known from [6, Theorem 1] that R is a unit-regular ring. Assume that $(2 \times 2)(xR) \lesssim 2(yR)$, and let $x \neq 0$. Then we have that $2(xR) \prec 2(yR)$ from the directly finiteness of R, and so $xR \prec yR$ by Lemma 1.7. Thus $xR \lesssim yR$. This corollary holds from Theorem 1.5.

Definition. Let R be a regular ring, and let s be a positive integer. Then R is said to satisfy *s-comparability* provided that for any $x, y \in R$, either $xR \lesssim s(yR)$ or $yR \lesssim s(xR)$ holds.

Lemma 1.9. *Let R be a unit-regular ring, and let $x, y \in R$. If $(2n)(xR) \lesssim 2(yR)$ for some positive integer n, then we have a decomposition $xR = x_1R \oplus \dots \oplus x_{2^n}R$ such that $n(x_iR) \lesssim yR$ for $i = 1, \dots, 2^n$.*

Proof. Noting that $xR \lesssim (2n)(xR) \lesssim 2(yR)$, there exists a decomposition $xR \cong a_1R \oplus a_1'R$ for some right ideals a_1R and $a_1'R$ of yR. Putting $c_1R = a_1R \cap a_1'R (\leq yR)$, there exist decompositions $a_1R = b_1R \oplus c_1R$ and $a_1'R = b_1'R \oplus c_1R$, and so $a_1R + a_1'R = b_1R \oplus b_1'R \oplus c_1R (\leq yR)$. Hence there exists a direct summand y_1R of yR such that $yR = (a_1R + a_1'R) \oplus y_1R$. Noting again that $(2n)(xR) \lesssim 2(yR)$, we see that

$$\begin{aligned} &2(n-1)(xR) \oplus 2(b_1R \oplus b_1'R \oplus c_1R \oplus c_1R) \\ &\quad \cong 2(n-1)(xR) \oplus 2(xR) \lesssim 2(yR) \end{aligned}$$

$$= 2(b_1R \oplus b_1'R \oplus c_1R) \oplus 2(y_1R).$$

Using the cancellation property [3, Theorem 4.14], we have that $xR \lesssim 2(n-1)(xR) \lesssim 2(y_1R)$. Continuing this procedure, we have decompositions

$$\begin{aligned} xR &\cong a_1R \oplus a_1'R \\ &\cong a_2R \oplus a_2'R \\ &\dots \qquad\qquad\qquad\qquad (\sharp) \\ &\cong a_nR \oplus a_n'R \end{aligned}$$

such that

$$\begin{aligned} &y_iR = (a_{i+1}R + a_{i+1}'R) \oplus y_{i+1}R, \\ &2(n-i-1)(xR) \lesssim 2(y_{i+1}R) \text{ for } i = 1, \dots, n-1 \text{ and} \\ &a_1^*R \oplus \dots \oplus a_n^*R \leq yR, \end{aligned}$$

where a_i^*R equals a_iR or $a_i'R$ for each i. Thus, from [3,Theorem 2.8], we have a decomposition $xR = x_1R \oplus \dots \oplus x_{2^n}R$ such that $n(x_iR) \lesssim yR$ for $i = 1, \dots, 2^n$.

Theorem 1.10. *Let R be a unit-regular ring satisfying s-comparability. Then for all $x, y \in R$,*

$$2(s+1)(xR) \lesssim 2(yR) \ \textit{implies} \ xR \lesssim yR.$$

Proof. Putting $D = (2s)(xR)$, we have $2(xR) \oplus D \lesssim 2(yR)$. Note that $xR \lesssim 2(yR)$, and using Lemma 1.1, there exists a decomposition $xR = x_1^*R \oplus x_1^{**}R$ such that $yR \gtrsim x_1^*R \gtrsim x_1^{**}R$. Hence we have a decomposition $yR = y_1^*R \oplus y_1^{**}R$ such that $y_1^*R \cong x_1^*R$, and so

$$\begin{aligned} &2(x_1^*R) \oplus 2(x_1^{**}R) \oplus D = 2(xR) \oplus D \\ &\lesssim 2(yR) = 2(y_1^*R) \oplus 2(y_1^{**}R). \end{aligned}$$

Using the cancellation property [3, Theorem 4.14], we have

$$\begin{aligned} &2(x_1^{**}R) \oplus D \lesssim 2(y_1^{**}R) \text{ and} \\ &2(x_1^{**}R) \lesssim x_1^{**}R \oplus x_1^*R = xR. \end{aligned}$$

Noting that $2(x_1^{**}R) \oplus D \lesssim 2(y_1^{**}R)$, we can use above discussion to this equation. We put $x^*R, x^{**}R, y^*R$ and $y^{**}R$ as following:

$$x^*R \ = x_1^*R \oplus \dots \oplus x_n^*R$$

$$\begin{aligned} x^{**}R &= x_n^{**}R, \\ y^{*}R &= y_1^{*}R \oplus \ldots \oplus y_n^{*}R \text{ and} \\ y^{**}R &= y_n^{**}R. \end{aligned}$$

Then, we have decompositions $xR = x^*R \oplus x^{**}R$ and $yR = y^*R \oplus y^{**}R$ such that

$$\begin{aligned} & x^*R \cong y^*R, \\ & 2^n(x^{**}R) \lesssim xR \text{ and} \\ & D = (2s)(xR) \lesssim 2(y^{**}R) \text{ for } n = 1, 2, \ldots . \end{aligned}$$

From Lemma 1.9, we have a decomposition $xR = x_1R \oplus \ldots \oplus x_{2^s}R$ such that $s(x_iR) \lesssim y^{**}R$ for $i = 1, \ldots, 2^s$. We may assume that $x^{**}R \neq 0$. Then there exists a positive integer $i (1 \leq i \leq 2^s)$ such that $x^{**}R \lesssim s(x_iR)$. Otherwise $x^{**}R \not\lesssim s(x_iR)$ for $i = 1, \ldots, 2^s$, we have that $x_iR \lesssim s(x^{**}R)$ from s-comparability. Taking a positive integer n such that $(s \times 2^s) \leq 2^{n-1}$, we have that

$$\begin{aligned} xR = x_1R \oplus \ldots \oplus x_{2^s}R &\lesssim (s \times 2^s)(x^{**}R) \\ &\lesssim 2^{n-1}(x^{**}R) \prec 2^n(x^{**}R) \lesssim xR, \end{aligned}$$

which contradicts the directly finiteness of xR. Thus we see that $xR = x^*R \oplus x^{**}R \lesssim x^*R \oplus s(x_iR) \lesssim y^*R \oplus y^{**}R = yR$ as desired.

Combining Theorems 1.5 and 1.10, we have the following.

Corollary 1.11. *Let R be a unit-regular ring satisfying s-comparability. Then R/I has the property special* (DF) *for all two-sided ideals I of R.*

§2. The criterion of the property special (DF).

In this section, we shall give the criterion of the property special (DF) for a unit-regular ring.

Let R be a regular ring. For a nonzero finitely generated projective R-module P, we consider the following condition:
(**) For each nonzero finitely generated submodule X of P, and each decompositions

$$X = A_1 \oplus B_1,$$

$$A_i = A_{2i} \oplus B_{2i} \text{ and}$$
$$B_i = A_{2i+1} \oplus B_{2i+1}$$

with

$$A_i \gtrsim B_i \text{ for each } i = 1, 2, \dots ,$$

there exists a nonzero finitely generated submodule Y of P such that $Y \lesssim \oplus_{i=n}^{\infty} A_i$ for all positive integers n.

Note 1. We can take above Y as a nonzero right ideal of R.

Note 2. Let P be a nonzero finitely generated projective R-module over a regular ring R satisfying the condition (**), and let Q be a nonzero direct summand of P. Then Q satisfies the condition (**).

Notation. Let P be a finitely generated projective R-module over a regular ring R. We put $L(P)$ to denote the lattice of all finitely generated submodules of P, partially ordered by inclusion.

Lemma 2.1([3, Proposition 2.4] and [4, Lemma 5]). *Let P be a finitely generated projective R-module over a regular ring R, and set $T = End_R(P)$. Then*

(a) *There exists a lattice isomorphism $F : L(T_T) \to L(P)$, given by the rule $F(J) = JP$. For $A \in L(P)$, we have $F^{-1}(A) = \{f \in T \mid fP \leq A\}$.*

(b) *For $J, K \in L(T_T)$, we have $J \cong K$ if and only if $F(J) \cong F(K)$.*

(c) *For $J, K \in L(T_T)$, we have $J \lesssim K$ if and only if $F(J) \lesssim F(K)$.*

(d) *For $J, K \in L(T_T)$ such that $J \oplus K \in L(T_T)$, we have that $F(J \oplus K) = F(J) \oplus F(K)$. For $A, B \in L(P)$ such that $A \oplus B \in L(P)$, we have $F^{-1}(A \oplus B) = F^{-1}(A) \oplus F^{-1}(B)$.*

Theorem 2.2. *Let P be a finitely generated projective R-module over a unit-regular ring R, and set $T = End_R(P)$. Then the following conditions $(a) \sim (c)$ are equivalent:*

(a) *T has the property special* (DF).

(b) *T_T satisfies the condition* (**).

(c) *P satisfies the condition* (**).

Proof. (b) $\Leftrightarrow$ (c) follows from Lemma 2.1. (a) $\Rightarrow$ (b). We assume that

(a) holds and T_T does not satisfy the condition (**). Then there exist a nonzero principal right ideal X of T and decompositions

$$X = A_1 \oplus B_1,$$
$$A_i = A_{2i} \oplus B_{2i} \text{ and}$$
$$B_i = A_{2i+1} \oplus B_{2i+1}$$

with

$$A_i \gtrsim B_i \text{ for } i = 1, 2, \ldots$$

such that there exists a positive integer n with $Y \not\lesssim \oplus_{i=n}^{\infty} A_i$ for each nonzero right ideal Y of T. Hence we have that $2(\oplus_{i=1}^{\infty} A_i) \;{}_{\oplus}\!\gtrsim (\oplus_{i=1}^{\infty} A_i) \oplus (\oplus_{i=1}^{\infty} B_i) \cong \aleph_0 X$, and so $2(\oplus_{i=1}^{\infty} A_i)$ is directly infinite, which contradicts the directly finiteness of $\oplus_{i=1}^{\infty} A_i$ by Lemma 1.2. (b) $\Rightarrow$ (a). Assume that (b) holds and T_T does not satisfy the property special (DF), i.e., there exists a directly finite projective T-module Q such that $2Q$ is directly infinite. Let $Q = \oplus_{i \in I} Q_i$ be a cyclic decomposition of Q. We see that I is an infinite set by [3,Proposition 5.2 and Corollary 4.7]. Noting that $2Q$ is directly infinite, there exists a nonzero principal right ideal X of T such that $X \lesssim 2(\oplus_{n=1}^{n(1)} Q_i)$ for some positive integer $n(1)$, and so we have a decomposition $X = A_1 \oplus B_1$ such that $\oplus_{i=1}^{n(1)} Q_i \gtrsim A_1 \gtrsim B_1$ by Lemma 1.1. Likewise, noting that there exist positive integers $n(2)$ and $n(3)$ with $n(1) < n(2) < n(3)$ such that $A_1 \lesssim 2(\oplus_{i=n(1)+1}^{n(2)} Q_i)$ and $B_1 \lesssim 2(\oplus_{i=n(2)+1}^{n(3)} Q_i)$, we have decompositions $A_1 = A_2 \oplus B_2$ and $B_1 = A_3 \oplus B_3$ such that

$$\oplus_{i=n(1)+1}^{n(2)} Q_i \gtrsim A_2 \gtrsim B_2 \text{ and}$$
$$\oplus_{i=n(2)+1}^{n(3)} Q_i \gtrsim A_3 \gtrsim B_3.$$

Continuing this procedure, we have that $\oplus_{i=1}^{\infty} A_i \lesssim_{\oplus} Q$. From the directly finiteness of Q, we see that $\oplus_{i=1}^{\infty} A_i$ is directly finite, and so there exists a positive integer m such that $Y \not\lesssim \oplus_{i=m}^{\infty} A_i$ for each nonzero principal right ideal Y of T by Lemma 1.2, which contradicts the condition (**). Thus this theorem is complete.

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